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Some remarks on $b_v(s)$-metric spaces and fixed point results with an application*

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Abstract. We compare the newly defined $b_v(s)$-metric spaces with several other abstract spaces like metric spaces, b-metric spaces and show that some well-known results, which hold in the latter class of spaces, may not hold in $b_v(s)$-metric spaces. Besides, we introduce the notions of sequential compactness and bounded compactness in the framework of $b_v(s)$-metric spaces. Using these notions, we prove some fixed point results involving Nemytzki–Edelstein type mappings in this setting, from which several comparable fixed point results can be deduced. In addition to these, we find some existence and uniqueness criteria for the solution to a certain type of mixed Fredholm–Volterra integral equations.

Keywords: $b_v(s)$-metric space, boundedly compact space, sequentially compact space, contractive mapping.

1 Introduction

There are many interesting extensions of the notion of metric spaces available in the literature where several classical fixed point results have been studied. One of such extensions is the concept of b-metric spaces, which was introduced by Bakhtin [3] in 1989, and later on, in 1993, it had been further investigated by Czerwik [5]. Afterward, in the year of 2000, Branciari [4] coined the notion of rectangular metric spaces or generalized metric spaces by modifying the triangle inequality of the usual metric spaces. As a generalization of b-metric spaces and rectangular metric spaces, George et al. [11] introduced the concept

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of rectangular b-metric spaces. Following this direction, in [4], Branciari introduced the concept of v-generalized metric spaces. Taking into account all these concepts, many mathematicians have elaborated several fixed point results in these settings, some of which meliorated and improved the original fixed point theories in usual metric spaces and some others give new results in the literature. In the literature, there is a huge amount of relevant texts available for intent readers (see [2, 7, 11, 13, 18] and the references therein).

In an attempt to extend all kinds of above mentioned generalizations of metric spaces, Mitrović and Radenović [16] introduced the concept of $b_v(s)$-metric spaces. Before going further, we recall the definition of such abstract spaces.

Definition 1. (See [16, Def. 1.8].) Let X be a nonempty set, $v \in \mathbb{N}$ and $s \geqslant 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is said to be a $b_v(s)$-metric if for all $x, y \in X$ and for all distinct points $u_1, u_2, \dots, u_v \in X$, each of them different from x and y, the following conditions hold:

(i) $d(x, y) \geqslant 0$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leqslant s[d(x, u_1) + d(u_1, u_2) + \dots + d(u_v, y)]$.

If d is a $b_v(s)$-metric on X, then the pair (X, d) is said to be a $b_v(s)$-metric space.

It is easy to observe that the class of $b_v(s)$-metric spaces is larger than that of all other metric spaces. Thus, it is a natural question to ask whether all properties of the above mentioned metric spaces remain invariant in case of $b_v(s)$-metric spaces or not. So in this paper, one of our main motivations is to compare some properties of sequences, the metric function d in $b_v(s)$-metric spaces with that of usual metric spaces, b-metric spaces. To be specific, we show that in a $b_v(s)$-metric space (X, d), a convergent sequence may not be Cauchy, the metric function d need not be continuous. Moreover, we show that a sufficient condition for a sequence to be Cauchy in the standard metric space, as well as in a b-metric space, may not work in this structure.

On the other hand, many mathematicians have achieved some interesting fixed point results in the setting of $b_v(s)$-metric spaces, such as the authors of [16] proved fixed point results associated to Banach and Reich contractions, the authors of [1] obtained a common fixed point theorem due to Jungck, the authors of [15] achieved fixed point result due to Sehgal–Guseman. It may be observed that the existing results in this structure are concerned with a different type of contraction mappings only. But at the same time, there are some important results regarding contractive type mappings in the standard metric spaces; see [8–10, 12]. So it is a natural question to ask whether the fixed point results related to different contractive conditions can be proved in the setting of $b_v(s)$-metric spaces or not. In fact, we are able to find some fixed point results concerning contractive type maps in this setting. We first recall the definition of a contractive mapping in metric spaces and then highlight some salient points regarding the existence of fixed points. A self-mapping T on a metric space (X, d) is said to be a contractive mapping if $d(Tx, Ty) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. It may be noted that completeness of the underlying space X does not give the guaranty of the existence of a fixed point of this map, but if X is compact, it is guaranteed; see [8]. In this article, we also try to find a (mild) additional criteria on the underlying $b_v(s)$-metric space X, which confirms the existence of a fixed point for a contractive mapping. To proceed in this direction, we introduce the notions of sequential compactness and bounded compactness of $b_v(s)$-metric spaces and establish correlations between them. In such spaces, we establish some fixed point theorems related to contractive mapping, which improve and generalize some standard fixed point results due to Nemytzki [17], Edelstein [8] and Suzuki [19].

Another importance of the (metric) fixed point theory is that it is an invaluable tool for finding existence and/or uniqueness criteria of solution(s) of several types of differential equations, integral equations, fractional integral equations, matrix equations, etc. In most of the cases, the fixed point results of usual metric spaces are applied to find the criteria as mentioned above. But the fixed point results of $b_v(s)$-metric spaces are yet to be employed to investigate for such tools. So at the end of this paper, we utilize one of our obtained results to find some criteria for the existence and uniqueness of solution of a special type of integral equation.

2 Preliminaries

It is well known that Nemytzki's result in [17], regarding contractive mappings, was first in the literature. Later on, Edelstein proved in [8] that a contractive self-mapping on a compact metric space has a unique fixed point. One can easily verify that, in this result, the compactness of X cannot be replaced by completeness. Therefore, some additional conditions have to be imposed on X or on T together with the completeness of X to ensure the existence of fixed point of T. Many researchers tried to find such additional conditions. One of such conditions is given by the next theorem, which was proved by Ćirić.

Theorem 1. (*See* [6].) *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and let for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon < d(x, y) < \epsilon + \delta \quad \Longrightarrow \quad d(Tx, Ty) \leqslant \epsilon$$

for any $x, y \in X$. Then T has a unique fixed point z, and for any $x \in X$, the sequence of iterates $(T^n x)$ converges to z.

Suzuki introduced another additional weaker assumption with the completeness of (X, d) to assure fixed point of T in the following theorem.

Theorem 2. (*See* [19, *Thm.* 5].) *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and let the following hold: For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d\big(T^i x, T^j x\big) < \epsilon + \delta \quad \Longrightarrow \quad d\big(T^{i+1} x, T^{j+1} x\big) \leqslant \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T has a unique fixed point z, and for any $x \in X$, the sequence of iterates $(T^n x)$ converges to z.

On the other hand, Mitrović and Radenović introduced the notions of Cauchy sequences and completeness in $b_v(s)$-metric space.

Definition 2. (See [16, Def. 1.9].) Let (X, d) be a $b_v(s)$-metric space, (x_n) be a sequence in X and $x \in X$.

(i) The sequence (x_n) is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > N$ and for all $p \in \mathbb{N}$.
(ii) The sequence (x_n) is said to be convergent to x if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$, and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(iii) (X, d) is said to be a complete $b_v(s)$-metric space if every Cauchy sequence in X converges to some $x \in X$.

In this manuscript, we now introduce the concepts of sequential compactness and bounded compactness of a $b_v(s)$-metric space.

Definition 3. Let (X, d) be a $b_v(s)$-metric space, and let (x_n) be a sequence in X. Then the sequence (x_n) is said to be a bounded sequence if there exists a real number $M > 0$ such that $d(x_n, x_m) \leqslant M$ for all $n, m \in \mathbb{N}$ or, equivalently, $d(x_n, x_{n+k}) \leqslant M$ for all $n, k \in \mathbb{N}$.

Definition 4. Let (X, d) be a $b_v(s)$-metric space. Then X is said to be sequentially compact if every sequence (x_n) in X has a subsequence, which converges to some point of X. Again, a subset A of X is said to be sequentially compact if every sequence (x_n) in A has a subsequence, which converges to some point of A.

Definition 5. Let (X, d) be a $b_v(s)$-metric space. Then X is said to be boundedly compact if every bounded sequence (x_n) in X has a subsequence, which converges to some point of X. Again, a subset A of X is said to be boundedly compact if every bounded sequence (x_n) in A has a subsequence, which converges to some point of A.

Clearly, it follows from the definition that every sequentially compact $b_v(s)$-metric space is boundedly compact but not conversely. To investigate this, we frame the following example.

Example 1. Consider the set $X = X_1 \cup X_2$, where $X_1 = \{1/n\colon\ n \in \mathbb{N} \text{ and } n \geqslant 2\}$ and $X_2 = \{0, 1, 2\}$. We define a function $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} |n-m| & \text{if } x, y \in X_1,\ x \neq y,\ x = \frac{1}{n},\ y = \frac{1}{m} \text{ and } |n-m| \neq 1, 3; \\ \frac{1}{2} & \text{if } x, y \in X_1,\ x \neq y,\ x = \frac{1}{n},\ y = \frac{1}{m} \text{ and } |n-m| = 1, 3; \\ n & \text{if } x \in X_1,\ y \in X_2 \text{ and } x = \frac{1}{n} \text{ or } y \in X_1,\ x \in X_2 \text{ and } y = \frac{1}{n}; \\ 5 & \text{if } x, y \in X_2 \text{ and } x \neq y; \\ 0 & \text{if } x, y \in X \text{ and } x = y. \end{cases}$$

Then it is an easy task to verify that (X, d) is a $b_3(2)$-metric space.

Note that the sequence $(1/(n+2))$ has no subsequence, which converges to some point of X. So, (X,d) is not sequentially compact. But one can easily check that a sequence (x_n) in X is bounded if and only if the range of the sequence (x_n) is finite. Thus every bounded sequence in X has a subsequence, which converges to some point of X, i.e, (X,d) is boundedly compact.

3 Comparison of $b_v(s)$-metric space with other spaces

In this section, we point out some properties of usual metric spaces and b-metric spaces, which are not true in case of $b_v(s)$-metric spaces. The first one is given in the following remark.

Remark 1. We know that a convergent sequence in usual metric spaces is always a Cauchy sequence. But this fact is not true in case of $b_v(s)$-metric spaces. To show this, we illustrate the following example in the line of [7, Ex. 1].

Example 2. Let us take $X = \{0, 1/n\colon n \in \mathbb{N}\}$. We define a function $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ \frac{1}{n} & \text{if } x = \frac{1}{n},\ y = 0 \text{ or } y = \frac{1}{n}, y = 0; \\ 1 + \frac{1}{n} + \frac{1}{m} & \text{if } x = \frac{1}{n},\ y = \frac{1}{m} \text{ and } n \neq m. \end{cases}$$

Then it is easy to check that (X,d) is a $b_2(1)$-metric space. Now we consider the sequence (x_n) in X where $x_n = 1/n$ for all $n \in \mathbb{N}$. Then we have

$$d\left(\frac{1}{n}, 0\right) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty,$$

but

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = 1 + \frac{1}{n} + \frac{1}{m} \to 1 \quad \text{as } n, m \to \infty.$$

Therefore, (x_n) is convergent in X but not a Cauchy sequence.

In the next part of this section, we state a lemma in $b_v(s)$-metric space, which also occurs in usual metric spaces.

Lemma 1. *Let (X,d_1) and (Y,d_2) be two $b_v(s)$-metric spaces. Then $(X \times Y, d_3)$ is also a $b_v(s)$-metric space, where d_3 is the product metric on $(X \times Y)$, i.e., $d_3 : (X \times Y) \times (X \times Y) \to \mathbb{R}$ is defined by*

$$d_3\big((x_1,y_1),(x_2,y_2)\big) = \max\big\{d_1(x_1,x_2), d_2(y_1,y_2)\big\}$$

for all $(x_1,y_1),(x_2,y_2) \in X \times Y$.

Proof. The proof of this lemma is similar to that of usual metric spaces. □

Remark 2. If (X, d) is a metric space, then we know that the function d is continuous on $X \times X$ with respect to the product metric on $X \times X$, but this is not true in case of $b_v(s)$-metric spaces. This fact can be substantiated from [7, Ex. 1] in case of rectangular metric spaces of Branciari, i.e., in $b_2(1)$-metric spaces.

Remark 3. We know that in a metric space (X, d) for a sequence (x_n), if there exists a real number μ such that $0 \leqslant \mu < 1$ and $d(x_n, x_{n+1}) \leqslant \mu \cdot d(x_{n-1}, x_n)$ holds for all $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence in X. This result also holds in b-metric spaces, i.e., in $b_1(s)$-metric spaces, which is proved in [14] by Miculescu and Mihail. Now in the next example, we show that this result may not hold in arbitrary $b_v(s)$-metric spaces.

Example 3. We consider the set $X = \mathbb{N}$. Define a function $d : X \times X \to \mathbb{R}$ by

$$d(n, m) = \begin{cases} 0 & \text{if } n = m; \\ \frac{1}{2^{\max\{n,m\}}} & \text{if } n \neq m \text{ and exactly one of } n, m \text{ is even;} \\ 1 & \text{if } n \neq m \text{ and } n, m \text{ both are even or both are odd.} \end{cases}$$

Then, clearly, (X, d) is not a b-metric space, i.e., $b_1(s)$-metric space, but (X, d) is a $b_2(1)$-metric space. Now we consider the sequence (x_n) in X where $x_n = n$ for all natural numbers n. Note that for any natural number n,

$$d(x_n, x_{n+1}) = \frac{1}{2^n}, \qquad d(x_{n-1}, x_n) = \frac{1}{2^{n-1}}.$$

Therefore,

$$d(x_n, x_{n+1}) \leqslant \mu d(x_{n-1}, x_n).$$

holds for $\mu = 1/2$. But (x_n) is not a Cauchy sequence, since $d(x_n, x_{n+2}) = 1$ for all natural numbers n.

4 Fixed point results

To begin with, we prove a fixed point theorem related to contractive mappings in the structure of sequentially compact $b_v(s)$-metric spaces.

Theorem 3. *Let (X, d) be a sequentially compact $b_v(s)$-metric space, and $T : X \to X$ a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Assume that the function d is continuous on $X \times X$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

Proof. At first, we fixed an element x_0 in X and then we consider a sequence (x_n), which is defined by $x_n = T^n x_0$ for every natural number n. Now we show that the sequence of real numbers (s_n), defined by $s_n = d(x_n, x_{n+1})$, converges to 0. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then clearly (s_n) converges to 0. So we now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we have

$$s_{n+1} = d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) = s_n.$$

This proves that (s_n) is a decreasing sequence of nonnegative real numbers and hence convergent to some $a \geqslant 0$. Again, by the sequential compactness of (X, d), there exists a convergent subsequence of (x_n), say, (x_{n_k}). Further, let this subsequence converges to $z \in X$. From the contractivity condition of T, it is continuous on X, and hence the subsequences (x_{n_k+1}) and (x_{n_k+2}) converge to Tz and T^2z respectively. Then we have

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(z, Tz).$$

Again, we have

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) = d(Tz, T^2z).$$

Now if $a > 0$, then $z \neq Tz$ and so we have $d(Tz, T^2z) < d(z, Tz)$, i.e., $a < a$, which is a contradiction. So we must have $a = 0$, i.e., (s_n) converges to 0. Further, since $a = 0$, we have $Tz = z$. So z is a fixed point of T.

Now we examine the uniqueness of this fixed point. To do this, let z_1 be another fixed point of T. Then we have

$$d(z, z_1) = d(Tz, Tz_1) < d(z, z_1),$$

which is a contradiction. Hence, z is the only fixed point of T.

Finally, we prove that (x_n) converges to z. If $x_n = z$ for some $n \in \mathbb{N}$, then clearly (x_n) converges to z. Let us now, assume that $x_n \neq z$ for all $n \in \mathbb{N}$. Since, (x_n) contains a subsequence (x_{n_k}), which converges to z, z is a cluster point of the sequence (x_n). Let z_1 be another cluster point of (x_n), then (x_n) contains a subsequence, which converges to z_1. Then by similar arguments as above we can prove that z_1 is a fixed point of T and this will again lead to a contradiction. Henceforth, z is the only cluster point of (x_n). Next, we consider the sequence (t_n) of real numbers given by $t_n = d(x_n, z)$ for all $n \in \mathbb{N}$. Therefore,

$$t_{n_k} = d(x_{n_k}, z) \to 0 \quad \text{as } n \to \infty.$$

This shows that the sequence (t_n) contains the subsequence (t_{n_k}), which converges to 0 and so 0 is a cluster point of $\{t_n\}$.

Again,

$$t_{n+1} = d(x_{n+1}, z) = d(Tx_n, Tz) < d(x_n, z) = t_n.$$

Consequently, (t_n) is a decreasing sequence of nonnegative real numbers and hence convergent. But 0 is a cluster point of the convergent sequence (t_n), so (t_n) must converge to 0. Therefore, $t_n \to 0$ as $n \to \infty$, i.e., $d(x_n, z) \to 0$ as $n \to \infty$. Thus, (x_n) converges to z, i.e., $(T^n x_0)$ converges to z. Since $x_0 \in X$ was arbitrary, it follows that $(T^n x)$ converges to the fixed point z for any $x \in X$. □

From Theorem 3, we can deduce the following corollary by taking $s = 1$ and $v = 1$.

Corollary 1. *Let (X, d) be a compact metric space and $T : X \to X$ a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point z, and for any $x \in X$, the sequence $(T^n x)$ converges to z.*

Remark 4. Corollary 1 extends Remark 3.1 of [8].

In the above theorem sequential compactness condition cannot be replaced by bounded compactness of the space, which follows from the following example.

Example 4. Let $X = \mathbb{N}$ and define a function $d : X \times X \to \mathbb{R}$ by

$$d(n, m) = \begin{cases} 0 & \text{if } n = m; \\ |n - m| + 1 + \frac{1}{n} + \frac{1}{m} & \text{if } n \neq m. \end{cases}$$

Then it is clear that (X, d) is a $b_1(1)$-metric space and d is continuous on $X \times X$. Also one can easily verify that (X, d) is boundedly compact but not sequentially compact. Next, we define a mapping $T : X \to X$ by

$$T(n) = n + 1$$

for all $n \in \mathbb{N}$. Then for any $n, m \in \mathbb{N}$ with $n \neq m$, we have

$$\begin{aligned} d(Tn, Tm) = d(n + 1, m + 1) &= |n - m| + 1 + \frac{1}{n + 1} + \frac{1}{m + 1} \\ &< |n - m| + 1 + \frac{1}{n} + \frac{1}{m} = d(n, m). \end{aligned}$$

Thus all the conditions of Theorem 3 are satisfied but T is fixed point free.

Therefore, we are in search of an additional condition either on X or on T with the bounded compactness of X to get a unique fixed point of T. Here in the next theorem we deal with one such additional condition.

Theorem 4. *Let (X, d) be a boundedly compact $b_v(s)$-metric space, and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Assume that the function d is continuous on $X \times X$. Further, assume that, for any $x \in X$ and for any $k \in \mathbb{N}$ with $k \geqslant v$, there exists a real number $M > 0$ (depending on x) such that $d(x, T^{k-v} x) \leqslant M$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.*

Proof. We choose an element x_0 from X and then we consider the sequence of iterates (x_n) where $x_n = Tx_{n-1}$ for all $n \geqslant 1$. If $x_n = x_{n+1}$ for some natural number n, then it is easy to notice that T has a fixed point, the fixed point is unique and the sequence (x_n) converges to that fixed point. So now we assume that $x_n \neq x_{n+1}$ for all natural numbers n. Then we claim that all terms of (x_n) are distinct. To prove our claim, we presume that $x_n = x_m$ for some natural numbers n, m with $m > n$. Then $Tx_n = Tx_m$ i.e. $x_{n+1} = x_{m+1}$. Therefore,

$$d(x_{n+1}, x_n) = d(x_{m+1}, x_m) < d(x_m, x_{m-1}) < \cdots < d(x_{n+1}, x_n),$$

which is a contradiction. So our claim is correct.

Now for any $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) < d(x_{n-2}, x_{n-1}) < \cdots < d(x_0, x_1).$$

Let $k, n \in \mathbb{N}$ be arbitrary but fixed. First suppose that $k \geqslant v$. Then by hypothesis we get a real number $M > 0$ such that $d(x_0, x_{k-v}) \leqslant M$. Therefore,

$$\begin{aligned} d(x_n, x_{n+k}) &\leqslant s\big\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+v}, x_{n+k})\big\} \\ &< s\big\{d(x_0, x_1) + d(x_0, x_1) + \cdots + d(x_{n+v-1}, x_{n+k-1})\big\} \\ &< \cdots < s\big\{vd(x_0, x_1) + d(x_0, x_{k-v})\big\} < s\big\{vd(x_0, x_1) + M\big\} \\ &= M_1, \quad \text{say.} \end{aligned}$$

Now suppose that $k < v$. Then we have

$$\begin{aligned} &d(x_n, x_{n+k}) \\ &\quad \leqslant s\big\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+v-1}, x_{n+v}) + d(x_{n+v}, x_{n+k})\big\} \\ &\quad < s\big\{d(x_0, x_1) + d(x_0, x_1) + \cdots + d(x_0, x_1) + d(x_{n+v-1}, x_{n+k-1})\big\} \\ &\quad < \cdots < s\big\{d(x_0, x_1) + d(x_0, x_1) + \cdots + d(x_0, x_1) + d(x_v, x_k)\big\}. \end{aligned} \tag{1}$$

Let $M_2 = \max_{k<v}\{d(x_v, x_k)\}$. Then, clearly, M_2 is finite. Thus by using equation (1), we get

$$d(x_n, x_{n+k}) < s\big\{vd(x_0, x_1) + M_2\big\} = M_3, \quad \text{say.}$$

Therefore, from our above discussions, we see that

$$d(x_n, x_{n+k}) < \max\{M_1, M_3\}$$

for all $n, k \in \mathbb{N}$, which shows that the sequence (x_n) is bounded. So by bounded compactness of X, there exists a convergent subsequence of (x_n), let it be (x_{n_k}). Let $\lim_{k\to\infty} x_{n_k} = z$. Then proceeding as Theorem 3, we can show that z is the unique fixed point of T and the sequence of iterates (x_n) converges to z. □

As a consequence of Theorem 4, we obtain the following corollary.

Corollary 2. *Let (X, d) be a boundedly compact $b_v(s)$-metric space such that d is continuous on $X \times X$, and let $T : X \to X$ be a bounded mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point, and for any $x \in X$, the sequence $(T^n x)$ converges to that fixed point.*

Taking $s = 1$ and $v = 1$ in Theorem 4, we get the following corollary.

Corollary 3. *Let (X, d) be a boundedly compact metric space, and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Further, assume that, for any $x \in X$ and for any $k \in \mathbb{N}$ with $k \geqslant 1$ there exists a real number $M > 0$ (depending on x) such that $d(x, T^{k-1}x) \leqslant M$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.*

Now we cite the following example, which supports the above theorem.

Example 5. Consider the set $X = \{1/n\colon\ n \in \mathbb{N},\ n \geqslant 2\}$. We define a mapping $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} |n - m| & \text{if } |n - m| \neq 1; \\ \frac{1}{2} & \text{if } |n - m| = 1. \end{cases}$$

Then it is an easy task to verify that (X, d) is a $b_3(3)$-metric space. Also, it is simply noticeable that (X, d) is boundedly compact but not sequentially compact.

Now we define a mapping $T : X \to X$ by

$$Tx = \frac{1}{4} \quad \text{for all } x \in X.$$

Then, clearly, $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

Further, for any $x \in X$ and for any $k \in \mathbb{N}$ with $k \geqslant 3$, there exists a real number $M > 0$ (depending on x) such that $d(x, T^{k-3}x) < M$ (here $M = 1/x + 4$). Then by Theorem 4, T has a unique fixed point. Note that $x = 1/4$ is the unique fixed point of T.

From the definitions of bounded compactness and completeness of $b_v(s)$ metric spaces we see that every boundedly compact $b_v(s)$-metric space is complete. Thus Example 4 also shows that if (X, d) is a complete $b_v(s)$-complete metric space and $T : X \to X$ is a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in$ with $x \neq y$, then T may not have a fixed point. So we again need some additional assumption with the completeness of X to warrant the fixed point of T.

Next, we consider the following example.

Example 6. Let (x_n^i) be a sequence whose ith term is 1 and all other terms are 0. Consider the set $X = \{(x_n^i)\colon\ i \in \mathbb{N}\}$. Now we define a function $d : X \times X \to \mathbb{R}$ by

$$d\big((x_n^i), (x_n^j)\big) = \begin{cases} 0, & \text{if } i = j; \\ 1 + \frac{100}{\sum_{n=1}^{\infty} |ix_n^i - jx_n^j|} & \text{if } i, j \leqslant 10; \\ 1 + \frac{10}{\sum_{n=1}^{\infty} |ix_n^i - jx_n^j|} & \text{elsewhere.} \end{cases}$$

Then it is trivial to check that (X, d) is a $b_2(10)$-metric space. Further, (X, d) is complete but not boundedly compact. We now define $T : X \to X$ by

$$T\big((x_n^i)\big) = x_n^{i+11}$$

for all $(x_n^i) \in X$. Let $(x_n^i), (x_n^j) \in X$ be arbitrary with $i \neq j$. Then the following three cases may arise.

Case 1. Let $i, j \leqslant 10$. Then

$$d\big(T(x_n^i), T(x_n^j)\big) = d\big((x_n^{i+11}), (x_n^{j+11})\big) = \frac{10}{i+j+22} + 1 < \frac{100}{i+j} + 1 = d\big((x_n^i), (x_n^j)\big).$$

Case 2. Let $i, j > 10$. Then

$$d\big(T(x_n^i), T(x_n^j)\big) = d\big((x_n^{i+11}), (x_n^{j+11})\big) = \frac{10}{i+j+22} + 1 < \frac{10}{i+j} + 1 = d\big((x_n^i), (x_n^j)\big).$$

Case 3. Let exactly any one of i and j is greater than 10. Then

$$d\big(T(x_n^i), T(x_n^j)\big) = d\big((x_n^{i+11}), (x_n^{j+11})\big) = \frac{10}{i+j+22} + 1 < \frac{10}{i+j} + 1 = d\big((x_n^i), (x_n^j)\big).$$

Thus, $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

Note that for any $x \in X$ and for any $k \in \mathbb{N}$ with $k \geqslant v$ (here $v = 2$), there exists a real number $M > 0$ (here we may take $M = 200$) such that $d(x, T^{k-v}x) \leqslant M$ but still T has no fixed point.

Thus we see that the additional condition, which is considered in Theorem 4 together with the completeness of X, does not deliver fixed point of T. It means that we have to find out some different additional condition with the completeness of X so as to ensure the existence of fixed point of T. In the following theorem, we consider such an additional condition in case of $b_v(1)$-metric spaces, which was firstly given by Suzuki [19].

Theorem 5. *Let (X, d) be a complete $b_v(1)$-metric space, and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. Further, assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d\big(T^i x, T^j x\big) < \epsilon + \delta \quad \Longrightarrow \quad d\big(T^{i+1} x, T^{j+1} x\big) \leqslant \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

Proof. First, we choose an arbitrary but fixed element x_0 in X, and then we consider a sequence (x_n), which is defined by $x_n = T^n x_0$ for every natural number n. If $x_n = x_{n+1}$ for some natural number n, then x_n is the unique fixed point of T. So, now we assume that $x_n \neq x_{n+1}$ for every natural numbers n. Under this assumption, following Theorem 4, we can show that all terms of (x_n) are distinct.

Next, we consider the sequence of real numbers (s_n), where $s_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then

$$s_{n+1} = d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) = s_n$$

for all $n \in \mathbb{N}$. Therefore, (s_n) is a decreasing sequence of nonnegative real numbers and hence convergent to some $a \geqslant 0$. If $a > 0$, then by given condition there exists $\delta > 0$ such that

$$d(x_n, x_{n+1}) < a + \delta \quad \Longrightarrow \quad d(x_{n+1}, x_{n+2}) \leqslant a$$

for all $n \in \mathbb{N}$. Again, by definition of a, for the above $\delta > 0$, there exists $n \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < a + \delta$.

Therefore, $d(x_{n+1}, x_{n+2}) \leqslant a$, and this leads to a contradiction. So we must have $a = 0$, i.e.,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2}$$

In a similar manner, we can show that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{3}$$

Now let $\epsilon > 0$ be arbitrary. Then by given condition we get a $\delta > 0$ such that

$$d(x_n, x_{n+1}) < \frac{\epsilon}{2} + \delta \quad \Longrightarrow \quad d(x_{n+1}, x_{n+2}) \leqslant \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$.

On the other hand, by equations (2) and (3), for the above $\delta > 0$, there exists a natural number N such that

$$d(x_n, x_{n+1}) < \frac{\delta}{2v}, \qquad d(x_n, x_{n+2}) < \frac{\delta}{2v} \tag{4}$$

for all $n \geqslant N$.

Now for each $n \geqslant N$, we show by mathematical induction on j that

$$d(x_{n+2v+1}, x_{n+2v+j}) \leqslant \frac{\epsilon}{2}$$

holds for all $j \in \mathbb{N}$. The result is obviously true for $j = 1$. Assume that the result is true for some $j \in \mathbb{N}$. So, $d(x_{n+2v+1}, x_{n+2v+j}) \leqslant \epsilon/2$, which implies

$$d(x_{n+3v+1}, x_{n+3v+j}) \leqslant \frac{\epsilon}{2}. \tag{5}$$

Then

$$\begin{aligned}
&d(x_{n+2v}, x_{n+2v+j}) \\
&\quad \leqslant \bigl\{d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + d(x_{n+2v+3}, x_{n+2v+4}) \\
&\qquad + \cdots + d(x_{n+2v+v}, x_{n+2v+v+1}) + d(x_{n+2v+v+1}, x_{n+2v+j})\bigr\} \\
&\quad = \bigl\{d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + d(x_{n+2v+3}, x_{n+2v+4}) \\
&\qquad + \cdots + d(x_{n+3v}, x_{n+3v+1})\bigr\} + d(x_{n+2v+j}, x_{n+3v+1}) \\
&\quad \leqslant \bigl\{d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + d(x_{n+2v+3}, x_{n+2v+4}) \\
&\qquad + \cdots + d(x_{n+3v}, x_{n+3v+1})\bigr\} \\
&\qquad + \bigl\{d(x_{n+2v+j}, x_{n+2v+j+1}) + (x_{n+2v+j+1}, x_{n+2v+j+2}) \\
&\qquad + \cdots + d(x_{n+2v+j+v-1}, x_{n+2v+j+v}) + d(x_{n+2v+j+v}, x_{n+3v+1})\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&= \big\{d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) \\
&\quad + d(x_{n+2v+3}, x_{n+2v+4}) + \cdots + d(x_{n+3v}, x_{n+3v+1})\big\} \\
&\quad + \big\{d(x_{n+2v+j}, x_{n+2v+j+1}) + d(x_{n+2v+j+1}, x_{n+2v+j+2}) \\
&\quad + \cdots + d(x_{n+3v+j-1}, x_{n+3v+j})\big\} + d(x_{n+3v+j}, x_{n+3v+1}).
\end{aligned} \tag{6}
$$

Using equations (4) and (5) in equation (6), we get

$$
\begin{aligned}
&d(x_{n+2v}, x_{n+2v+j}) < \left(\frac{\delta}{2v} + \frac{\delta}{2v} + \cdots + \frac{\delta}{2v}\right) + \left(\frac{\delta}{2v} + \frac{\delta}{2v} + \cdots + \frac{\delta}{2v}\right) + \frac{\epsilon}{2} \\
&\implies \quad d(x_{n+2v}, x_{n+2v+j}) < \frac{\epsilon}{2} + \delta
\end{aligned}
$$

for all $n \geqslant N$. This implies that

$$d(x_{n+2v+1}, x_{n+2v+j+1}) \leqslant \frac{\epsilon}{2}$$

for all $n \geqslant N$. Thus by induction it follows that

$$d(x_{n+2v+1}, x_{n+2v+j}) \leqslant \frac{\epsilon}{2} < \epsilon$$

for all $n \geqslant N$ and for all $j \in \mathbb{N}$, which proves that (x_n) is a Cauchy sequence. So, by the completeness of X, (x_n) converges to some $z \in X$. Again, since T is contractive, the sequence (x_n) converges to Tz also. So z is a fixed point of T. The uniqueness of the fixed point can be analogously proved from Theorem 3.

Since we choose $x_0 \in X$ arbitrarily, it follows that $(T^n x)$ converges to the unique fixed point z for all $x \in X$. □

It is very interesting to verify whether the additional criteria, used in the above theorem, will work for $s > 1$ or not. If not, then it is also necessary to find an additional assumption, which will ensure the existence of a fixed point in the complete $b_v(s)$-metric structure. In this respect, we pose the following open problem.

Open question. *Let (X, d) be a complete $b_v(s)$-metric space, and let T be a self-mapping on X such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. If $s > 1$, then find out a weaker additional assumption on T, which will ensure that T has a fixed point.*

From Theorem 5 we can derive several important corollaries. We present a number of selected ones, which extend several well-known results in the literature.

Corollary 4. (*See* [19, *Thm.* 5].) *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Further, assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(T^i x, T^j x) < \epsilon + \delta \quad \implies \quad d(T^{i+1} x, T^{j+1} x) \leqslant \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

Corollary 5. *Let (X,d) be a complete rectangular metric space, and let $T : X \to X$ be a mapping such that $d(Tx,Ty) < d(x,y)$ for all $x, y \in X$ with $x \neq y$. Further, assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d\bigl(T^i x, T^j x\bigr) < \epsilon + \delta \quad \Longrightarrow \quad d\bigl(T^{i+1}x, T^{j+1}x\bigr) \leqslant \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

Now we present an example in order to endorse Theorem 5.

Example 7. Consider the set $X = C[0,1]$ and let $X_1 = \{x \in C[0,1]\colon \sup_{0\leqslant t\leqslant 1}|x(t)| \leqslant 1\}$. Let us define a function $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} \sup_{0\leqslant t\leqslant 1}|x(t)-y(t)| & \text{if } x,y \in X_1;\\ \frac{2}{3} & \text{if } x,y \in X\setminus X_1 \text{ and } x\neq y;\\ 3 & \text{if any one of } x \text{ and } y \text{ lies in } X\\ & \text{and the other in } X\setminus X_1;\\ 0 & \text{elsewhere.}\end{cases}$$

Then it is easy to check that (X,d) is a $b_2(1)$-metric space. Next, we define a mapping $T : X \to X$ by

$$(Tx)(t) = \begin{cases} \frac{t}{2}x(t) & \text{if } x \in X_1;\\ 0 & \text{if } x \in X\setminus X_1.\end{cases}$$

Let $x, y \in X$ be arbitrary with $x \neq y$. Then the following three cases may arise.

Case 1. Let $x, y \in X_1$. Then

$$\begin{aligned} d(Tx,Ty) &= \sup_{0\leqslant t\leqslant 1}\left|\frac{t}{2}x(t) - \frac{t}{2}y(t)\right| \leqslant \frac{1}{2}\sup_{0\leqslant t\leqslant 1}\bigl|x(t)-y(t)\bigr| \\ &< d(x,y). \end{aligned}$$

Case 2. Let $x, y \in X \setminus X_1$. Then

$$d(Tx,Ty) = 0 < d(x,y).$$

Case 3. Let $x \in X_1$ and $y \in X \setminus X_1$. Then

$$d(Tx,Ty) = \sup_{0\leqslant t\leqslant 1}\left|\frac{t}{2}x(t)\right| < 3 = d(x,y).$$

Thus $d(Tx,Ty) < d(x,y)$ for all $x, y \in X$ with $x \neq y$. Next, assume that $x \in X$ and $\epsilon > 0$ be arbitrary. Here we choose $\delta = \epsilon$. Therefore, if $x \in X \setminus X_1$, then, clearly,

$$d\bigl(T^i x, T^j x\bigr) < \epsilon + \delta \quad \Longrightarrow \quad d\bigl(T^{i+1}x, T^{j+1}x\bigr) \leqslant \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Now we assume that $x \in X_1$ and $d(T^i x, T^j x) < \epsilon + \delta$ for some $i, j \in \mathbb{N} \cup \{0\}$. Therefore,

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant 1} \left| \left(\frac{t}{2}\right)^i x(t) - \left(\frac{t}{2}\right)^j x(t) \right| < \epsilon + \epsilon \\ \implies \quad &\sup_{0 \leqslant t \leqslant 1} \left| \left(\frac{t}{2}\right)^{i+1} x(t) - \left(\frac{t}{2}\right)^{j+1} x(t) \right| < \epsilon \\ \implies \quad &d\big(T^{i+1}x, T^{j+1}x\big) < \epsilon. \end{aligned}$$

Thus we see that all conditions of Theorem 5 hold good. So by this theorem, T has a unique fixed point in X. Indeed, $x : [0,1] \to \mathbb{R}$, defined by $x(t) = 0$ for all $t \in [0,1]$, is the fixed point of T in X.

5 Application to an integral equation

In this section, we will apply our results, established in previous section, to a mixed Fredholm–Volterra type integral equation of the following form:

$$y(t) = f(t) + \int_0^t G_1(x,t) f_1\big(x, y(x)\big)\,\mathrm{d}x + \int_0^1 G_2(x,t) f_2\big(x, y(x)\big)\,\mathrm{d}x, \quad 0 \leqslant t \leqslant 1, \tag{7}$$

where $f : [0,1] \to \mathbb{R}^+$, $G_1, G_2 : [0,1] \times [0,1] \to \mathbb{R}^+$ and $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions with $f(0) = G_1(t,0) = G_2(t,0) = f_1(t,0) = f_2(t,0) = 0$.

We utilize Theorem 3 to develop some conditions on the functions f, f_1, f_2, G_1 and G_2, which will ensure the existence and uniqueness of solution of equation (7). We present these conditions in the following theorem.

Theorem 6. *Let us consider the integral equation given by equation* (7)*, and assume that the following conditions hold*:

(i) *There exists five positive constants α_1, α_2, β_1, β_2 and β such that* $|f(t_1 - f(t_2)| \leqslant \beta|t_1 - t_2|$, $|f_1(x,y_1) - f_1(x,y_2)| \leqslant \beta_1|y_1 - y_2|$, $|f_2(x,y_1) - f_2(x,y_2)| \leqslant \beta_2|y_1 - y_2|$, $|G_1(x,t_1) - G_1(x,t_2)| \leqslant \alpha_1|t_1 - t_2|$ *and* $|G_2(x,t_1) - G_2(x,t_2)| \leqslant \alpha_2|t_1 - t_2|$ *hold for all* $x, t_1, t_2 \in [0,1]$ *and* $y_1, y_2 \in \mathbb{R}$;

(ii) $\beta + (7/2)\alpha_1\beta_1 + \alpha_2\beta_2 \leqslant 1$.

Then the integral equation acquires a unique nonnegative solution $y(t)$ with $|\int_0^1 y(t)\mathrm{d}t| \leqslant 1$.

Proof. Let $C[0,1]$ be the set of all real-valued continuous functions, which are defined on $[0,1]$, and consider a function $d : C[0,1] \times C[0,1] \to \mathbb{R}$ defined by

$$d(x,y) = \sup_{0 \leqslant t \leqslant 1} \big|x(t) - y(t)\big|^2$$

for all $x, y \in C[0,1]$. Then one can comfortably certify that d is a $b_1(2)$-metric on $C[0,1]$ and d is continuous on $X \times X$. Let us take $A = \{y \in C[0,1]\colon\ |y(t_1) - y(t_2)| \leqslant |t_1 - t_2|$ for all $t_1, t_2 \in [0,1],\ y(t) \geqslant 0$ for all $t \in [0,1]$ and $|\int_0^1 y(t)\,\mathrm{d}t| \leqslant 1\}$. Then by utilizing Arzela–Ascoli theorem, we can easily verify that A is a sequentially compact subset of the $b_1(2)$-metric space $(C[0,1], d)$.

Now we define a mapping T on A as follows:

$$Ty(t) = f(t) + \int_0^t G_1(x,t) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x + \int_0^1 G_2(x,t) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x$$

for all $y(t) \in A$. Let $t_1, t_2 \in [0,1]$ be arbitrary. Without loss of generality, we assume that $t_2 \geqslant t_1$. Then we have

$$\begin{aligned}
&\bigl|Ty(t_1) - Ty(t_2)\bigr| \\
&\quad = \Biggl| f(t_1) + \int_0^{t_1} G_1(x,t_1) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x + \int_0^1 G_2(x,t_1) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x \\
&\qquad - f(t_2) - \int_0^{t_2} G_1(x,t_2) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x - \int_0^1 G_2(x,t_2) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| \\
&\quad \leqslant \bigl|f(t_1) - f(t_2)\bigr| + \Biggl| \int_0^{t_1} G_1(x,t_1) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x - \int_0^{t_2} G_1(x,t_2) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| \\
&\qquad + \Biggl| \int_0^1 G_2(x,t_1) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x - \int_0^1 G_2(x,t_2) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| \\
&\quad \leqslant \beta|t_1 - t_2| + \Biggl| \int_0^{t_1} G_1(x,t_1) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x - \int_0^{t_1} G_1(x,t_2) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| \\
&\qquad + \Biggl| \int_{t_1}^{t_2} v G_1(x,t_2) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| + \Biggl| \int_0^1 f_2\bigl(x, y(x)\bigr)\bigl\{G_2(x,t_1) - G_2(x,t_2)\bigr\}\,\mathrm{d}x \Biggr|. \qquad (8)
\end{aligned}$$

Now

$$\begin{aligned}
&\Biggl| \int_0^{t_1} G_1(x,t_1) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x - \int_0^{t_1} G_1(x,t_2) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x \Biggr| \\
&\quad \leqslant \int_0^{t_1} \bigl|G_1(x,t_1) - G_1(x,t_2)\bigr| \bigl|f_1\bigl(x, y(x)\bigr)\bigr|\,\mathrm{d}x \leqslant \int_0^{t_1} \alpha_1\beta_1|t_1 - t_2| \bigl|y(x)\bigr|\,\mathrm{d}x \\
&\quad = \alpha_1\beta_1|t_1 - t_2| \Biggl| \int_0^{t_1} y(x)\,\mathrm{d}x \Biggr| \leqslant \alpha_1\beta_1|t_1 - t_2|.
\end{aligned}$$

Similarly, we can show that

$$\left|\int_0^1 f_2\bigl(x,y(x)\bigr)\bigl\{G_2(x,t_1)-G_2(x,t_2)\bigr\}\,\mathrm{d}x\right| \leqslant \alpha_2\beta_2|t_1-t_2|.$$

Again,

$$\begin{aligned}
&\left|\int_{t_1}^{t_2} G_1(x,t_2)f_1\bigl(x,y(x)\bigr)\,\mathrm{d}x\right| \\
&\quad\leqslant \int_{t_1}^{t_2} \bigl|G_1(x,t_2)\bigr|\bigl|f_1\bigl(x,y(x)\bigr)\bigr|\,\mathrm{d}x \leqslant \alpha_1\beta_1\int_{t_1}^{t_2}\bigl|y(x)\bigr|\,\mathrm{d}x \\
&\quad\leqslant \alpha_1\beta_1\int_{t_1}^{t_2}\bigl|y(x)\bigr|\,\mathrm{d}x \leqslant \frac{5}{2}\alpha_1\beta_1|t_1-t_2| \quad \left(\text{since } y\in A,\ \text{so } \bigl|y(x)\bigr|\leqslant\frac{5}{2}\right).
\end{aligned}$$

Employing the above three facts in equation (8), we get

$$\bigl|Ty(t_1)-Ty(t_2)\bigr| \leqslant \left(\beta+\frac{7}{2}\alpha_1\beta_1+\alpha_2\beta_2\right)|t_1-t_2| \leqslant |t_1-t_2|. \tag{9}$$

Next, $|\int_0^1 f(t)\,\mathrm{d}t| \leqslant \int_0^1 \beta t\,\mathrm{d}t = \beta/2$,

$$\begin{aligned}
&\left|\int_0^1\left\{\int_0^t G_1(x,t)f_1\bigl(x,y(x)\bigr)\,\mathrm{d}x\right\}\mathrm{d}t\right| \\
&\quad\leqslant \int_0^1\int_0^t \bigl|G_1(x,t)f_1\bigl(x,y(x)\bigr)\bigr|\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^1\int_0^t \bigl|\alpha_1\beta_1 t y(x)\bigr|\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{\alpha_1\beta_1}{2}.
\end{aligned}$$

Similarly, we can show that

$$\left|\int_0^1\left\{\int_0^1 G_2(x,t)f_2\bigl(x,y(x)\bigr)\,\mathrm{d}x\right\}\mathrm{d}t\right| \leqslant \frac{\alpha_2\beta_2}{2}.$$

Therefore, we have

$$\begin{aligned}
&\left|\int_0^1 f(t)\,\mathrm{d}t\right| + \left|\int_0^1\left\{\int_0^t G_1(x,t)f_1\bigl(x,y(x)\bigr)\,\mathrm{d}x\right\}\mathrm{d}t\right| \\
&\quad + \left|\int_0^1\left\{\int_0^1 G_2(x,t)f_2\bigl(x,y(x)\bigr)\,\mathrm{d}x\right\}\mathrm{d}t\right| \leqslant \frac{1}{2}(\beta+\alpha_1\beta_1+\alpha_2\beta_2)
\end{aligned}$$

$$\implies \quad \left|\int_0^1 Ty(t)\right| \leqslant \frac{1}{2}(\beta+\alpha_1\beta_1+\alpha_2\beta_2) \leqslant 1 \quad \bigl(\text{using assumption (ii)}\bigr).$$

Also, it can be easily verified that $Ty(t) \geqslant 0$ for all $t \in [0,1]$. Therefore, $Ty \in A$, and hence T is a self-map on A.

Next, let $y_1, y_2 \in A$ be arbitrary with $y_1 \neq y_2$. Then for any $t \in [0,1]$, we have

$$\begin{aligned}
&\bigl|Ty_1(t) - Ty_2(t)\bigr| \\
&\quad = \Biggl|f(t) + \int_0^t G_1(x,t) f_1\bigl(x, y_1(x)\bigr)\,\mathrm{d}x + \int_0^1 G_2(x,t) f_2\bigl(x, y_1(x)\bigr)\,\mathrm{d}x \\
&\qquad - f(t) - \int_0^t G_1(x,t) f_1\bigl(x, y_2(x)\bigr)\,\mathrm{d}x - \int_0^1 G_2(x,t) f_2\bigl(x, y_2(x)\bigr)\,\mathrm{d}x\Biggr| \\
&\quad \leqslant \int_0^t \bigl|G_1(x,t)\bigr|\bigl|f_1\bigl(x,y_1(x)\bigr) - f_1\bigl(x,y_2(x)\bigr)\bigr|\,\mathrm{d}x \\
&\qquad + \int_0^1 \bigl|G_2(x,t)\bigr|\bigl|f_2\bigl(x,y_1(x)\bigr) - f_2\bigl(x,y_2(x)\bigr)\bigr|\,\mathrm{d}x \\
&\quad \leqslant \int_0^1 \alpha_1\beta_1 t\bigl|y_1(x) - y_2(x)\bigr|\,\mathrm{d}x + \int_0^1 \alpha_2\beta_2 t\bigl|y_1(x) - y_2(x)\bigr|\,\mathrm{d}x \\
&\quad \leqslant \int_0^1 (\alpha_1\beta_1 + \alpha_2\beta_2)\bigl|y_1(x) - y_2(x)\bigr|\,\mathrm{d}x.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl|Ty_1(t) - Ty_2(t)\bigr|^2 \\
&\quad \leqslant \Biggl\{\int_0^1 (\alpha_1\beta_1 + \alpha_2\beta_2)\bigl|y_1(x) - y_2(x)\bigr|\,\mathrm{d}x\Biggr\}^2 \\
&\quad \leqslant \int_0^1 (\alpha_1\beta_1 + \alpha_2\beta_2)^2\bigl|y_1(x) - y_2(x)\bigr|^2\,\mathrm{d}x \leqslant (\alpha_1\beta_1 + \alpha_2\beta_2) d(y_1, y_2).
\end{aligned}$$

The above relation is true for all $t \in [0,1]$, and hence we have

$$\begin{aligned}
&d(Ty_1, Ty_2) \leqslant (\alpha_1\beta_1 + \alpha_2\beta_2) d(y_1, y_2) \\
&\implies \quad d(Ty_1, Ty_2) < d(y_1, y_2).
\end{aligned}$$

Thus we see that all the conditions of Theorem 3 hold good here, and so we can assure that T has a unique fixed point in A, say, y. By the formulation T and A we see that y is the unique solution of equation (7), and the solution y is nonnegative and satisfies the condition $|\int_0^1 y(t)\,\mathrm{d}t| \leqslant 1$. □

It is well known that the mixed Fredholm–Volterra integral equations arise from the mathematical model of the spatio-temporal developments of an epidemic model and also from several parabolic boundary value problems. It may be noted that these are all associated to some physical problems. Here we demonstrate a specific example, which validates the aforementioned result.

Example 8. Let us consider the integral equation

$$y(t) = f(t) + \int_0^t G_1(x,t) f_1\bigl(x, y(x)\bigr)\,\mathrm{d}x + \int_0^1 G_2(x,t) f_2\bigl(x, y(x)\bigr)\,\mathrm{d}x, \quad 0 \leqslant t \leqslant 1, \tag{10}$$

where we take $f(t) = t/25 - t^7/3000$, $G_1(x,t) = (x^2t^2)/30$, $G_2(x,t) = (12/35)xt$, $f_1(x,y) = x|y|$ and $f_2(x,y) = (x+1)|y|$.

If we choose $\beta = 127/3000$, $\alpha_1 = 1/15$, $\alpha_2 = 12/35$, $\beta_1 = 1$ and $\beta_2 = 2$, then one can easily verify that all the assumptions of Theorem 6 are satisfied. So by the same theorem, the integral equation (10) has a unique nonnegative solution $y(t)$ satisfying $|\int_0^1 y(t)\,\mathrm{d}t| \leqslant 1$. Further, note that $y(t) = t/20$ is the unique solution of equation (10).

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