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Fractional integrals, derivatives and integral equations with weighted Takagi–Landsberg functions*

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Abstract. In this paper, we find fractional Riemann–Liouville derivatives for the Takagi–Landsberg functions. Moreover, we introduce their generalizations called weighted Takagi–Landsberg functions, which have arbitrary bounded coefficients in the expansion under Schauder basis. The class of weighted Takagi–Landsberg functions of order $H > 0$ on $[0, 1]$ coincides with the class of H-Hölder continuous functions on $[0, 1]$. Based on computed fractional integrals and derivatives of the Haar and Schauder functions, we get a new series representation of the fractional derivatives of a Hölder continuous function. This result allows us to get a new formula of a Riemann–Stieltjes integral. The application of such series representation is a new method of numerical solution of the Volterra and linear integral equations driven by a Hölder continuous function.

Keywords: Takagi–Landsberg functions, fractional derivatives, Schauder basis, Volterra equation.

1 Introduction

Our aim is to get a broad class of continuous functions on $[0, 1]$, which are nowhere differentiable but have fractional derivatives. The prominent example is the Takagi–Landsberg function with Hurst parameter $H > 0$ introduced in [10], given by

$$x^H(t) = \sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} e_{m,k}(t), \quad t \in [0, 1],$$

where $\{e_{m,k},\, m \in \mathbb{N}_0,\, k = 0, \ldots, 2^m - 1\}$ are the Faber–Schauder functions on $[0, 1]$. In the present paper, we find the fractional derivatives of the Takagi–Landsberg functions, and for other properties, we refer to the surveys [2] and [9]. In the case $H = 1/2$, the function x^H is known as the Takagi function.

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There are several generalizations of the function x^H. In the paper of Mishura and Schied [13], the signed Takagi–Landsberg functions of the form

$$\sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t), \quad t \in [0,1] \text{ with } \theta_{m,k} \in \{-1,+1\}$$

are considered. Their results concern the maximum, the maximizers, and the modulus of continuity. Particularly, it was shown that $\max_{t\in[0,1]} x^H(t) = 1/(3(1-2^{-H}))$. The case of $H = 1/2$ is considered in [16], where the connections to the Fölmer's pathwise Itô calculus (e.g. [5]) is also described. The signed Takagi–Landsberg functions form the wide class of continuous nondifferentiable functions with finite pth variations.

We want to extend this class further and introduce so-called weighted Takagi–Landsberg functions for which we let $\theta_{m,k}$ be arbitrary bounded coefficients. We study the continuity properties of such functions and show that they are H-Hölder continuous on $[0,1]$. Moreover, we prove that every Hölder continuous function is a weighted Takagi–Landsberg function, which immediately gives a new series representation for the Hölder continuous functions, which we call a Takagi–Landsberg representation. Then we compute the fractional Riemann–Liouville derivatives and integrals of the Faber–Schauder functions, and therefore we obtain the fractional derivatives of the (weighted) Takagi–Landsberg functions. Such a new series representation of the fractional derivative for Hölder continuous functions is very promising for further development of the continuous functions without derivatives. Particularly, the Takagi–Landsberg representation gives a new method for numerical solution of the integral equations involving Hölder continuous functions.

As an example, we consider the Volterra integral equation with fractional noise, called also fractional Langevin equation, e.g. [4,12]. This equation is of interest for modelling of anomalous diffusion in physics (e.g. [8,11]) and financial markets (e.g. [17]). Our method of its numerical solution allows us to reduce it to the system of linear algebraic equations, which is computationally effective. We prove that the numerical solution of the fractional Langevin equation, due to our method, approaches the theoretical solution, and illustrate this by numerical examples.

We also obtain the series expansion of the Riemann–Stieltjes integral applying methodology based on fractional Rieman–Liouville integrals introduced in [18] and developed in [14]. As an illustration, we consider the linear differential equation driven by Hölder continuous function and prove that its numerical solution, due to our method, tends to the exact solution in the specific norm. Moreover, the method gives directly the coefficients in the Takagi–Landsberg expansion of the solution in contrast to other procedures. This result is also supported by numerical examples. Nonlinear equations can also be solved by the application of the Takagi–Landsberg representation and will be covered by further research.

The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and Schauder basis. In Section 3, we compute fractional Riemann–Liouville integrals and derivatives of the Haar functions (Section 3.1) and the Faber–Schauder functions (Section 3.2).

In Section 4, we introduce the weighted Takagi–Landsberg functions and obtain the series representations of their Riemann–Liouville derivatives. The series expansion of the Riemann–Stieltjes integral is given in Section 5. In Section 6, we consider the application of the Takagi–Landsberg representation for the solution of the Volterra integral (Section 6.1) and linear differential (Section 6.2) equations. The numerical results are presented in Sections 6.3 and 6.4.

2 Preliminaries

First, we recall the definitions of fractional Riemann–Liouville integrals and derivatives and their basic properties. Let $f \in L_1([0,T])$. We define left- and right-sided fractional integrals of order $\alpha > 0$ on $(0,T)$ by

$$\big[I^\alpha_{0+}f\big](t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u,$$

$$\big[I^\alpha_{T-}f\big](t) := \frac{1}{\Gamma(\alpha)}\int_t^T (u-t)^{\alpha-1} f(u)\,\mathrm{d}u,$$

respectively (cf. [15, Def. 2.1]).

Define the spaces of functions that can be represented as fractional integrals:

$$I^\alpha_+\big(L_p\big([0,T]\big)\big) := \big\{f \in L_1\big([0,T]\big)\colon \exists \varphi \in L_p\big([0,T]\big) \text{ such that } f = I^\alpha_{0+}\varphi\big\},$$
$$I^\alpha_-\big(L_p\big([0,T]\big)\big) := \big\{f \in L_1\big([0,T]\big)\colon \exists \varphi \in L_p\big([0,T]\big) \text{ such that } f = I^\alpha_{T-}\varphi\big\}.$$

From [15, formula (2.19)] it follows that $I^\alpha_+(L_p([0,T])) = I^\alpha_-(L_p([0,T]))$ for $1 < p < 1/\alpha$.

For the functions from $I^\alpha_+(L_1([0,T])) = I^\alpha_-(L_1([0,T]))$, we define the left- and right-sided fractional Riemann–Liouville derivatives on $(0,T)$ of order α by

$$\big[D^\alpha_{0+}f\big](t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-u)^{-\alpha} f(u)\,\mathrm{d}u,$$

$$\big[D^\alpha_{T-}f\big](t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^T (u-t)^{-\alpha} f(u)\,\mathrm{d}u.$$

Recall that the Faber–Schauder functions are defined as

$$e_\emptyset(t) := t, \quad e_{0,0}(t) := \big(\min\{t,\, 1-t\}\big)^+, \quad e_{m,k}(t) := 2^{-m/2} e_{0,0}\big(2^m t - k\big)$$

for $t \in \mathbb{R}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$. They can be expressed in terms of Haar functions $H_{m,k}$ as

$$e_{m,k}(t) = \int_0^t H_{m,k}(s)\,\mathrm{d}s = \big[I^1_{0+}H_{m,k}\big](t),$$

where $H_{m,k}(s) = 2^{m/2}\mathbf{1}_{J_{m,k}}(s) - 2^{m/2}\mathbf{1}_{J_{m,\,k+0.5}}(s)$, and $J_{m,t} := (t/2^m, (t+0.5)/2^m]$, $m \in \mathbb{N}_0$, $k = 0, \dots, 2^m - 1$, $t \in [0,1]$.

The Faber–Schauder functions form a Schauder basis in $C([0,1])$ and produce the following expansion of a function $f \in C([0,1])$ (e.g. [7]):

$$f(t) = f(0) + \big(f(1) - f(0)\big)t + \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} 2^{m/2} a_{m,k} e_{m,k}(t), \quad t \in [0,1], \tag{1}$$

with coefficients

$$a_{m,k} = 2f\left(\frac{k+0.5}{2^m}\right) - f\left(\frac{k+1}{2^m}\right) - f\left(\frac{k}{2^m}\right).$$

3 Fractional derivatives of the Takagi–Landsberg function

3.1 Haar functions

In this section, we calculate the fractional integrals and derivatives of the Haar functions.

Lemma 1. *Let $\alpha > 0$, $T > 0$, $k, m \in \mathbb{N}_0$ and $0 \leqslant k < 2^m$. Then for $t \in (0,1)$, we have*

$$I_{0+}^{\alpha} H_{m,k}(t) = \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(\left(t - \frac{k}{2^m}\right)_+^{\alpha} - \left(t - \frac{k+0.5}{2^m}\right)_+^{\alpha} + \left(t - \frac{k+1}{2^m}\right)_+^{\alpha} \right), \tag{2}$$

and for $t \in (0,T)$,

$$\begin{aligned} I_{T-}^{\alpha} H_{m,k}(t) = \frac{2^{m/2}}{\Gamma(1+\alpha)} &\left(2\left(T \wedge \frac{k+0.5}{2^m} - t\right)_+^{\alpha} - \left(T \wedge \frac{k}{2^m} - t\right)_+^{\alpha} \right. \\ &\left. - \left(T \wedge \frac{k+1}{2^m} - t\right)_+^{\alpha} \right). \end{aligned} \tag{3}$$

Proof. If $t < k/2^m$, then $I_{0+}^{\alpha} H_{m,k}(t) = 0$. Let $t \in J_{m,k}$, then

$$I_{0+}^{\alpha} H_{m,k}(t) = \frac{2^{m/2}}{\Gamma(\alpha)} \int_{k/2^m}^{t} (t-u)^{\alpha-1}\,\mathrm{d}u = \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(t - \frac{k}{2^m}\right)^{\alpha}. \tag{4}$$

Let $t \in J_{m,\,k+0.5}$, then

$$\begin{aligned} I_{0+}^{\alpha} H_{m,k}(t) &= \frac{2^{m/2}}{\Gamma(\alpha)} \left(\int_{k/2^m}^{(k+0.5)/2^m} (t-u)^{\alpha-1}\,\mathrm{d}u - \int_{(k+0.5)/2^m}^{t} (t-u)^{\alpha-1}\,\mathrm{d}u \right) \\ &= \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(\left(t - \frac{k}{2^m}\right)^{\alpha} - 2\left(t - \frac{k+0.5}{2^m}\right)^{\alpha} \right). \end{aligned} \tag{5}$$

If $t > (k+1)/2^m$, then

$$I_{0+}^{\alpha} H_{m,k}(t) = \frac{2^{m/2}}{\Gamma(\alpha)} \left(\int\limits_{k/2^m}^{(k+0.5)/2^m} (t-u)^{\alpha-1}\,\mathrm{d}u - \int\limits_{(k+0.5)/2^m}^{(k+1)/2^m} (t-u)^{\alpha-1}\,\mathrm{d}u \right)$$
$$= \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(\left(t - \frac{k}{2^m}\right)^{\alpha} - 2\left(t - \frac{k+0.5}{2^m}\right)^{\alpha} + \left(t - \frac{k+1}{2^m}\right)^{\alpha} \right). \quad (6)$$

Summarizing (4)–(6), we get statement (2).

Now prove relation (3). Obviously, if $T < k/2^m$ or $t > (k+1)/2^m$, then $I_{T-}^{\alpha} H_{m,k}(t) = 0$. Let $t \in J_{m,\,k+0.5}$, then

$$I_{T-}^{\alpha} H_{m,k}(t) = -\frac{2^{m/2}}{\Gamma(\alpha)} \int\limits_{t}^{T\wedge(k+1)/2^m} (u-t)^{\alpha-1}\,\mathrm{d}u = -\frac{2^{m/2}}{\Gamma(1+\alpha)} \left(T \wedge \frac{k+1}{2^m} - t\right)^{\alpha}. \quad (7)$$

Let $t \in J_{m,k}$, then

$$I_{T-}^{\alpha} H_{m,k}(t) = \frac{2^{m/2}}{\Gamma(\alpha)} \left(\int\limits_{t}^{T\wedge(k+0.5)/2^m} (u-t)^{\alpha-1}\,\mathrm{d}u - \int\limits_{T\wedge(k+0.5)/2^m}^{T\wedge(k+1)/2^m} (u-t)^{\alpha-1}\,\mathrm{d}u \right)$$
$$= \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(2\left(T \wedge \frac{k+0.5}{2^m} - t\right)^{\alpha} - \left(T \wedge \frac{k+1}{2^m} - t\right)^{\alpha} \right). \quad (8)$$

If $t < k/2^m$, then

$$I_{T-}^{\alpha} H_{m,k}(t)$$
$$= \frac{2^{m/2}}{\Gamma(\alpha)} \left(\int\limits_{T\wedge k/2^m}^{T\wedge(k+0.5)/2^m} (u-t)^{\alpha-1}\,\mathrm{d}u - \int\limits_{T\wedge(k+0.5)/2^m}^{T\wedge(k+1)/2^m} (u-t)^{\alpha-1}\,\mathrm{d}u \right)$$
$$= \frac{2^{m/2}}{\Gamma(1+\alpha)} \left(-\left(T \wedge \frac{k}{2^m} - t\right)^{\alpha} + 2\left(T \wedge \frac{k+0.5}{2^m} - t\right)^{\alpha} - \left(T \wedge \frac{k+1}{2^m} - t\right)^{\alpha} \right). \quad (9)$$

Summarizing (7)–(9), we get statement (3). □

For $m \in \mathbb{N}_0$, $k = 0, \dots, 2^m - 1$, $H > 0$, denote by

$$\tau_{1,2^m+k}^{\alpha}(t) = \frac{(t - \frac{k}{2^m})_+^{\alpha} - 2(t - \frac{k+0.5}{2^m})_+^{\alpha} + (t - \frac{k+1}{2^m})_+^{\alpha}}{\Gamma(1+\alpha)}, \quad t \in [0,1],\ \alpha \geqslant 0, \quad (10)$$

$$\tau_{2,2^m+k}^{\alpha}(t,T)$$
$$= \frac{2(T \wedge \frac{k+0.5}{2^m} - t)_+^{\alpha} - (T \wedge \frac{k}{2^m} - t)_+^{\alpha} - (T \wedge \frac{k+1}{2^m} - t)_+^{\alpha}}{\Gamma(1+\alpha)}, \quad t \in [0,1],\ \alpha \geqslant 0.$$

Then $I_{0+}^{\alpha} H_{m,k}(t) = 2^{m/2} \tau_{1,2^m+k}^{\alpha}(t)$ and $I_{T-}^{\alpha} H_{m,k}(t) = 2^{m/2} \tau_{2,2^m+k}^{\alpha}(t,T)$.

Remark 1. We give immediate bounds for $\tau^\alpha_{1,2^m+k}$ and $\tau^\alpha_{2,2^m+k}$. For instance, for any $m \in \mathbb{N}_0$ and $k = 0, \dots, 2^{m-1}$, we have

$$\begin{aligned}\left|\tau^\alpha_{1,2^m+k}(t)\right| &= \left|I^\alpha_{0+}2^{-m/2}H_{m,k}(t)\right| \leqslant I^\alpha_{0+}\left[\mathbf{1}_{J_{m,k}} + \mathbf{1}_{J_{m,\,k+0.5}}\right](t) \\ &\leqslant I^\alpha_{0+}\left[\mathbf{1}_{J_{m,k}} + \mathbf{1}_{J_{m,\,k+0.5}}\right](1) = \frac{1}{\Gamma(\alpha)}\int\limits_{k/2^m}^{(k+1)/2^m}(1-u)^{\alpha-1}\,\mathrm{d}u \\ &= \frac{1}{\Gamma(1+\alpha)}\left(\left(1-\frac{k+1}{2^m}\right)^\alpha - \left(1-\frac{k}{2^m}\right)^\alpha\right) \leqslant \frac{2^{-m\alpha}}{\Gamma(1+\alpha)}.\end{aligned}$$

Similarly, we get that $|\tau^\alpha_{1,2^m+k}(t)| \leqslant I^\alpha_{T-}\mathbf{1}_{J_{m,k}\cup J_{m,\,k+0.5}}(t) \leqslant I^\alpha_{1-}\mathbf{1}_{J_{m,k}\cup J_{m,\,k+0.5}}(0) \leqslant 2^{-m\alpha}/\Gamma(1+\alpha)$.

Remark 2. One can observe that functions $\tau^\alpha_{1,2^m+k}$ and $\tau^\alpha_{2,2^m+k}$ can be written in terms of a fractional Gaussian noise with Hurst index $H \in (0,1)$, that is a centered Gaussian process with the covariance function

$$\mathbf{E}\left[Y^H(t)Y^H(0)\right] = C_H(t) = \frac{1}{2}\left(|t+1|^{2H} - 2|t|^{2H} + |t-1|^{2H}\right), \quad t \in \mathbb{R}.$$

Indeed,

$$\tau^\alpha_{1,2^m+k}(t) = \frac{2^{1-\alpha}}{2^{m\alpha}\Gamma(1+\alpha)}C_{\alpha/2}\left(2^{m+1}t - 2k - 1\right)$$

if $t \geqslant (k+1)/2^m$, and

$$\tau^\alpha_{2,2^m+k}(t,T) = -\frac{2^{1-\alpha}}{2^{m\alpha}\Gamma(1+\alpha)}C_{\alpha/2}\left(2k + 1 - 2^{m+1}t\right)$$

if $t \leqslant k/(2^m)$ and $T \geqslant (k+1)/(2^m)$.

Since $\alpha/2 \in (0,1/2)$, if $\alpha \in (0,1)$, we can study properties of the integrals $I^\alpha_{0+}H_{m,k}$ and $I^\alpha_{T-}H_{m,k}$ using the known results about C_H with $H < 1/2$. For instance, it is known that $C_H(t) < 0$ if $t \geqslant 1$ and $H \in (0,1/2)$. Further, we use the fact that function C_H in the case $H < 1/2$ is absolutely integrable and monotonically increasing on $[1,+\infty)$, e.g. [3, Sect. 3.2]

Remark 3. We provide some auxiliary bounds for functions $\tau^\alpha_{1,\,2^m+k}$ and $\tau^\alpha_{2,\,2^m+k}$. Let $\alpha \in (0,1)$, then $C_{\alpha/2}(x)$ is negative and monotonically increasing for $x \geqslant 1$, which gives that $|C_{\alpha/2}(x)| \leqslant |C_{\alpha/2}(\lfloor x\rfloor)|$, $x \geqslant 1$. Therefore, $\tau^\alpha_{1,\,2^m+k}(t)$ is negative for $k \leqslant \lfloor 2^m t\rfloor - 1$, and

$$\frac{2^{m\alpha}\Gamma(1+\alpha)}{2^{1-\alpha}}\left|\tau^\alpha_{1,2^m+k}(t)\right| = \left|C_{\alpha/2}\left(2^{m+1}t-2k-1\right)\right| \leqslant \left|C_{\alpha/2}\left(2\lfloor 2^m t\rfloor-2k-1\right)\right|.$$

Similarly, if $\lfloor 2^m t\rfloor + 1 \leqslant k \leqslant \lfloor 2^m T\rfloor - 1$, then

$$\frac{2^{m\alpha}\Gamma(1+\alpha)}{2^{1-\alpha}}\left|\tau^\alpha_{2,2^m+k}(t,T)\right| = \left|C_{\alpha/2}\left(1+2k-2^{m+1}t\right)\right| \leqslant \left|C_{\alpha/2}\left(2k-\lfloor 2^m t\rfloor-1\right)\right|.$$

3.2 The Faber–Schauder functions

Here, we find the fractional integrals and derivatives of the Faber–Schauder functions.

Lemma 2. *Let* $\alpha \in (0,1)$, $T > 0$, $k, m \in \mathbb{N}_0$ *and* $0 \leqslant k < 2^m$. *Then for* $t \in (0,1)$, *we have* $I^\alpha_{0+} e_{m,k}(t) = I^{1+\alpha}_{0+} H_{m,k}(t)$ *and* $I^\alpha_{T-} e_{m,k}(t) = e_{m,k}(T) I^\alpha_{T-} \mathbf{1}_{[0,1]}(t) - I^{1+\alpha}_{T-} H_{m,k}(t)$, $t \in (0,T)$.

Proof. It follows from [15, formula (2.65)] that $I^\alpha_{0+} e_{m,k} = I^\alpha_{0+} I^1_{0+} e_{m,k} = I^{1+\alpha}_{0+} H_{m,k}$. Consider $I^\alpha_{T-} e_{m,k} = I^\alpha_{T-} I^1_{0+} H_{m,k}$. It equals

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_t^{\mathrm{T}} \left(\int_0^s H_{m,k}(z)\,\mathrm{d}z \right) (s-t)^{\alpha-1}\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{T}} H_{m,k}(z) \left(\int_{z\vee t}^{\mathrm{T}} (s-t)^{\alpha-1}\,\mathrm{d}s \right) \mathrm{d}z \\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^{\mathrm{T}} H_{m,k}(z) \bigl((T-t)^\alpha - (z-t)_+^\alpha \,\mathrm{d}s \bigr)\,\mathrm{d}z \\
&= \frac{e_{m,k}(T)}{\Gamma(1+\alpha)} (T-t)^\alpha - I^{1+\alpha}_{T-} H_{m,k}(t).
\end{aligned}$$

Finally, we note that $(1/\Gamma(1+\alpha))(T-t)^\alpha = I^\alpha_{T-} \mathbf{1}_{[0,1]}(t)$ □

Proposition 1. *Let* $\alpha \in (0,1)$, $T > 0$, $k, m \in \mathbb{N}_0$ *and* $0 \leqslant k < 2^m$. *Then for* $t \in (0,1)$, *we have*

$$\begin{aligned}
D^\alpha_{0+} e_{m,k}(t) = \frac{2^{m(\alpha-1/2)}}{\Gamma(2-\alpha)} \bigl(\bigl(2^m t - k\bigr)_+^{1-\alpha} - 2\bigl(2^m t - k - 0.5\bigr)_+^{1-\alpha} \\
+ \bigl(2^m t - k - 1\bigr)_+^{1-\alpha} \bigr),
\end{aligned} \tag{11}$$

and

$$D^\alpha_{T-} e_{m,k}(t) = e_{m,k}(T) D^\alpha_{T-} \mathbf{1}_{[0,1]}(t) - I^{1-\alpha}_{T-} H_{m,k}(t), \quad t \in (0,T). \tag{12}$$

Proof. Formula (11) follows directly from Lemma 2 and formula (2) since $D^\alpha_{0+} e_{m,k} = D^\alpha_{0+} I^1_{0+} H_{m,k} = I^{1-\alpha}_{0+} H_{m,k}$, e.g. [15, formula 2.65].

We obtain the derivative D^α_{T-} from the relation $D^\alpha_{T-} e_{m,k}(t) = -(\mathrm{d}/\mathrm{d}t)[I^{1-\alpha}_{T-} e_{m,k}](t)$. Thus, we have from Lemma 2 and formula (3) that

$$\begin{aligned}
D^\alpha_{T-} e_{m,k}(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} \bigl(e_{m,k}(T) I^{1-\alpha}_{T-} \mathbf{1}_{[0,1]}(t) - I^{2-\alpha}_{T-} H_{m,k}(t) \bigr) \\
&= \frac{e_{m,k}(T)}{\Gamma(1-\alpha)} (T-t)^{-\alpha} + \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}_{T-} I^1_{T-} H_{m,k}(t) \\
&= \frac{e_{m,k}(T)}{\Gamma(1-\alpha)} (T-t)^{-\alpha} - D^\alpha_{T-} I^1_{T-} H_{m,k}(t) \\
&= e_{m,k}(T) D^\alpha_{T-} \mathbf{1}_{[0,1]}(t) - I^{1-\alpha}_{T-} H_{m,k}(t).
\end{aligned}$$

□

Remark 4. We can write the fractional derivatives $D^{\alpha}_{0+}e_{m,k}$ and $D^{\alpha}_{T-}e_{m,k}$ as

$$D^{\alpha}_{0+}e_{m,k}(t) = 2^{m/2}\tau^{1-\alpha}_{1,2^m+k}(t), \tag{13}$$

$$D^{\alpha}_{T-}e_{m,k}(t) = \frac{e_{m,k}(T)}{\Gamma(1-\alpha)}(T-t)^{-\alpha} - 2^{m/2}\tau^{1-\alpha}_{2,2^m+k}(t,T). \tag{14}$$

Lemma 3.

(i) *Let a series $\sum_{n=0}^{\infty} a_n(t)$, $t \in [0,T]$, be uniformly bounded by a nonnegative function $A \in L_1[0,T]$, then*

$$I^{\alpha}_{0+}\left(\sum_{n=0}^{\infty} a_n\right)(t) = \sum_{n=0}^{\infty}\bigl(I^{\alpha}_{0+}a_n\bigr)(t), \quad t \in [0,T].$$

(ii) *Let $\sum_{n=0}^{\infty} a_n(t)$, $t \in [0,T]$, be a convergent in $L_1[0,T]$ series, $a_n \in I^{\alpha}_{+}(L_1[0,T])$, $n \geqslant 0$. If the exists a summable sequence $b_n \geqslant 0$, $n \geqslant 0$, such that $|(D^{\alpha}_{0+}a_n)(t)| \leqslant b_n$ for all $t \in [0,T]$, then*

$$D^{\alpha}_{0+}\left(\sum_{n=0}^{\infty} a_n\right)(t) = \sum_{n=0}^{\infty}\bigl(D^{\alpha}_{0+}a_n\bigr)(t), \quad t \in [0,T].$$

Proof. (i) The first statement follows from the Lebesgue dominated convergence theorem, that is

$$\int_0^t (t-z)^{\alpha-1}\sum_{n=0}^{\infty}\bigl|a_n(z)\bigr|\,\mathrm{d}z \leqslant \int_0^t (t-z)^{\alpha-1}A(z)\,\mathrm{d}z = \Gamma(\alpha)\bigl(I^{\alpha}_{0+}A\bigr)(t),$$

where $I^{\alpha}_{0+}A(t)$ is finite for almost all $t \in (0,T)$ due to $\|I^{\alpha}_{0+}A\|_{L_1[0,T]} < \infty$, e.g [15, Thm. 2.6].

(ii) Note that $(D^{\alpha}_{0+}a_n)(t) = (\mathrm{d}/\mathrm{d}t)(I^{1-\alpha}_{0+}a_n)(t)$. Since $\sum_{n=0}^{\infty} a_n \in L_1[0,T]$, we have from the first part that $I^{1-\alpha}_{0+}(\sum_{n=0}^{\infty} a_n)(t) = \sum_{n=0}^{\infty}(I^{1-\alpha}_{0+}a_n)(t)$. Then

$$D^{\alpha}_{0+}\left(\sum_{n=0}^{\infty} a_n\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha}_{0+}\left(\sum_{n=0}^{\infty} a_n\right)\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=0}^{\infty}\bigl(I^{1-\alpha}_{0+}a_n\bigr)\right)(t).$$

Since $\sum_{n=0}^{\infty} |(\mathrm{d}/\mathrm{d}t)(I^{1-\alpha}_{0+}a_n)(t)| \leqslant \sum_{n=0}^{\infty} b_n < \infty$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=0}^{\infty}\bigl(I^{1-\alpha}_{0+}a_n\bigr)\right)(t) = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\bigl(I^{1-\alpha}_{0+}a_n\bigr)(t) = \sum_{n=0}^{\infty}\bigl(D^{\alpha}_{0+}a_n\bigr)(t), \quad t \in [0,T]. \quad \square$$

Consider the partial sums of the fractional derivatives of the Faber–Schauder functions $D^{\alpha}_{0+}[\sum_{k=0}^{2^m-1} e_{m,k}](t)$ and $D^{\alpha}_{T-}[\sum_{k=0}^{2^m-1} e_{m,k}](t)$. Due to (12)–(14), we have

$$D^{\alpha}_{0+}\left[\sum_{k=0}^{2^m-1} e_{m,k}\right](t) = 2^{m/2}\sum_{k=0}^{\lfloor 2^m t\rfloor-1} \tau^{1-\alpha}_{1,2^m+k}(t) + 2^{m/2}\tau^{1-\alpha}_{1,2^m+\lfloor 2^m t\rfloor}(t)$$

and

$$D^{\alpha}_{T-}\left[\sum_{k=0}^{2^m-1} e_{m,k}\right](t) - \left(\sum_{k=0}^{2^m-1} e_{m,k}(T)\right) D^{\alpha}_{T-}\mathbf{1}_{[0,1]}(t) = -2^{m/2}\sum_{k=\lfloor 2^m t\rfloor}^{\lfloor 2^m T\rfloor} \tau^{1-\alpha}_{2,2^m+k}(t,T).$$

Proposition 2. *If $m \geqslant 1$, then*

$$\left|\sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{1,2^m+k}(t)\right| \leqslant c_1(\alpha)2^{m(\alpha-1)} \quad \textit{uniformly on } [0,1], \tag{15}$$

$$\left|\sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{2,2^m+k}(t,T)\right| \leqslant c_1(\alpha)2^{m(\alpha-1)} \quad \textit{uniformly on } [0,T], \tag{16}$$

where $c_1(\alpha) = (2^{\alpha}/\Gamma(2-\alpha))(\sum_{k\geqslant 1}|C_{(1-\alpha)/2}(k)| + 2)$.

Proof. From Remarks 1 and 3, it follows that

$$\begin{aligned}\left|\sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{1,2^m+k}(t)\right| &\leqslant \frac{2^{m(\alpha-1)+\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{\lfloor 2^m t\rfloor-1}\left|C_{(1-\alpha)/2}\big(2\lfloor 2^m t\rfloor - 2k - 1\big)\right| + \frac{2^{m(\alpha-1)}}{\Gamma(2-\alpha)} \\ &\leqslant \frac{2^{m(\alpha-1)+\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k\geqslant 1}\left|C_{(1-\alpha)/2}(k)\right| + 2^{-\alpha}\right) \leqslant c_1(\alpha)2^{m(\alpha-1)},\end{aligned}$$

where the series $\sum_{k\geqslant 1}|C_{(1-\alpha)/2}(k)|$ converges due to integrability of $C_{(1-\alpha)/2}$, e.g. [3, Sect. 3.2].

Consider the case of D^{α}_{T-}. Let $\lfloor 2^m t\rfloor + 1 \leqslant \lfloor 2^m T\rfloor - 1$, then it follows from Remarks 1 and 3 that

$$\begin{aligned}\left|\sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{2,2^m+k}(t,T)\right| &\leqslant \sum_{k=\lfloor 2^m t\rfloor+1}^{\lfloor 2^m T\rfloor-1}\left|\tau^{1-\alpha}_{2,2^m+k}(t,T)\right| + 2\frac{2^{m(\alpha-1)}}{\Gamma(2-\alpha)} \\ &\leqslant \frac{2^{m(\alpha-1)+\alpha}}{\Gamma(2-\alpha)}\sum_{k=\lfloor 2^m t\rfloor+1}^{\lfloor 2^m T\rfloor-1}\left|C_{(1-\alpha)/2}\big(2k-2\lfloor 2^m t\rfloor-1\big)\right| + 2\frac{2^{m(\alpha-1)}}{\Gamma(2-\alpha)} \\ &\leqslant \frac{2^{m(\alpha-1)+\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k\geqslant 1}\left|C_{(1-\alpha)/2}(k)\right| + 2^{1-\alpha}\right).\end{aligned}$$

Let $\lfloor 2^m t\rfloor + 1 \geqslant \lfloor 2^m T\rfloor$. Then there are at most two nonzero $\tau^{1-\alpha}_{2,2^m+k}(t,T)$. Thus, we get the upper bound $|\sum_{k=0}^{2^m-1}\tau^{1-\alpha}_{2,2^m+k}(t,T)| \leqslant 2(2^{m(\alpha-1)}/\Gamma(2-\alpha))$. $\square$

It follows from Proposition 2 that the series $\sum_{m\geqslant 0} 2^{m(1/2-H)}|D^{\alpha}_{0+}[\sum_{k=0}^{2^m-1} e_{m,k}](t)|$ converges uniformly on $[0,1]$ for $\alpha < H$. This ensures that Lemma 3 holds for the Takagi–Landsberg function x^H and yields

$$D^{\alpha}_{0+}x^H(t) = \sum_{m=0}^{\infty} 2^{m(1-H)} \sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{1,2^m+k}(t).$$

Take the expansion $x^H(t)-x^H(T) = \sum_{m\geqslant 0} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1}(e_{m,k}(t)-e_{m,k}(T))$. Since the series $\sum_{m\geqslant 0} 2^{m(1/2-H)}|D^{\alpha}_{T-}[\sum_{k=0}^{2^m-1}(e_{m,k} - e_{m,k}(T)](t)|$ converges uniformly on $[0,T]$ for $\alpha < H$, then it holds by Lemma 3 that

$$D^{\alpha}_{T-}x^H(t) = x^H(T)D^{\alpha}_{T-}\mathbf{1}_{[0,1]}(t) - \sum_{m=0}^{\infty} 2^{m(1-H)} \sum_{k=0}^{2^m-1} \tau^{1-\alpha}_{2,2^m+k}(t,T).$$

Now consider the special case $\alpha = H$ and the values of $D^{\alpha}_{0+}x^H(t)$ at points of the m_0th dyadic partition of $[0,1]$, that is the set $\mathbb{T}_{m_0} := \{k2^{-m_0} | k = 0,\dots,2^{m_0}\}$.

Proposition 3. *Let* $k_0, m_0 \in \mathbb{N}_0$ *and* $k_0 \leqslant 2^{m_0} - 1$. *Then*

$$\sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} D^H_{0+}e_{m,k}\left(\frac{k_0}{2^{m_0}}\right) = -\infty.$$

Proof. In the case $\alpha = H$, $m \geqslant m_0$, it follows from Remark 2 that

$$\begin{aligned} d_m := {}& 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} D^H_{0+}e_{m,k}\left(\frac{k_0}{2^{m_0}}\right) \\ ={}& \frac{2^H}{\Gamma(2-H)} \sum_{k=0}^{2^{m-m_0}k_0-1} C_{(1-H)/2}\left(\frac{2^{m+1}k_0}{2^{m_0}} - 2k - 1\right). \end{aligned} \tag{17}$$

For all $k \leqslant 2^{m-m_0}k_0 - 1$, we have $2^{m-m_0+1}k_0 - 2k - 1 \geqslant 1$ and

$$C_{(1-H)/2}\big(2^{m-m_0+1}k_0 - 2k - 1\big) < 0, \tag{18}$$

which gives that the right-hand side of (17) is negative.

Now we show that the sequence d_m is monotonically decreasing if $m \geqslant m_0$. Consider the difference $d_{m+1} - d_m$, which equals

$$\begin{aligned} &\frac{2^H}{\Gamma(2-H)}\Bigg[\sum_{k=0}^{2^{m+1}k_0/2^{m_0}-1} C_{(1-H)/2}\left(\frac{2^{m+1+1}k_0}{2^{m_0}} - 2k - 1\right) \\ &\qquad - \sum_{k=0}^{2^m k_0/2^{m_0}-1} C_{(1-H)/2}\left(\frac{2^{m+1}k_0}{2^{m_0}} - 2k - 1\right)\Bigg] \\ &\quad = \frac{2^H}{\Gamma(2-H)} \sum_{k=0}^{2^{m-m_0}k_0-1} C_{(1-H)/2}\big(2^{m-m_0+2}k_0 - 2k - 1\big). \end{aligned}$$

We get from the last relation and (18) that $d_{m+1} - d_m < 0$, so $d_{m+1} < d_m < d_{m_0} < 0$ for all $m > m_0$. This means that $\sum_{m=m_0}^{\infty} d_m < \sum_{m=m_0}^{\infty} d_{m_0} = -\infty$. □

4 A weighted Takagi–Landsberg function

In this section, we consider the extension of the class of the Takagi–Landsberg functions. Namely, for constants $c_{m,k} \in [-L, L]$, $k, m \in \mathbb{N}_0$, we define a *weighted Takagi–Landsberg function* as $y_{c,H} : [0,1] \to \mathbb{R}$ via

$$y_{c,H}(t) = \sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} c_{m,k} e_{m,k}(t), \quad t \in [0,1]. \tag{19}$$

Since $|y_{c,H}(t)| \leqslant L x^H(t), t \in [0,1]$, the series in (19) converges uniformly and $y_{c,H} \in L_1([0,1])$.

Lemma 4. *Let $H > 0$. Any H-Hölder continuous function f on $[0,1]$ can be expanded as*

$$f(t) = f(0)(1-t) + f(1)t + \sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} c_{m,k} e_{m,k}(t), \quad t \in [0,1]. \tag{20}$$

We call formula (20) *the Takagi–Landsberg representation of function f.*

Proof. To show this, we first provide the relation between coefficients $a_{m,k}$ in expansion (1) and $c_{m,k}$ in (19), that is

$$c_{m,k} = a_{m,k} 2^{mH} = 2^{mH}\left[2f\left(\frac{k+0.5}{2^m}\right) - f\left(\frac{k+1}{2^m}\right) - f\left(\frac{k}{2^m}\right)\right]. \tag{21}$$

Theorem 3 on p. 191 in [7] states that f is H-Hölder continuous if and only if coefficients $a_{m,k}$ in expansion (1) satisfy $|a_{m,k}| \leqslant C(2^m + k)^{-H}$, $m \geqslant 0$, for a constant $C > 0$. Thus, if f is H-Hölder continuous, then $|c_{m,k}| = |a_{m,k}| 2^{mH} \leqslant C := L$ and f is a weighed Takagi–Landsberg function. If $y_{c,H}$ admits representation (19), i.e. $c_{m,k} \in [-L, L]$, then $a_{m,k} = c_{m,k} 2^{-mH}$ from (21) satisfy $|a_{m,k}| \leqslant L 2^{-mH} \leqslant 2L \times (2^m + k)^{-H}$, $m \geqslant 0$. Hence, $y_{c,H}$ is H-Hölder continuous. □

Now let us establish that $y_{c,H}$ admit fractional derivatives of order $\alpha < H$.

Theorem 1. *Let $0 < \alpha < H$ then*

$$D_{0+}^{\alpha} y_{c,H}(t) = \sum_{m=0}^{\infty} 2^{m(1-H)} \sum_{k=0}^{2^m-1} c_{m,k} \tau_{1,2^m+k}^{1-\alpha}(t), \tag{22}$$

$$D_{T-}^{\alpha}\bigl[y_{c,H} - y_{c,H}(T)\bigr](t) = -\sum_{m=0}^{\infty} 2^{m(1-H)} \sum_{k=0}^{2^m-1} c_{m,k} \tau_{2,2^m+k}^{1-\alpha}(t,T). \tag{23}$$

Proof. Due to (11), the fractional derivatives of summands in (19) equal

$$D_{0+}^{\alpha}\left[2^{m(1/2-H)}\sum_{k=0}^{2^m-1} c_{m,k}e_{m,k}\right](t) = 2^{m(1-H)}\sum_{k=0}^{2^m-1} c_{m,k}\tau_{1,2^m+k}^{1-\alpha}(t).$$

From (15) we have the following uniform bound

$$\begin{aligned}\left|D_{0+}^{\alpha}\left[2^{m(1/2-H)}\sum_{k=0}^{2^m-1} c_{m,k}e_{m,k}\right](t)\right| &\leqslant L2^{m(1-H)}\sum_{k=0}^{2^m-1}\left|\tau_{1,2^m+k}^{1-\alpha}(t)\right| \\ &\leqslant Lc_1(\alpha)2^{m(\alpha-H)}.\end{aligned}$$

Analogously,

$$\begin{aligned}\left|D_{T-}^{\alpha}\left[2^{m(1/2-H)}\sum_{k=0}^{2^m-1} c_{m,k}\big(e_{m,k}-e_{m,k}(T)\big)\right](t)\right| &\leqslant L2^{m(1-H)}\sum_{k=0}^{2^m-1}\left|\tau_{2,2^m+k}^{1-\alpha}(t)\right| \\ &\leqslant Lc_1(\alpha)2^{m(\alpha-H)}.\end{aligned}$$

Thus, from Lemma 3 we get the existence of $D_{0+}^{\alpha}y_{c,H}$ and $D_{T-}^{\alpha}y_{c,H}$. Consequently, the statement of the theorem holds. □

5 The Riemann–Stieltjes integral in terms of weighted Takagi–Landsberg functions

Let $\alpha \in (0,1)$. Denote by $\mathbf{H}^{\alpha}[0,1]$ the space of α-Hölder continuous function on $[0,1]$. In this section, we consider the Riemann–Stieltjes integral of $f \in \mathbf{H}^{H_1}[0,1]$ with respect to $g \in \mathbf{H}^{H_2}[0,1]$ if $H_1 + H_2 > 1$, which can be defined as

$$\int_0^t f\,\mathrm{d}g = -\int_0^t D_{0+}^{\alpha}f(s)D_{t-}^{1-\alpha}\big[g(\cdot)-g(t)\big](s)\,\mathrm{d}s$$

for any $\alpha \in (0,1)$ such that $\alpha < H_1$, $1-\alpha < H_2$, see, e.g. [18].

We use the Takagi–Landsberg representation of functions f and g (19) to give the series expansion of integral $\int_0^t f\,\mathrm{d}g$. Denote by

$$\begin{aligned}\varDelta_{2^m+k,2^n+l}^{\alpha}(t) &= \tau_{1,2^m+k}^{\alpha}\left(t\wedge\frac{l}{2^n}\right) - 2\tau_{1,2^m+k}^{\alpha}\left(t\wedge\frac{l+0.5}{2^n}\right) \\ &\quad + \tau_{1,2^m+k}^{\alpha}\left(t\wedge\frac{l+1}{2^n}\right), \quad t\in[0,1],\end{aligned} \tag{24}$$

for $\alpha > 0$, $n, m \in \mathbb{N}_0$, $l = 0,\dots,2^n-1$, $k = 0,\dots,2^m-1$.

Theorem 2. *Let* $f \in \mathbf{H}^{H_1}[0,1]$ *and* $g \in \mathbf{H}^{H_2}[0,1]$ *with* $H_1 + H_2 > 1$ *possess the following Takagi–Landsberg representations*:

$$f(t) = \sum_{m=0}^{\infty} 2^{m(1/2-H_1)} \sum_{k=0}^{2^m-1} c^{(1)}_{m,k} e_{m,k}(t), \quad t \in [0,1],$$

$$g(t) = \sum_{m=0}^{\infty} 2^{m(1/2-H_2)} \sum_{k=0}^{2^m-1} c^{(2)}_{m,k} e_{m,k}(t), \quad t \in [0,1],$$

where $|c^{(1)}_{m,k}|, |c^{(2)}_{m,k}| \leqslant L$ *for some* $L > 0$. *If* $1 - H_2 < \alpha < H_1$, *then*

$$\begin{aligned}\int_0^t f(s)\,\mathrm{d}g(s) &= -\int_0^t D^{\alpha}_{0+} f(s) D^{1-\alpha}_{t-}\bigl[g(\cdot) - g(t)\bigr](s)\,\mathrm{d}s \\ &= -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1}\sum_{l=0}^{2^n-1} 2^{m(1-H_1)+n(1-H_2)} c^{(1)}_{m,k} c^{(2)}_{n,l} \Delta^2_{2^m+k,2^n+l}(t). \qquad (25)\end{aligned}$$

Proof. Due to Theorem 1, we have that $D^{\alpha}_{0+} f$ and $D^{1-\alpha}_{t-}(g(\cdot) - g(t))$ exist and converge uniformly as series (22) and (23). Therefore, $D^{\alpha}_{0+} f(s) D^{1-\alpha}_{t-}[g(\cdot) - g(t)](s)$ converges uniformly on $s \in (0,t)$ as well with the following bound

$$\bigl|D^{\alpha}_{0+} f(s) D^{1-\alpha}_{t-}\bigl[g(\cdot) - g(t)\bigr](s)\bigr| \leqslant \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{m(\alpha-H_1)+n(1-\alpha-H_2)} L^2 c_1(\alpha) c_1(1-\alpha)$$

for all $s \in (0,t)$. So, we apply the Lebesgue dominated convergence theorem to the integral $\int_0^t D^{\alpha}_{0+} f(s) D^{1-\alpha}_{t-}[g(\cdot) - g(t)]\,\mathrm{d}s$, which equals now

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1}\sum_{l=0}^{2^n-1} 2^{m(1/2-H_1)+n(1/2-H_2)} c^{(1)}_{m,k} c^{(2)}_{n,l} \\ &\quad\times \int_0^t D^{\alpha}_{0+} e_{m,k}(s) D^{1-\alpha}_{t-}\bigl[e_{n,l} - e_{n,l}(t)\bigr](s)\,\mathrm{d}s. \qquad (26)\end{aligned}$$

Compute the integral in (26) using Proposition 1:

$$\begin{aligned}&\int_0^t D^{\alpha}_{0+} e_{m,k}(s) D^{1-\alpha}_{t-}\bigl[e_{n,l}(\cdot) - e_{n,l}(t)\bigr](s)\,\mathrm{d}s \\ &\quad= -\int_0^t I^{1-\alpha}_{0+} H_{m,k}(s) I^{\alpha}_{t-} H_{n,l}(s)\,\mathrm{d}s \\ &\quad= -\frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s \int_s^t (s-u_1)^{-\alpha} H_{m,k}(u_1)(u_2-s)^{\alpha-1} H_{n,l}(u_2)\,\mathrm{d}u_2\,\mathrm{d}u_1\,\mathrm{d}s\end{aligned}$$

$$= -\frac{1}{B(1-\alpha,\alpha)} \int_0^t H_{n,l}(u_2) \int_0^{u_2} H_{m,k}(u_1) \int_{u_1}^{u_2} (s-u_1)^{-\alpha}(u_2-s)^{\alpha-1}\,\mathrm{d}s\,\mathrm{d}u_1\,\mathrm{d}u_2$$
$$= -\int_0^t H_{n,l}(u_2) \int_0^{u_2} H_{m,k}(u_1)\,\mathrm{d}u_1\,\mathrm{d}u_2 = -\int_0^t H_{n,l}(u) e_{m,k}(u)\,\mathrm{d}u.$$

Obviously, if $t < k/2^m \vee l/2^m$, the last integral equals zero.

Let $t \in J_{n,l}$, then

$$\int_0^t H_{n,l}(u) e_{m,k}(u)\,\mathrm{d}u = 2^{n/2} \int_{l/2^n}^t e_{k,m}(u)\,\mathrm{d}u = 2^{n/2}\left(I_{0+}^2 H_{m,k}(t) - I_{0+}^2 H_{m,k}\left(\frac{l}{2^n}\right)\right).$$

If $t \in J_{n,\,l+0.5}$, then

$$\int_0^t H_{n,l}(u) e_{m,k}(u)\,\mathrm{d}u = 2^{n/2} \int_{l/2^n}^{(l+0.5)/2^n} e_{m,k}(u)\,\mathrm{d}u - \int_{(l+0.5)/2^n}^t e_{m,k}(u)\,\mathrm{d}u$$
$$= 2^{n/2}\left(2I_{0+}^2 H_{m,k}\left(\frac{l+0.5}{2^n}\right) - I_{0+}^2 H_{m,k}\left(\frac{l}{2^n}\right) - I_{0+}^2 H_{m,k}(t)\right).$$

The case $t > (l+1)/2^n$ is similar. Thus, we have

$$\int_0^t H_{n,l}(u) e_{m,k}(u)\,\mathrm{d}u$$
$$= 2^{n/2}\left(2I_{0+}^2 H_{m,k}\left(t \wedge \frac{l+0.5}{2^n}\right) - I_{0+}^2 H_{m,k}\left(t \wedge \frac{l}{2^n}\right) - I_{0+}^2 H_{m,k}\left(t \wedge \frac{l+1}{2^n}\right)\right).$$

Note that

$$\int_0^t H_{n,l}(u) e_{m,k}(u)\,\mathrm{d}u$$
$$= \int_0^t H_{n,l}(u_2) \int_0^{u_2} H_{m,k}(u_1)\,\mathrm{d}u_1\,\mathrm{d}u_2 = \int_0^t H_{m,k}(u_1) \int_{u_1}^t H_{n,l}(u_2)\,\mathrm{d}u_2\,\mathrm{d}u_1$$
$$= \int_0^t H_{m,k}(u)\big[e_{n,l}(t) - e_{n,l}(u)\big]\,\mathrm{d}u.$$

Then the statement follows from Lemma 1, relations (10) and (24). □

Remark 5. The Riemann–Stieltjes integral in Theorem 2 can be written as

$$\begin{aligned}
\int_0^t f(s)\,\mathrm{d}g(s) &= -\int_0^t D_{0+}^{\alpha} f(s) D_{t-}^{1-\alpha}\bigl[g(\cdot)-g(t)\bigr](s)\,\mathrm{d}s \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{n(1/2-H_2)} c_{n,l}^{(2)} \int_0^t H_{n,l}(u) f(u)\,\mathrm{d}u \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1} 2^{m(1/2-H_1)} c_{m,k}^{(1)} \int_0^t H_{m,k}(u)\bigl[g(t)-g(u)\bigr]\,\mathrm{d}u \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l_1=0}^{2^{n_1}-1}\sum_{l_2=0}^{2^{n_2}-1} 2^{n_1(1/2-H_1)+n_2(1/2-H_2)} c_{n_1,l_1}^{(1)} c_{n_2,l_2}^{(2)} \\
&\quad\times \int_0^t H_{n_2,l_2}(u) e_{n_1,l_1}(u)\,\mathrm{d}u. \qquad (27)
\end{aligned}$$

Remark 6. Particularly, we have

$$\begin{aligned}
\int_0^t f(s)\,\mathrm{d}s &= -\int_0^t D_{0+}^{\alpha} f(s) D_{t-}^{1-\alpha}\bigl[(\cdot)-t\bigr](s)\,\mathrm{d}s = I_{0+}^{1} f(t), \\
\int_0^t \mathrm{d}g(s) &= -\int_0^t D_{0+}^{\alpha} \mathbf{1}_{[0,t]}(s) D_{t-}^{1-\alpha}\bigl[g(\cdot)-g(t)\bigr](s)\,\mathrm{d}s = g(t)-g(0), \\
\int_0^t s\,\mathrm{d}g(s) &= -\int_0^t D_{0+}^{\alpha}\bigl[(\cdot)\bigr](s) D_{t-}^{1-\alpha}\bigl[g(\cdot)-g(t)\bigr](s)\,\mathrm{d}s = tg(t) - I_{0+}^{1} g(t).
\end{aligned}$$

From [18, Prop. 4.4.1] it follows that $\int_0^{\cdot} f\,\mathrm{d}g \in \mathbf{H}^{H_2}[0,1]$.

Corollary 1. *The coefficients x_0^R, x_1^R, c^R in Takagi–Landsberg representation of the Riemann–Stieltjes integral in Theorem 2 equal $x_0^R = 0$,*

$$\begin{aligned}
x_1^R &= -\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{l_1=0}^{2^{n_1}-1}\sum_{l_2=0}^{2^{n_2}-1} 2^{n_1(1-H_1)+n_2(1-H_2)} c_{n_1,l_1}^{(1)} c_{n_2,l_2}^{(2)} \Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}(1), \\
c_{m,k}^R &= \sum_{n_1=0}^{\infty}\sum_{l_1=0}^{2^{n_1}-1} c_{n_1,l_1}^{(1)} \sum_{n_2=0}^{\infty} 2^{n_1(1/2-H_1)+(n_2-m)(1/2-H_2)} \\
&\quad\times \sum_{l_2=0}^{2^{n_2}-1} c_{n_2,l_2}^{(2)} \int_0^1 e_{n_1,l_1}(u) H_{n_2,l_2}(u) H_{m,k}(u)\,\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&= 2^{mH_2} \sum_{n_1=0}^{\infty} \sum_{l_1=0}^{2^{n_1}-1} c_{n_1,l_1} \sum_{n_2=0}^{\infty} \frac{2^{n_1(1/2-H_1)}}{2^{(m-n_2)(1/2-H_2)}} \\
&\quad \times \sum_{l_2=0}^{2^{n_2}-1} c^{(2)}_{n_2,l_2} \left(\Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k}{2^m}\right) - 2\Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k+0.5}{2^m}\right) \right. \\
&\quad \left. + \Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k+1}{2^m}\right)\right).
\end{aligned}
$$

Proof. The value of x_1^R follows from (25). Denote by $R(t)$ the value of the integral $\int_0^t f\,\mathrm{d}g$. The function $R \in \mathbf{H}^{H_2}[0,1]$ possesses the representation as a weighted Takagi–Landsberg function with coefficients c^R given by $c^R_{m,k} = 2^{mH_2}[2R((k+0.5)/2^m) - R((k+1)/2^m) - R(k/2^m)]$. Then for $m \in \mathbb{N}_0$ and $k = 0, \dots, 2^m - 1$, we have from (27) that

$$
\begin{aligned}
c^R_{m,k} &= 2^{mH_2} \sum_{n_1=0}^{\infty} \sum_{l_1=0}^{2^{n_1}-1} c^{(1)}_{n_1,l_1} \sum_{n_2=0}^{\infty} 2^{n_1(1/2-H_1)+n_2(1/2-H_2)} \\
&\quad \times \sum_{l_2=0}^{2^{n_2}-1} c^{(2)}_{n_2,l_2} \left(2 \int_0^{(k+0.5)/2^m} e_{n_1,l_1}(u) H_{n_2,l_2}(u)\,\mathrm{d}u \right. \\
&\quad \left. - \int_0^{k/2^m} e_{n_1,l_1}(u) H_{n_2,l_2}(u)\,\mathrm{d}u - \int_0^{(k+1)/2^m} e_{n_1,l_1}(u) H_{n_2,l_2}(u)\,\mathrm{d}u \right) \\
&= \sum_{n_1=0}^{\infty} \sum_{l_1=0}^{2^{n_1}-1} c_{n_1,l_1} \sum_{n_2=0}^{\infty} \frac{2^{n_1(1/2-H_1)}}{2^{(m-n_2)(1/2-H_2)}} \\
&\quad \times \sum_{l_2=0}^{2^{n_2}-1} c^g_{n_2,l_2} \int_0^1 e_{n_1,l_1}(u) H_{n_2,l_2}(u) H_{m,k}(u)\,\mathrm{d}u.
\end{aligned}
$$

We can rewrite the last integral as

$$
\begin{aligned}
&2^{(k+0.5)/2^m} \left(\Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k}{2^m}\right) - 2\Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k+0.5}{2^m}\right) \right. \\
&\quad \left. + \Delta^2_{2^{n_1}+l_1,2^{n_2}+l_2}\left(\frac{k+1}{2^m}\right)\right).
\end{aligned}
$$

□

Remark 7. Let $g(0) = g(1) = 0$. The integral $\int_0^t s\,\mathrm{d}g(s)$ possesses the following Takagi–Landsberg representation:

$$
\int_0^t s\,\mathrm{d}g(s) = tg(t) - I^1_{0+} g(t),
$$

$$\int_0^t s\,\mathrm{d}g(s) = -t\sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{-n(1+H_2)-2}c_{n,l}^{(2)} + \sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1} 2^{m(1/2-H_2)}e_{m,k}(t)$$
$$\times\left[\frac{kc_{m,k}^{(2)}}{2^m} + 2^{mH_2}\sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{n(1/2-H_2)}c_{n,l}^{(2)}D_{2^n+l,2^m+k}\right],$$

where $D_{2^n+l,2^m+k} := (1/2^m)e_{n,l}((k+0.5)/2^m) - (1/2^m)e_{n,l}((k+1)/2^m) + 2^{n/2} \times \Delta^2_{2^n+l,2^m+k}(1)$.

6 Applications to fractional integral equations

In this section, we solve integral equations, involving fractional integrals and derivatives, with the help of the Takagi–Landsberg representations of the Hölder continuous functions. To do so, we use the uniqueness of the Schauder expansion.

Let $H \in (0,1)$ and $g \in \mathbf{H}^H[0,1]$ have the Takagi–Landsberg representation (20) with coefficients $g_0, g_1, c^g = \{c^g_{m,k}\}$. Denote by S_m the operator that gives the partial sums of the Takagi–Landsberg expansion of g by

$$[S_m g](t) := g_0(1-t) + g_1 t + \sum_{n=0}^{m}\sum_{l=0}^{2^n-1} 2^{n(1/2-H)}c^g_{n,l}e_{n,l}(t), \quad t \in [0,1].$$

From the properties of the Schauder system we get that $g(k/2^m) = [S_{m-1}g](k/2^m)$, $0 \leqslant k \leqslant 2^m - 1$. In this section, it is also convenient to make the new indexation of c^g. We write c^g_n for $c^g_{m,k}$ if $n = 2^m + k$, $m \geqslant 0$, $k = 0, \dots, 2^m - 1$.

Remark 8. Let $X : [0,1] \to \mathbb{R}$ be a H_1-Hölder continuous function. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(X) \in \mathbf{H}^{H_2}$. If X admits representation (19) with coefficients $c^x_{m,k}$, then $f(X)$ has representation with coefficients $c^f_{m,k}$, where

$$c^x_{m,k} = 2^{mH_1}\left[2X\left(\frac{k+0.5}{2^m}\right) - X\left(\frac{k+1}{2^m}\right) - X\left(\frac{k}{2^m}\right)\right] \tag{28}$$

and

$$\begin{aligned} c^f_{m,k} &= 2^{mH_2}\left[2f\left(X\left(\frac{k+0.5}{2^m}\right)\right) - f\left(X\left(\frac{k+1}{2^m}\right)\right) - f\left(X\left(\frac{k}{2^m}\right)\right)\right] \\ &= 2^{mH_2}\left[2f\left(S_m X\left(\frac{k+0.5}{2^m}\right)\right) - f\left(S_{m-1}X\left(\frac{k+1}{2^m}\right)\right)\right. \\ &\qquad \left. - f\left(S_{m-1}X\left(\frac{k}{2^m}\right)\right)\right]. \end{aligned}$$

Thus, coefficients c^f_m are determined by coefficients $\{c^x_n, n \leqslant m\}$.

6.1 Volterra integral equation

Let $H < \alpha \in (0,1)$, $\theta \neq 0$ and $g \in \mathbf{H}^H[0,1]$, that is g has the Takagi–Landsberg representation with bounded coefficients $c^g = \{c^g_{m,k}\}$. Consider the Volterra integral equation given by

$$X(t) = x_0 + \theta\big[I^\alpha X\big](t) + g(t), \quad t \in [0,1]. \tag{29}$$

Equation (29) is called also as the fractional Langevin equation, e.g. [4].

It follows from the general theory of integral equations that (29) has a unique solution in $C[0,1]$, e.g. [6, Sect. XII.6.2]. Indeed, the operator I^α_{0+} has the norm $\|I^\alpha_{0+}\|_\infty = (1/\Gamma(\alpha))\max_{t\in[0,1]}(\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s) = 1/\Gamma(1+\alpha)$. Moreover, by [15, formula (2.21)] its powers equal $[I^\alpha_{0+}]^n = I^{\alpha n}_{0+}$ with $\|[I^\alpha_{0+}]^n\|_\infty = 1/\Gamma(\alpha n+1)$. Denote by $\tilde g = x_0 + g$. A solution X of equation (29) can be expanded as a power series $X = \tilde g + \theta I^\alpha_{0+}\tilde g + \cdots + \theta^n I^{n\alpha}_{0+}\tilde g + \cdots$, which converges for all θ with

$$\begin{aligned}|\theta| &< \lim_{n\to\infty}\big\|\big[I^\alpha_{0+}\big]^n\big\|_\infty^{-1/n} = \lim_{n\to\infty}\big(\Gamma(\alpha n+1)\big)^{1/n}\\ &= \lim_{n\to\infty}(2\pi)^{1/(2n)}\mathrm{e}^{-\alpha}(n\alpha)^{\alpha-1/(2n)} = \infty,\end{aligned}$$

where the asymptotic behavior of the Gamma function is given by [1, formula 6.1.39]. Since operator I^α_{0+} maps $C[0,1]$ into $H^\alpha[0,1]$ (e.g. [15, p. 58, Cor. 2]), the solution of (29) belongs to $H^H[0,1]$.

Thus, X posses the Takagi–Landsberg representation (20) with $x_1 \in \mathbb{R}$ and bounded coefficients $c^x = \{c^x_m,\, m \geqslant 0\}$

$$X(t) = x_0 + (x_1 - x_0)t + \sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{n(1/2-H)} c^x_{2^n+l} e_{n,l}(t), \quad t \in [0,1].$$

Then we apply Lemma 3 and formula (22) to get that $[I^\alpha X](t)$ has the following series representation:

$$\big[I^\alpha X\big](t) = \frac{x_0}{\Gamma(1+\alpha)}t^\alpha + \frac{x_1-x_0}{\Gamma(\alpha+2)}t^{1+\alpha} + \sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{n(1-H)} c^x_{2^n+l}\tau^{1+\alpha}_{1,2^n+l}(t), \quad t \in [0,1].$$

We introduce a truncated fractional integral $I^\alpha_{0+}S_p : \mathbf{H}^\alpha[0,1] \to \mathbf{H}^\alpha[0,1]$ of order $p \in \mathbb{N}$ as

$$\begin{aligned}\big[I^\alpha_{0+}S_p X\big](t) &= \frac{x_0}{\Gamma(1+\alpha)}t^\alpha + \frac{x_1-x_0}{\Gamma(\alpha+2)}t^{1+\alpha}\\ &\quad + \sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c^x_{2^n+l} 2^{n(1-H)}\tau^{1+\alpha}_{1,2^n+l}(t), \quad t \in [0,1].\end{aligned}$$

Denote by X_p the solution of the following truncated equation:

$$X_p(t) = x_0 + \theta\big[I^\alpha_{0+}S_p X_p\big](t) + g(t), \quad t \in [0,1]. \tag{30}$$

Obviously, $\|I^\alpha_{0+}S_p\|_\infty \leqslant \|I^\alpha_{0+}\|_\infty$, thus (30) has a unique solution in $C[0,1]$. By construction $I^\alpha_{0+}S_p X_p \in \mathbf{H}^\alpha[0,1]$, so X_p is H-Hölder continuous on $[0,1]$ as well.

Here we give the solution of (30) by finding the coefficients c^p and x_1^p in the Takagi–Landsberg expansion (20) of X_p.

Denote by

$$a_{2^m+k,2^n+l} = -\theta 2^{mH+n(1-H)} \Delta^{1+\alpha}_{2^n+l,2^m+k}(1), \tag{31}$$

$$a_{2^m+k,0} = 2^{mH}\theta\tau^{1+\alpha}_{2,2^m+k}(0,1), \tag{32}$$

$$b_{2^m+k} = c^g_{2^m+k} + 2^{mH}\theta x_0\tau^{\alpha}_{2,2^m+k}(0,1) - 2^{mH}\theta x_0\tau^{1+\alpha}_{2,2^m+k}(0,1), \tag{33}$$

$$b_0 = x_0 + g_1 + \frac{\theta x_0}{\Gamma(1+\alpha)} - \frac{\theta x_0}{\Gamma(\alpha+2)}, \tag{34}$$

$$a_{0,0} = \frac{\theta}{\Gamma(\alpha+2)}, \qquad a^p_{0,2^n+l} = \theta 2^{n(1-H)}\tau^{1+\alpha}_{1,2^n+l}(1). \tag{35}$$

Lemma 5. *Let* $p \geqslant 1$, $P = 2^{p+1} - 1$ *and denote by* $A_p = (a_{k,l})^P_{k,l=0}$, $\mathbf{C}_p = (x_1^p, c_1^p, \dots, c_P^p)^{\mathrm{T}}$, *and* $\mathbf{b}_p = (b_0, \dots, b_P)^{\mathrm{T}}$, *where* $a_{k,l}$ *and* b_k *are given by* (31)–(35). *Let* $\mathbf{C}_p$ *be a solution of*

$$\mathbf{C}_p = A_p\mathbf{C}_p + \mathbf{b}_p,$$

and let $c^p_m = b_m + x^p_1 a_{m,0} + \sum_{n=1}^P c^p_n a_{m,n}$, $m > P$. *Then the function*

$$X_p = x_0(1-t) + x_1^p t + \sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1} 2^{m(1/2-H)} c^p_{2^m+k} e_{m,k}(t), \quad t \in [0,1],$$

is the solution of equation (30).

Proof. Since the Takagi–Landsberg expansion is unique and its coefficients are determined by (28), we have the following relation:

$$\begin{aligned}
c^p_{2^m+k} &= c^g_{2^m+k} + 2^{mH}\theta x_0 \tau^{\alpha}_{2,2^m+k}(0,1) + 2^{mH}\theta(x^p_1 - x_0)\tau^{1+\alpha}_{2,2^m+k}(0,1) \\
&\quad + 2^{mH}\theta \sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c^p_{2^n+l} 2^{n(1-H)}\Biggl(2\tau^{1+\alpha}_{1,2^n+l}\biggl(\frac{k+0.5}{2^m}\biggr) \\
&\quad - \tau^{1+\alpha}_{1,2^n+l}\biggl(\frac{k+1}{2^m}\biggr) - \tau^{1+\alpha}_{1,2^n+l}\biggl(\frac{k}{2^m}\biggr)\Biggr) \\
&= b_{2^m+k} + x^p_1 a_{2^m+k,0} + \sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c^p_{2^n+l} a^p_{2^m+k,2^n+l}.
\end{aligned} \tag{36}$$

At point $t = 1$, equation (30) gives the next relation:

$$\begin{aligned}
x^p_1 &= x_0 + g_1 + \frac{\theta x_0}{\Gamma(1+\alpha)} + \frac{\theta(x^p_1 - x_0)}{\Gamma(\alpha+2)} + \theta\sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c^p_{2^n+l}2^{n(1-H)}\tau^{1+\alpha}_{1,2^n+l}(1) \\
&= b_0 + x^p_1 a_{0,0} + \sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c^p_{2^n+l} a^p_{0,2^n+l}.
\end{aligned} \tag{37}$$

Then relations (36) and (37) yield the statement of the lemma. $\square$

Lemma 6. *Let X_p be the solution of equation* (30)*, then X_p tends to the solution of* (29) *in the supremum norm on* $[0,1]$.

Proof. Let X be the solution of (29). Denote by $err_p = X_p - X$. Note that

$$err_p(t) = \theta I^\alpha_{0+} S_p X_p(t) - \theta I^\alpha_{0+} X(t) = \theta I^\alpha_{0+} err_p(t) + \theta I^\alpha_{0+}\big[S_p X_p - X_p\big](t). \tag{38}$$

Due to the power series expansion of err_p as a solution of equation (38), we have

$$\|err_p\|_\infty \leqslant \left(1 + \sum_{n=1}^\infty |\theta|^n \big\|I^{\alpha n}_{0+}\big\|\right)\big\|\theta I^\alpha_{0+}\big[S_p X_p - X_p\big]\big\|_\infty.$$

Then $|X_p(t) - S_p X_p(t)| \leqslant \sum_{m=p+1}^\infty \sum_{k=0}^{2^m-1} 2^{m(1/2-H)}|c_{m,k}|e_{m,k}(t) \leqslant L(x^H(t) - S_p x^H(t))$, where x^H is a Takagi–Landsberg function. The second term in the right-hand side of (38) is bounded by

$$\begin{aligned}
&\frac{|\theta|}{\Gamma(\alpha)}\left|\int_0^t \frac{S_p X_p(u) - X(u)}{(t-u)^{1-\alpha}}\,\mathrm{d}u\right| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|S_p X_p(u) - X_p(u)|}{(t-u)^{1-\alpha}}\,\mathrm{d}u \leqslant \frac{L}{\Gamma(\alpha)}\int_0^t \frac{x^H(u) - S_p x^H(u)}{(t-u)^{1-\alpha}}\,\mathrm{d}u \\
&\quad\leqslant \frac{Lt^\alpha}{\Gamma(1+\alpha)}\sup_{u\in[0,1]}\big(x^H(u) - S_p x^H(u)\big).
\end{aligned} \tag{39}$$

Thus, $\|I^\alpha_{0+}[S_p X_p - X_p](t)\|_\infty \to 0$ as $p\to\infty$. This yields that $\|X - X_p\|_\infty \to 0$, $p\to\infty$. $\square$

6.2 A linear differential equation

Let $\beta,\gamma\in\mathbb{R}$ and $\beta\neq 0$, $\gamma\neq 0$. Let $g : [0,1]\to\mathbb{R}$ be a Hölder continuous of order $H > 1/2$ with $g(0) = g(1) = 0$, that is g be a weighted Takagi–Landsberg function with bounded coefficients $c^g = \{c^g_{m,k},\ m\geqslant 0,\ k = 0,\dots,2^m-1\}$. Let $\alpha\in(1-H, H)$. Consider the linear equation

$$\begin{aligned}
X(t) &= x_0 + \beta I^1_{0+} X(t) + \gamma\int_0^t X(z)\,\mathrm{d}g(z) \\
&= x_0 + \beta\int_0^t X(z)\,\mathrm{d}z - \gamma\int_0^t \big[D^\alpha_{0+}X\big](z)\big[D^{1-\alpha}_{t-}\big(g(\cdot) - g(t)\big)\big](z)\,\mathrm{d}z,
\end{aligned} \tag{40}$$

where $t\in[0,1]$.

Denote by $U : \mathbf{H}^H \to \mathbf{H}^H$ the operator $U(x) = \beta I^1_{0+}x + \gamma\int_0^\cdot x\,\mathrm{d}g$. It was shown in [14] that U is a compact linear operator on Banach space $W^{\alpha,\infty}_0$ with respect to the norm $\|f\|_{\alpha,\infty} := \sup_{t\in[0,1]}(|f(t)| + \int_0^t |f(t) - f(s)|/(t-s)^{\alpha+1}\,\mathrm{d}s)$, and for $\lambda \geqslant 0$, an equivalent norm is defined by $\|f\|_{\alpha,\lambda} := \sup_{t\in[0,1]} \mathrm{e}^{-\lambda t}(|f(t)| + \int_0^t |f(t) - f(s)|/(t-s)^{\alpha+1}\,\mathrm{d}s)$. Moreover, there exists $\lambda_0 > 0$ such that $\|U(x) - U(y)\|_{\alpha,\lambda_0} \leqslant (1/2) \times \|x - x\|_{\alpha,\lambda_0}$ for all $x, y \in U(B_0) \subset B_0 = \{u \in W_0^{\alpha,\infty}\colon \|u\|_{\alpha,\lambda_0} \leqslant 2(1 + |x_0|)\}$. This ensures that there exists a unique solution $X \in W_0^{\alpha,\infty}$ of equation (40), e.g. [14, Thm. 5.1].

Let us apply the Takagi–Landsberg expansion to solve (40). Using notation (10), we get that the first integral in the right-hand side of (40) has the following representation:

$$\int_0^t X(s)\,\mathrm{d}s = x_0 t + \frac{x_1 - x_0}{2}t^2 - \sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1} 2^{n(1-H)} c^x_{2^n+l}\tau^2_{1,2^n+l}(t), \quad t \in [0,1].$$

The Riemann–Stieltjes integral in (40) is H-Hölder continuous and admits the following representation due to Theorem 2 and Remark 7:

$$\begin{aligned}\int_0^t X(s)\,\mathrm{d}g(s) &= x_0 g(t) + (x_1 - x_0)\bigl(tg(t) - I^1_{0+}g(t)\bigr)\\ &\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{2^m-1}\sum_{l=0}^{2^n-1} 2^{(m+n)(1-H)} c^x_{m,k} c^g_{n,l} \varDelta^2_{2^m+k,2^n+l}(t),\end{aligned}$$

where $t \in [0,1]$.

Denote by X_p the solution of the following truncated equation:

$$\begin{aligned}X_p(t) &= x_0 + \beta\bigl[I^1_{0+}S_p X_p\bigr](t)\\ &\quad - \gamma\int_0^t \bigl[D^\alpha_{0+}S_p X\bigr](z)\bigl[D^{1-\alpha}_{t-}S_p\bigl(g - g(t)\bigr)\bigr](z)\,\mathrm{d}z, \quad t \in [0,1]. \qquad (41)\end{aligned}$$

Denote by

$$\begin{aligned}a_{2^m+k,2^n+l} &= -2^{mH}\beta 2^{n(1-H)}\varDelta^2_{2^n+l,2^m+k}(1)\\ &\quad + \gamma\sum_{n_2=0}^{p} 2^{(n+n_2)(1-H)+mH}\sum_{l_2=0}^{2^{n_2}-1} c^g_{n_2,l_2}\biggl(\varDelta^2_{2^n+l,2^{n_2}+l_2}\biggl(\frac{k}{2^m}\biggr)\\ &\quad \times -2\varDelta^2_{2^n+l,2^{n_2}+l_2}\biggl(\frac{k+0.5}{2^m}\biggr) + \varDelta^2_{2^n+l,2^{n_2}+l_2}\biggl(\frac{k+1}{2^m}\biggr)\biggr), \qquad (42)\\ a_{2^m+k,0} &= -\frac{\beta x_0 2^{m(H-2)}}{4} + \gamma\frac{kc^g_{m,k}}{2^m}\\ &\quad + \gamma\sum_{n_2=0}^{p}\sum_{l_2=0}^{2^{n_2}-1} 2^{mH}2^{n_2(1/2-H)} c^g_{n_2,l_2} D_{2^n_2+l_2,2^m+k}, \qquad (43)\end{aligned}$$

$$a_{0,2^n+l} = \beta 2^{-n(H+1)-2} - \gamma \sum_{n_2=0}^{p} 2^{(n_1+n_2)(1-H)} \sum_{l_2=0}^{2^{n_2}-1} c^g_{n_2,l_2} \Delta_{2^n+l,2^{n_2}+l_2}(1), \tag{44}$$

$$b_{2^m+k} = -a_{2^m+k,0} + \gamma x_0 c^g_{m,k}, \quad a_{0,0} = \frac{\beta}{2} - \gamma \sum_{n_2=0}^{p} \sum_{l_2=0}^{2^{n_2}-1} 2^{-n_2(1+H)-2} c^g_{n_2,l_2}, \tag{45}$$

$$b_0 = \frac{\beta x_0}{2} + \gamma x_0 \sum_{n_2=0}^{p} \sum_{l_2=0}^{2^{n_2}-1} 2^{-n_2(1+H)-1} c^g_{n_2,l_2}. \tag{46}$$

Lemma 7. *Let $p \geqslant 1$, $P = 2^{p+1} - 1$ and denote by $A_p = (a_{k,l})_{k,l=0}^{P}$, $\mathbf{C}_p = (x_1^p, c_1^p, \dots, c_P^p)^{\mathrm{T}}$, and $\mathbf{b}_p = (b_0, \dots, b_P)^{\mathrm{T}}$, where $a_{k,l}$ and b_k are given by* (42)–(46). *Let $\mathbf{C}_p$ be a solution of*

$$\mathbf{C}_p = A_p \mathbf{C}_p + \mathbf{b}_p$$

and $c_m^p = b_m + x_1^p a_{m,0} + \sum_{n=1}^{P} c_n^p a_{m,n}$, $m > P$. Then the function

$$X_p = x_0(1-t) + x_1^p t + \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} 2^{m(1/2-H)} c^p_{2^m+k} e_{m,k}(t), \quad t \in [0,1],$$

is the solution of equation (41).

Proof. From Remark 7, Lemma 5 and Corollary 1 we have the following relations for the coefficients c^p in the Takagi–Landsberg expansion of X_p:

$$\begin{aligned}
c^p_{2^m+k} &= \beta\bigl(x_0 - x_1^p\bigr) 2^{m(H-2)-2} \\
&\quad - 2^{mH} \beta \sum_{n=0}^{p} \sum_{l_1=0}^{2^{n_1}-1} c^p_{2^{n_1}+l_1} 2^{n_1(1-H)} \Delta^2_{2^{n_1}+l_1,2^m+k}(1) \\
&\quad + \gamma x_0 c^g_{m,k} + \gamma\bigl(x_1^p - x_0\bigr) \frac{k c^g_{m,k}}{2^m} \\
&\quad + \gamma\bigl(x_1^p - x_0\bigr) \sum_{n_2=0}^{p} \sum_{l_2=0}^{2^{n_2}-1} 2^{mH} 2^{n_2(1/2-H)} c^g_{n_2,l_2} D_{2_2^{n_2}+l_2,2^m+k} \\
&\quad + \gamma \sum_{n_1=0}^{p} \sum_{l_1=0}^{2^{n_1}-1} c^p_{2^{n_1}+l_1} \sum_{n_2=0}^{p} 2^{(n_1+n_2-m)(1/2-H)} \\
&\quad \times \sum_{l_2=0}^{2^{n_2}-1} c^g_{n_2,l_2} \int_0^1 e_{n_1,l_1}(u) H_{n_2,l_2}(u) H_{m,k}(u)\, \mathrm{d}u \\
&= b_{2^m+k} + x_1^p a_{2^m+k,0} + \sum_{n=0}^{p} \sum_{l=0}^{2^n-1} c^p_{2^n+l} a^p_{2^m+k,2^n+l},
\end{aligned}$$

and

$$\begin{aligned}
x_1^p &= x_0 + \beta\frac{x_0 + x_1^p}{2} + \beta\sum_{n=0}^{p}\sum_{l=0}^{2^n-1} c_{2^n+l}^p 2^{-n(H+1)-2} \\
&\quad - \gamma\bigl(x_1^p - x_0\bigr)\sum_{n_2=0}^{p}\sum_{l_2=0}^{2^{n_2}-1} 2^{-n_2(1+H)-2} c_{n_2,l_2}^g \\
&\quad - \gamma\sum_{n_1=0}^{p}\sum_{n_2=0}^{p}\sum_{l_1=0}^{2^{n_1}-1}\sum_{l_2=0}^{2^{n_2}-1} 2^{n_1(1-H)+n_2(1-H)} c_{n_1,l_1}^p c_{n_2,l_2}^g \varDelta^2_{2^{n_1}+l_1,2^{n_2}+l_2}(1) \\
&:= b_0 + x_1 a_{0,0} + \sum_{n_1=0}^{p}\sum_{l_1=0}^{2^{n_1}-1} c_{2^{n_1}+l_1}^p a_{0,2^{n_1}+l_1}^p. \qquad \square
\end{aligned}$$

Lemma 8. *Let X_p be the solution of equation* (41). *Then X_p tends to the solution of* (40) *in the norm $\|\cdot\|_{\alpha,\infty}$.*

Proof. Let X be the solution of (40). Recall the operator $U(x) = \beta I_{0+}^1 x + \gamma\int_0^\cdot x dg$, $x \in \mathbf{H}^H[0,1]$ and consider the norm $\|\cdot\|_{\alpha,\lambda}$ with $\lambda > 0$. Then

$$\begin{aligned}
\|X_p - X\|_{\alpha,\lambda} &= \biggl\|\beta I_{0+}^1 S_p X_p - \beta I_{0+}^1 X + \gamma\int_0^\cdot S_p X_p d[S_p g] - \gamma\int_0^\cdot X\,\mathrm{d}g\biggr\|_{\alpha,\lambda} \\
&\leqslant \bigl\|U(X_p) - U(X)\bigr\|_{\alpha,\lambda} \\
&\quad + \biggl\|\beta I_{0+}^1[S_p X_p - X_p] + \gamma\int_0^\cdot S_p X_p\,\mathrm{d}[S_p g] - \gamma\int_0^\cdot X_p\,\mathrm{d}g\biggr\|_{\alpha,\lambda} \\
&\leqslant \bigl\|U(X_p) - U(X)\bigr\|_{\alpha,\lambda} + \bigl\|U(S_p X_p) - U(X_p)\bigr\|_{\alpha,\lambda} \\
&\quad + \biggl\|\gamma\int_0^\cdot S_p X_p\,\mathrm{d}[S_p g] - \gamma\int_0^\cdot S_p X_p\,\mathrm{d}g\biggr\|_{\alpha,\lambda}.
\end{aligned}$$

Since $\|U(x) - U(y)\|_{\alpha,\lambda} \leqslant (1/2)\|x - y\|_{\alpha,\lambda}$, then

$$\frac{\|X_p - X\|_{\alpha,\lambda}}{2} \leqslant \frac{\|S_p X_p - X_p\|_{\alpha,\lambda}}{2} + |\gamma|\biggl\|\int_0^\cdot S_p X_p\,\mathrm{d}[S_p g - g]\biggr\|_{\alpha,\lambda}. \tag{47}$$

By [14, Props. 4.2 and 4.4] there exist constants d_1 and d_2 such that the second norm in the RHS of (47) is bounded above by

$$\begin{aligned}
&\frac{d_2}{\lambda^{1-2\alpha}}\frac{1 + \|S_p X_p\|_{\alpha,\lambda}}{\Gamma(1-\alpha)}\sup_{0<s<t<1}\bigl|D_{t-}^{1-\alpha}\bigl[S_p g - g - S_p(t) + g(t)\bigr](s)\bigr| \\
&\quad \leqslant \frac{d_2}{\lambda^{1-2\alpha}}\frac{1 + \|S_p X_p\|_{\alpha,\lambda}}{\Gamma(1-\alpha)}\sum_{n=p}^{\infty} 2^{m(1-\alpha-H)} L c_1(1-\alpha) \to 0, \quad p \to \infty,
\end{aligned}$$

where the last inequality follows from (16).

Similarly to (39), consider the norm

$$\begin{aligned}
&\|S_pX_p - X_p\|_{\alpha,\lambda} \\
&\quad = \sup_{0<t<1} \mathrm{e}^{-\lambda t}\left(|S_pX_p - X_p|(t) + \int_0^t \frac{|[S_pX_p - X_p](t) - [S_pX_p - X_p](s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right) \\
&\quad \leqslant \sup_{0<t<1} \mathrm{e}^{-\lambda t}\left(\frac{Lt^\alpha}{\Gamma(1+\alpha)}\|x^H - S_px^H\|_\infty \right.\\
&\qquad \left. + \sum_{m=p}^{\infty} 2^{m(1/2-H)} L \int_0^t \frac{|\sum_{k=0}^{2^m-1}(e_{m,k}(t) - e_{m,k}(s))|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right).
\end{aligned}$$

Consider the last integral in more detail. At first, note that

$$\begin{aligned}
\sum_{k=0}^{2^m-1} e_{m,k}(t) &= 2^{-m/2}\big((2^mt-k)\wedge(1+k-2^mt)\big)_+ \\
&= 2^{-m/2}\big(2^mt - \lfloor 2^mt\rfloor\big)\wedge\big(1+\lfloor 2^mt\rfloor - 2^mt\big)\big) \\
&= 2^{-m/2}e_{0,0}\big(\{2^mt\}\big), \quad t\in[0,1],
\end{aligned}$$

$|e_{0,0}(\{x\}) - e_{0,0}(\{y\})| \leqslant 1$, $|e_{0,0}(\{x\}) - e_{0,0}(\{y\})| \leqslant |\{x\}-\{y\}|$, $x,y \geqslant 0$. Therefore,

$$\begin{aligned}
&\int_0^t \frac{|\sum_{k=0}^{2^m-1}(e_{m,k}(t) - e_{m,k}(s))\|}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\
&\quad = 2^{-m/2}\int_0^t \frac{|e_{0,0}(\{2^mt\}) - e_{0,0}(\{2^ms\})|}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\
&\quad = 2^{m\alpha-m/2}\int_0^{2^mt} \frac{|e_{0,0}(\{2^mt\}) - e_{0,0}(\{z\})|}{(2^mt-z)^{\alpha+1}}\,\mathrm{d}z \\
&\quad = \frac{(\int_0^{\lfloor 2^mt\rfloor-0.5} + \int_{\lfloor 2^mt\rfloor-0.5}^{\lfloor 2^mt\rfloor} + \int_{\lfloor 2^mt\rfloor}^{2^mt})(\cdots)\,\mathrm{d}z}{2^{m/2-m\alpha}}.
\end{aligned} \tag{48}$$

The first integral in the RHS of (48) is bounded by $\int_0^{\lfloor 2^mt\rfloor-0.5} 1/(2^mt-z)^{\alpha+1}\,\mathrm{d}z \leqslant (2^mt - \lfloor 2^mt\rfloor + 0.5)^{-\alpha}/\alpha \leqslant 2^\alpha/\alpha$. If $z \in (\lfloor 2^mt\rfloor - 0.5, \lfloor 2^mt\rfloor)$, then $z = \lfloor 2^mt\rfloor - 1 + \{z\}$, $2^mt - z = \{2^mt\} + 1 - \{z\}$, and the second integral in the RHS of (48) is less or equal than

$$\int_{\lfloor 2^mt\rfloor-0.5}^{\lfloor 2^mt\rfloor} \frac{|\{2^mt\}\wedge(1-\{2^mt\}) - (1-\{z\})|}{(\{2^mt\}+1-\{z\})^{\alpha+1}}\,\mathrm{d}z$$

$$\leqslant \int\limits_{\lfloor 2^m t\rfloor-0.5}^{\lfloor 2^m t\rfloor} \frac{\mathrm{d}z}{(\{2^m t\}+1-\{z\})^\alpha}$$
$$\leqslant \frac{1}{1-\alpha}\big((0.5+\{2^m t\})^{1-\alpha}-(\{2^m t\})^{1-\alpha}\big) \leqslant \frac{2^{\alpha-1}}{1-\alpha}. \tag{49}$$

If $z \in (\lfloor 2^m t\rfloor, 2^m t)$, then $2^m t - z = \{2^m t\} - \{z\}$ and $|e_{0,0}(\{2^m t\}) - e_{0,0}(\{z\})| \leqslant \{2^m t\} - \{z\} = 2^m t - z$. Thus, the third integral in the RHS of (48) equals

$$\int\limits_{\lfloor 2^m t\rfloor}^{2^m t} \frac{|e_{0,0}(\{2^m t\})-e_{0,0}(\{z\})|}{(2^m t-z)^{\alpha+1}}\,\mathrm{d}z \leqslant \int\limits_{\lfloor 2^m t\rfloor}^{2^m t} \frac{\mathrm{d}z}{(2^m t-z)^\alpha} = \frac{(\{2^m t\})^{1-\alpha}}{1-\alpha} \leqslant \frac{1}{1-\alpha}. \tag{50}$$

Hence, we get from (49) and (50) that the upper bound for the right-hand side of (48) is $2^{m\alpha-m/2}(2^\alpha/\alpha+(2^{\alpha-1}+1)/(1-\alpha))$. Finally, the norm $\|S_p X_p - X_p\|_{\alpha,\lambda}$ is bounded by

$$\sup_{0<t<1} \mathrm{e}^{-\lambda t}\left(\frac{Lt^\alpha}{\Gamma(1+\alpha)}\|x^H - S_p x^H\|_\infty + L\left(\frac{2^\alpha}{\alpha}+\frac{2^{\alpha-1}+1}{1-\alpha}\right)\sum_{m=p}^{\infty} 2^{m(\alpha-H)}\right). \tag{51}$$

Note that $\alpha < H$ in equation (40). Therefore, the right-hand side of (51) tends to 0 as $p \to \infty$. It was shown in the proof of Lemma 6 that the first term in (47) tends to 0 as $p \to \infty$. Thus, $\|X_p - X\|_{\alpha,\infty} \to 0, p \to \infty$. □

6.3 Numerical experiments: the Volterra integral equation

In this section, we illustrate our method of solution of (29) by numerical examples.

Let $0 < H < \alpha \in (0,1)$ and put $g(t) = t^H(1-t^\alpha)$, $t \in [0,1]$. Then the solution of equation $X(t) = (\Gamma(\alpha+H+1)/\Gamma(H+1))I_{0+}^\alpha X(t) + t^H(1-t^\alpha)$, $t \in [0,1]$, obviously equals $\{X(t) = t^H,\ t \in [0,1]\}$.

We solve truncated equation (30) by Lemma 5 for several combinations of α and H. For each case, we compute the norm of the error $\|X - X_p\|_\infty$, where X_p is the solution of truncated equation, and present them on Table 1.

Table 1. Volterra integral equation: norms of the error $\|X - X_p\|_\infty$.

p	$H=0.01$, $\alpha=0.05$	$H=0.2$, $\alpha=0.3$	$H=0.2$, $\alpha=0.5$	$H=0.2$, $\alpha=0.8$	$H=0.5$, $\alpha=0.51$	$H=0.5$, $\alpha=0.8$	$H=0.8$, $\alpha=0.81$	$H=0.8$, $\alpha=0.9$
3	$2.33e-01$	$6.76e-02$	$5.83e-02$	$5.04e-02$	$2.38e-02$	$2.04e-02$	$5.60e-03$	$5.39e-03$
4	$1.92e-01$	$4.32e-02$	$2.66e-02$	$2.25e-02$	$9.02e-03$	$7.56e-03$	$1.78e-03$	$1.71e-03$
5	$1.62e-01$	$2.83e-02$	$1.53e-02$	$9.96e-03$	$3.34e-03$	$2.76e-03$	$5.50e-04$	$5.28e-04$
6	$1.39e-01$	$1.89e-02$	$9.16e-03$	$4.37e-03$	$1.23e-03$	$9.97e-04$	$1.68e-04$	$1.61e-04$
7	$1.21e-01$	$1.28e-02$	$5.53e-03$	$1.91e-03$	$5.97e-04$	$3.58e-04$	$5.06e-05$	$4.85e-05$
8	$1.07e-01$	$8.75e-03$	$3.35e-03$	$8.35e-04$	$2.92e-04$	$1.28e-04$	$1.51e-05$	$1.45e-05$
9	$9.48e-02$	$6.02e-03$	$2.04e-03$	$3.64e-04$	$1.43e-04$	$4.54e-05$	$4.50e-06$	$4.31e-06$
10	$8.50e-02$	$4.17e-03$	$1.25e-03$	$1.71e-04$	$7.07e-05$	$1.61e-05$	$1.33e-06$	$1.27e-06$

6.4 Numerical experiments: linear integral equation

In this section, we consider the numerical solution of (40).

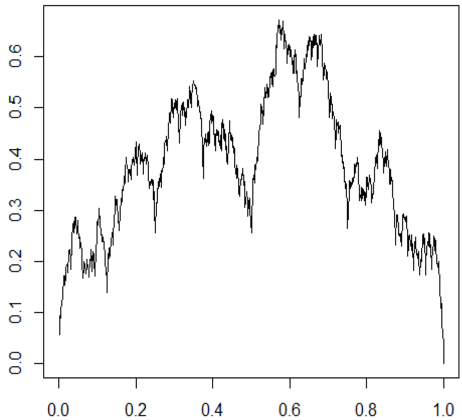
First, we put $g(t) = 0.5^H - |t-0.5|^H$, $t \in [0,1]$, for $H \in (0.5,1)$, and $\beta = -2$, $\gamma = 3$, $x_0 = 1$ in (40). We take $p = 6$, $H \in \{0.51, 0.6, 0.7, 0.8, 0.9\}$, solve truncated equation (41) by Lemma 7 and get the Takagi–Landsberg representation of the truncated solution X_p with coefficients x_0, x_1^p, c_p. We present the values of the error's norm $\|X - X_p\|_\infty$ in Table 2, where $X(t) = x_0 \exp(\beta t + \gamma g(t))$, $t \in [0,1]$, is the exact solution. Moreover, we compute the difference between the exact coefficients x_1, c^x in the representation of X and x_1^p, c^p. The values of $\max_{1 \leqslant n \leqslant 2^{p+1}} |c_n^x - c_n^p|$ are given in Table 2.

Second, we illustrate our method with the function $g(t) = \sum_{m=0}^{7} \sum_{k=0}^{2^m-1} c_{m,k}^g e_{m,k}(t)$, $t \in [0,1]$, where c^g are some bounded coefficients (we simulate them randomly). The example of function g, the corresponding exact X and truncated X_p ($p = 6$) solutions of (40) with $H = 0.51$ are presented on Fig. 1. One can observe that the small difference between the exact and truncated solution. Moreover, if we increase the value of $p = 7$, then the graphs of X and X_p for $H = 0.501$ become visually indistinguishable, and the computed norm of the error $\|X - X_p\|_\infty$ is 0.01888 for this example.

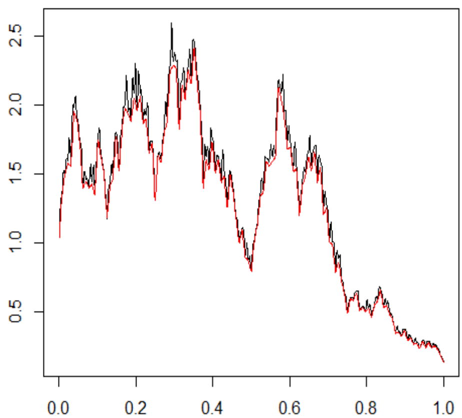
From the other hand, the wrong value of H, which is greater than the Hölder exponent of g, affects on solution X_p and the error between X_p and X increases. We illustrate such mis-specification of H on Fig. 2, where one clearly see the difference between the exact solution X and numerical solution X_p when H is significantly larger than true value 0.5.

Table 2. Linear equation: description of the error $X - X_p$.

	$H = 0.51$	$H = 0.6$	$H = 0.7$	$H = 0.8$	$H = 0.9$	$H = 0.99$
$\|X - X_p\|_\infty$	0.18934	0.08398	0.03218	0.01142	0.00325	0.00047
$\max_{1 \leqslant n \leqslant 2^{p+1}} \|c_n^x - c_n^p\|$	0.03701	0.01305	0.00409	0.00124	0.00043	0.00028

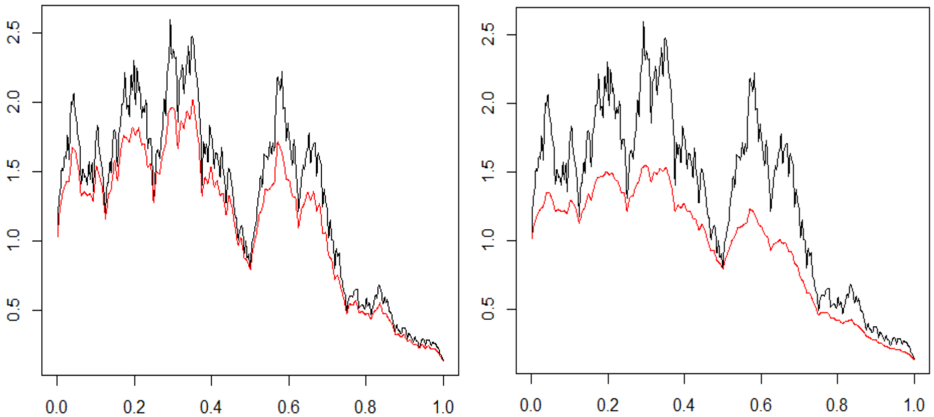


(a)



(b)

Figure 1. (a) Function g; (b) solutions X (black) and X_p (red) for $H = 0.51$.



(a) (b)

Figure 2. The mis-specification of (a) $H = 0.6$ and (b) $H = 0.8$: graphs of X (black) and X_p (red).

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