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A switching control for finite-time synchronization of memristor-based BAM neural networks with stochastic disturbances*

Liangchen Li[a], **Rui Xu**[b,1], **Qintao Gan**[a], **Jiazhe Lin**[a]

[a]Institute of Applied Mathematics, Army Engineering University,
Shijiazhuang 050003, Hebei, China
llc0610@126.com; ganqintao@sina.com; linjiazhe0413@163.com

[b]Complex Systems Research Center, Shanxi University,
Taiyuan 030006, Shanxi, China
xurui@sxu.edu.cn

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Abstract. This paper deals with the finite-time stochastic synchronization for a class of memristor-based bidirectional associative memory neural networks (MBAMNNs) with time-varying delays and stochastic disturbances. Firstly, based on the physical property of memristor and the circuit of MBAMNNs, a MBAMNNs model with more reasonable switching conditions is established. Then, based on the theory of Filippov's solution, by using Lyapunov–Krasovskii functionals and stochastic analysis technique, a sufficient condition is given to ensure the finite-time stochastic synchronization of MBAMNNs with a certain controller. Next, by a further discussion, an error-dependent switching controller is given to shorten the stochastic settling time. Finally, numerical simulations are carried out to illustrate the effectiveness of theoretical results.

Keywords: finite-time synchronization, BAM neural networks, stochastic disturbances, switching controller.

1 Introduction

In 1971, Chua postulated a new kind of passive circuit element called memristor, which can connect the charge and magnetic flux [7]. In [8], he further explained that the current-voltage characteristic of a memristor under a bipolar periodic electrical signal should be a pinched hysteretic line, which reflects that the memristance (resistance of memristor) of a memristor depends on how much charge has passed through the memristor in a particular direction. This feature is nonvolatile, i.e. the memristor can keep its memristance without excitation voltage. These two properties make memristor an attractive candidate for the synapses in artificial neural networks.

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[1]Corresponding author.

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Bidirectional associative memory neural networks (BAMNNs) were proposed in 1988 [15]. They display a two-way associative search for stored bipolar vector pairs and generalize the single-layer autoassociative Hebbian correlation to a two-layer pattern-matched heteroassociative circuits [37]. This kind of neural networks have been successfully applied in various fields, including automatic control, pattern recognition and associative memory. Memristor-based bidirectional associative memory neural networks (MBAMNNs) can be implemented by replacing the resistors with memristors in VLSI circuits of conventional BAMNNs. Memristor-based neural networks can achieve many brain-like functions. In [21], Pershin and Ventra demonstrated the formation of associative memory in a simple neural network consisting of three electronic neurons connected by two memristor-emulator synapses by experiments. The application of memristors can also make BAMNNs have more abundant dynamic behaviors and a broader application prospect.

The synchronization of BAMNNs is an interesting and meaningful topic. Some results about the synchronization of traditional BAMNNs can be found in [20,36]. Moreover, we find scholars are more interested in the synchronization of MBAMNNs recently. In [19] and [32], authors achieved the synchronization of MBAMNNs with impulsive control and sampled-data control, respectively. In [5], authors studied the adaptive synchronization of MBAMNNs with mixed delays. In [24], the anti-synchronization conditions for MBAMNNs with different memductance functions were analysed. These results are of infinite-time synchronization, which implies that synchronization may only be reached in infinite time and inherently requires persistent external control. In [4, 33], Chen et al. and Yuan et al. studied the finite-time synchronization of MBAMNNs, which means that the MBAMNNs can achieve synchronization in a finite time called the settling time. So comparing with infinite-time synchronization, finite-time synchronization is more practical in many real applications and is worthy of further study. Some other researches on finite-time synchronization of BAMNNs and other kinds of neural networks can be seen in [1, 25, 31, 35] and references therein.

When we study the synchronization of neural networks, there are some practical factors that should not be ignored in modeling. Firstly, time delays frequently occur in the response and communication of neurons, which may be caused by the limited transfer speed and information processing. The delays in neural networks may result in instability or oscillation, which has been pointed out in many articles [3, 16, 23, 28, 30]. Then, the real systems are always in an external noisy environment, so they are inevitably disturbed by stochastic disturbances. Therefore, in [17, 27], authors studied the synchronization of neural networks with stochastic disturbances. So far, to our best knowledge, results on finite-time synchronization of MBAMNNs with time-varying delays and stochastic disturbances has not been reported in the literature.

Motivated by the discussion above, in this paper, we study the finite-time synchronization of MBAMNNs with time-varying delays and stochastic disturbances. Based on the circuit structure of the MBAMNNs and the basic principle of memristors, we come up with a new kind of MBAMNNs models consisted of stochastic differential equations that discontinuous on the right-hand side. To analyse this kind of models, we use the stochastic differential inclusion theory under the framework of Filippov's solution [10].

By constructing suitable Lyapunov–Krasovskii functionals and using stochastic analysis technique, a sufficient condition is given to ensure the finite-time stochastic synchronization of MBAMNNs with a certain controller. Then, inspired by the work of Gao et al. in [11], we design an error-dependent switching controller to achieve the synchronization of MBAMNNs with a shorter stochastic settling time.

The rest of this paper is organized as follows. The model description, some preliminaries, necessary definitions and lemmas are presented in Section 2. In Section 3, the sufficient conditions to achieve finite-time synchronization of MBAMNNs are derived. In Section 4 numerical examples are given to demonstrate the feasibility of the theoretical results. Finally, we make a summary in Section 5.

Notations. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Given column vectors $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, where the superscript T represents the transpose operator. $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ denotes vector norm of x. For $\tau > 0$, $C([-\tau, 0], \mathbb{R}^n)$ denotes the family of continuous function ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|_c = \sup_{-\tau \leqslant s \leqslant 0} |\phi(s)|$. $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ denotes the family of all nonnegative functions on $\mathbb{R}^n \times \mathbb{R}^+$, which are twice differentiable in x and differentiable in t. $\mathrm{co}[a,b]$ represents closure of the convex hull generated by a and b. $\mathbf{E}(\cdot)$ stands for the mathematical expectation operator.

2 Model description and preliminaries

Considering the memristor manufactured by Hewlett–Packard laboratory [29], it is a $\mathrm{Pt}/\mathrm{TiO}_{2-\mathrm{x}}/\mathrm{Pt}$ structure in which doped TiO_2 and undoped TiO_2 are fabricated between two Pt electrodes. The memristor model was given by

$$v(t) = \left(R^{\mathrm{on}} \frac{w(t)}{D} + R^{\mathrm{off}} \left(1 - \frac{w(t)}{D}\right)\right) i(t), \qquad \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \mu_{\mathrm{V}} \frac{R^{\mathrm{on}}}{D} i(t),$$

which yields

$$R_{\mathrm{mem}}(t) = R^{\mathrm{off}} + \left(R^{\mathrm{on}} - R^{\mathrm{off}}\right)\left(w_0 + \mu_V \frac{R^{\mathrm{on}}}{D^2} \int_{t_0}^{t} i(s)\,\mathrm{d}s\right),$$
$$\frac{\mathrm{d}R_{\mathrm{mem}}(t)}{\mathrm{d}t} = \left(R^{\mathrm{on}} - R^{\mathrm{off}}\right)\mu_V \frac{R^{\mathrm{on}}}{D^2} \frac{v(t)}{R_{\mathrm{mem}}(t)},$$

where D is the full length of the device; $w(t)$ is the length of doped region, w_0 is the initial length of doped region; R^{off} and R^{on} represent the maximum and minimum memristance, respectively; $R_{\mathrm{mem}}(t)$ is the memristance of the memristor; μ_V is the average ion mobility rate in the memristor. The memductance of the memristor can be derived as

$$\begin{aligned} M_{\mathrm{mem}}(t) &= \frac{1}{R_{\mathrm{mem}}(t)}, \\ \frac{\mathrm{d}M_{\mathrm{mem}}(t)}{\mathrm{d}t} &= \left(R^{\mathrm{off}} - R^{\mathrm{on}}\right)\mu_V \frac{R^{\mathrm{on}}}{D^2} v(t)\left(M_{\mathrm{mem}}(t)\right)^3. \end{aligned} \tag{1}$$

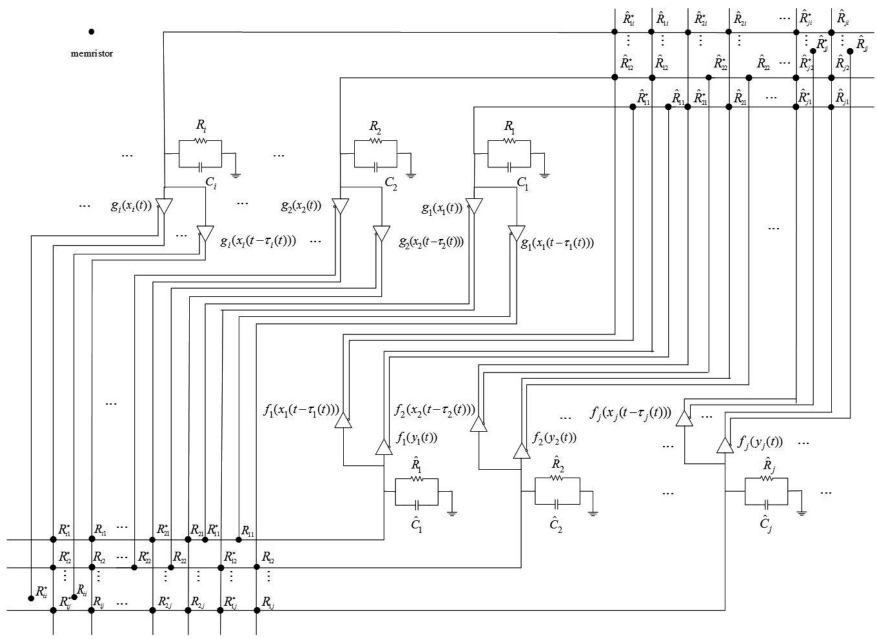

Figure 1. Circuit diagram of MBAMNNs (2).

In this paper, based on the circuit shown in Fig. 1, we consider the following memristor-based BAM neural networks:

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -d_i\big(x_i(t)\big)x_i(t) + \sum_{j=1}^{m} \mathrm{sign}_{ij}\, a_{ij}\big(x_i(t)\big) f_j\big(y_j(t)\big) \\
&\quad + \sum_{j=1}^{m} \mathrm{sign}_{ij}\, b_{ij}\big(x_i(t)\big) f_j\big(y_j\big(t-\tau_j(t)\big)\big), \\
\frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -r_j\big(y_j(t)\big)y_j(t) + \sum_{i=1}^{n} \mathrm{sign}_{ji}\, p_{ji}\big(y_j(t)\big) g_i\big(x_i(t)\big) \\
&\quad + \sum_{i=1}^{n} \mathrm{sign}_{ji}\, q_{ji}\big(y_j(t)\big) g_i\big(x_i\big(t-\tau_i(t)\big)\big),
\end{aligned} \tag{2}$$

where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $x_i(t)$ and $y_i(t)$ stand for the voltages of capacitors C_i and $\hat{C}_j$, respectively; $d_i(\cdot)$ and $r_j(\cdot)$ are the rates of neuron self-inhibition; $f_i(\cdot)$ and $g_j(\cdot)$ denote the activation functions. $\tau_i(t), \tau_j(t)$ denote time-varying delays satisfying $0 < \tau_{i,j}(t) \leqslant \tau$, $|\tau'_{i,j}(t)| \leqslant \lambda < 1$, λ is a positive constant; sign_{ji} is a sign function, and its value is given by

$$\mathrm{sign}_{ji} = \begin{cases} 1, & j \neq i, \\ -1, & j = i, \end{cases}$$

$a_{ij}(\cdot)$, $p_{ji}(\cdot)$ and $b_{ij}(\cdot)$, $q_{ji}(\cdot)$ are the feedback connection weights and the delayed feedback connection weights, which are given by

$$d_i\big(x_i(t)\big) = \frac{1}{C_i}\left[\sum_{j=1}^{m}(M_{ij}+M_{ij}^*) + \frac{1}{R_i}\right],$$
$$a_{ij}\big(x_i(t)\big) = \frac{M_{ij}}{C_i}, \qquad b_{ij}\big(x_i(t)\big) = \frac{M_{ij}^*}{C_i},$$
$$r_j\big(y_j(t)\big) = \frac{1}{\hat{C}_j}\left[\sum_{i=1}^{n}(\hat{M}_{ji}+\hat{M}_{ji}^*) + \frac{1}{\hat{R}_j}\right],$$
$$p_{ji}\big(y_j(t)\big) = \frac{\hat{M}_{ji}}{\hat{C}_j}, \qquad q_{ji}\big(y_j(t)\big) = \frac{\hat{M}_{ji}^*}{\hat{C}_j},$$

in which M_{ij}, M_{ij}^*, $\hat{M}_{ji}$, $\hat{M}_{ji}^*$ are the memductances of R_{ij}, R_{ij}^*, $\hat{R}_{ji}$, $\hat{R}_{ji}^*$, respectively. And R_{ij} represents the memristor between $x_i(t)$ and $f_j(y_j(t))$, R_{ij}^* represents the memristor between $x_i(t)$ and $f_j(y_j(t-\tau_j(t)))$, $\hat{R}_{ji}$ represents the memristor between $y_j(t)$ and $g_i(x_i(t))$, $\hat{R}_{ji}^*$ represents the memristor between $y_j(t)$ and $g_i(x_i(t-\tau_i(t)))$, R_i and $\hat{R}_j$ stand for the parallel-resistors. By the feature of the memductance derived in (1), M_{ij}, M_{ij}^* are bounded, and their derivatives satisfy

$$\frac{\mathrm{d}M_{ij}}{\mathrm{d}t} = \big(R_{ij}^{\mathrm{off}} - R_{ij}^{\mathrm{on}}\big)\mu_V \frac{R_{ij}^{\mathrm{on}}}{(D_{ij})^2}\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big)(M_{ij})^3,$$
$$\frac{\mathrm{d}M_{ij}^*}{\mathrm{d}t} = \big(R_{ij}^{*\mathrm{off}} - R_{ij}^{*\mathrm{on}}\big)\mu_V \frac{R_{ij}^{*\mathrm{on}}}{(D_{ij}^*)^2}\big(\mathrm{sign}_{ij}\, f_j\big(y_j\big(t-\tau_j(t)\big)\big) - x_i(t)\big)(M_{ij}^*)^3,$$

When the length of memristor is rather short, the memristor may switch between the On state and Off state in a very short time under usual voltage. So we can assume that the memristor's state depends on the sign of the potential difference between the two sides of the device. Discussing $\hat{M}_{ji}$, $\hat{M}_{ji}^*$ in the same way, then the state-dependent parameters $d_i(x_i(t))$, $a_{ij}(x_i(t))$, $b_{ij}(x_i(t))$, $r_j(y_j(t))$, $p_{ji}(y_j(t))$, $q_{ji}(y_j(t))$ abide by the following conditions:

$$a_{ij}\big(x_i(t)\big) = \begin{cases} a_{ij}^+, & x_i(t) < \mathrm{sign}_{ij}\, f_j(y_j(t)),\\ \text{unchanged}, & x_i(t) = \mathrm{sign}_{ij}\, f_j(y_j(t)),\\ a_{ij}^-, & x_i(t) > \mathrm{sign}_{ij}\, f_j(y_j(t)),\end{cases}$$

$$b_{ij}\big(x_i(t)\big) = \begin{cases} b_{ij}^+, & x_i(t) < \mathrm{sign}_{ij}\, f_j(y_j(t-\tau_j(t))),\\ \text{unchanged}, & x_i(t) = \mathrm{sign}_{ij}\, f_j(y_j(t-\tau_j(t))),\\ b_{ij}^-, & x_i(t) > \mathrm{sign}_{ij}\, f_j(y_j(t-\tau_j(t))),\end{cases}$$

$$d_i\big(x_i(t)\big) = \sum_{j=1}^{m}\big(a_{ij}\big(x_i(t)\big) + b_{ij}\big(x_i(t)\big)\big) + d_i^*,$$

$$p_{ji}(y_j(t)) = \begin{cases} p_{ji}^+, & y_j(t) < \operatorname{sign}_{ji} g_i(x_i(t)), \\ \text{unchanged}, & y_j(t) = \operatorname{sign}_{ji} g_i(x_i(t)), \\ p_{ji}^-, & y_j(t) > \operatorname{sign}_{ji} g_i(x_i(t)), \end{cases}$$

$$q_{ji}(y_j(t)) = \begin{cases} q_{ji}^+, & y_j(t) < \operatorname{sign}_{ji} g_i(x_i(t-\tau_i(t))), \\ \text{unchanged}, & y_j(t) = \operatorname{sign}_{ji} g_i(x_i(t-\tau_i(t))), \\ q_{ji}^-, & y_j(t) > \operatorname{sign}_{ji} g_i(x_i(t-\tau_i(t))), \end{cases}$$

$$r_j(y_j(t)) = \sum_{i=1}^{n} \left(p_{ji}(y_j(t)) + q_{ji}(y_j(t))\right) + r_j^*,$$

where d_i^*, a_{ij}^+, a_{ij}^-, b_{ij}^+, b_{ij}^-, r_j^*, p_{ji}^+, p_{ji}^-, q_{ji}^+, q_{ji}^- are known constants relating to the maximum and minimum memristance of memristors, the resistance of resistors, as well as the capacitance of capacitors. Moreover, unchanged means that the memristance keeps the current value. The initial values of network (2) is $x_i(s) = \phi_{xi}(s)$, $y_j(s) = \phi_{yj}(s)$ $(i = 1, 2, \dots, n, j = 1, 2, \dots, m)$, $(\phi_{x1}(s), \phi_{x2}(s), \dots, \phi_{xn}(s), \phi_{y1}(s), \phi_{y2}(s), \dots, \phi_{ym}(s))^{\mathrm{T}} \in C([-\tau, 0], \mathbb{R}^{n+m})$.

We regard network (2) as the drive network. Denote $\hat{x}_i(t), \hat{y}_j(t)$ as the states of response network, and $e_{xi}(t)$, $e_{yj}(t)$ are the error variables, which are defined as $e_{xi}(t) = x_i(t) - \hat{x}_i(t)$, $e_{yj}(t) = y_j(t) - \hat{y}_j(t)$. The structure of response network should be the same as that of the drive network, so that the modifications to $\hat{x}_i(t)$ come from the feedback of $\hat{y}_j(t)$ and $\hat{y}_j(t-\tau_j(t))$ $(j = 1, 2, \dots, m)$. Considering the combined effect of controller and the feedback, we assume the intensity of stochastic disturbances introduced to $\hat{x}_i(t)$ is a function of $e_{yj}(t)$ and $e_{yj}(t-\tau_j(t))$ $(j = 1, 2, \dots, m)$. Correspondingly, the intensity of stochastic disturbances on $\hat{y}_j(t)$ is a function of $e_{xi}(t)$ and $e_{xi}(t-\tau_i(t))$ $(i = 1, 2, \dots, n)$. Then the response network with stochastic disturbances can be described as

$$\begin{aligned} \mathrm{d}\hat{x}_i(t) = {} & \Bigg[- d_i(\hat{x}_i(t))\hat{x}_i(t) + \sum_{j=1}^{m} \operatorname{sign}_{ij} a_{ij}(\hat{x}_i(t)) f_j(\hat{y}_j(t)) \\ & + \sum_{j=1}^{m} \operatorname{sign}_{ij} b_{ij}(\hat{x}_i(t)) f_j(\hat{y}_j(t-\tau_j(t))) + u_i(t) \Bigg] \mathrm{d}t \\ & + \sum_{j=1}^{m} \sigma(t, e_{yj}(t), e_{yj}(t-\tau_j(t))) \,\mathrm{d}\Omega_j(t), \\ \mathrm{d}\hat{y}_j(t) = {} & \Bigg[-r_j(\hat{y}_j(t))\hat{y}_j(t) + \sum_{i=1}^{n} \operatorname{sign}_{ji} p_{ji}(\hat{y}_j(t)) g_i(\hat{x}_i(t)) \\ & + \sum_{i=1}^{n} \operatorname{sign}_{ji} q_{ji}(\hat{y}_j(t)) g_i(\hat{x}_i(t-\tau_i(t))) + v_j(t) \Bigg] \mathrm{d}t \\ & + \sum_{i=1}^{n} \sigma(t, e_{xi}(t), e_{xi}(t-\tau_i(t))) \,\mathrm{d}\Omega_i(t), \end{aligned} \tag{3}$$

where $u_i(t)$ and $v_j(t)$ are the control inputs designed in the following form:

$$u_i(t) = k_i^1 e_{xi}(t) + k_i^2 \operatorname{sign}(e_{xi}(t)) + \eta \left(\int_{t-\tau_i(t)}^{t} k_i \mathrm{e}_{xi}^2(s)\,\mathrm{d}s \right)^{(\alpha+1)/2} \frac{e_{xi}(t)}{|e_{xi}(t)|^2} + \eta \operatorname{sign}\bigl(e_{xi}(t)\bigr)\bigl|e_{xi}(t)\bigr|^{\alpha},$$

$$v_j(t) = w_j^1 e_{yi}(t) + w_j^2 \operatorname{sign}\bigl(e_{yj}(t)\bigr) + \eta \left(\int_{t-\tau_j(t)}^{t} w_j \mathrm{e}_{yj}^2(s)\,\mathrm{d}s \right)^{(\alpha+1)/2} \frac{e_{yj}(t)}{|e_{yj}(t)|^2} + \eta \operatorname{sign}\bigl(e_{yj}(t)\bigr)\bigl|e_{yj}(t)\bigr|^{\alpha}$$

in which k_i^1, k_i^2, k_i, w_j^1, w_j^2, w_j, η, α are the feedback gains to be designed. $\Omega_i(t)$ and $\Omega_j(t)$ are the Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$. $\sigma(\cdot)$ is the density function of stochastic effects.

For the error variables $e_{xi}(t)$, $e_{yj}(t)$, we have

$$\begin{aligned}
\mathrm{d}e_{xi}(t) = \Biggl\{ &-\bigl(d_i\bigl(x_i(t)\bigr)x_i(t) - d_i\bigl(\hat{x}_i(t)\bigr)\hat{x}_i(t)\bigr) \\
&+ \sum_{j=1}^{m} \bigl[\operatorname{sign}_{ij}\bigl(a_{ij}\bigl(x_i(t)\bigr)f_j\bigl(y_j(t)\bigr) - a_{ij}\bigl(\hat{x}_i(t)\bigr)f_j\bigl(\hat{y}_j(t)\bigr)\bigr) \\
&+ \operatorname{sign}_{ij}\bigl(b_{ij}\bigl(x_i(t)\bigr)f_j\bigl(y_j\bigl(t-\tau_j(t)\bigr)\bigr) - b_{ij}\bigl(\hat{x}_i(t)\bigr)f_j\bigl(\hat{y}_j\bigl(t-\tau_j(t)\bigr)\bigr)\bigr)\bigr] \\
&- u_i(t) \Biggr\}\,\mathrm{d}t + \sum_{j=1}^{m} \sigma\bigl(t, e_{yj}(t), e_{yj}\bigl(t-\tau_j(t)\bigr)\bigr)\,\mathrm{d}\Omega_j(t), \\
\mathrm{d}e_{yj}(t) = \Biggl\{ &-\bigl(r_j\bigl(y_j(t)\bigr)y_j(t) - r_j\bigl(\hat{y}_j(t)\bigr)\hat{y}_j(t)\bigr) \\
&+ \sum_{i=1}^{n} \bigl[\operatorname{sign}_{ji}\,\bigl(p_{ji}\bigl(y_j(t)\bigr)g_i\bigl(x_i(t)\bigr) - p_{ji}\bigl(\hat{y}_j(t)\bigr)g_i\bigl(\hat{x}_i(t)\bigr)\bigr) \\
&+ \operatorname{sign}_{ji}\bigl(q_{ji}\bigl(y_j(t)\bigr)g_i\bigl(x_i\bigl(t-\tau_i(t)\bigr)\bigr) - q_{ji}\bigl(\hat{y}_j(t)\bigr)g_i\bigl(\hat{x}_i\bigl(t-\tau_i(t)\bigr)\bigr)\bigr)\bigr] \\
&- v_j(t) \Biggr\}\,\mathrm{d}t + \sum_{i=1}^{n} \sigma\bigl(t, e_{xi}(t), e_{xi}\bigl(t-\tau_i(t)\bigr)\bigr)\,\mathrm{d}\Omega_i(t).
\end{aligned} \tag{4}$$

By applying the theories of set-valued map and differential inclusions [2,10], the Filippov solutions of error variables $e_{xi}(t)$ and $e_{yj}(t)$ satisfy

$$\begin{aligned}
\mathrm{d}e_{xi}(t) \in \Biggl\{ &-d_i^* e_{xi}(t) + \sum_{j=1}^{m} \bigl[\operatorname{co}\bigl[a_{ij}^-, a_{ij}^+\bigr]\bigl(\operatorname{sign}_{ij} f_j\bigl(y_j(t)\bigr) - x_i(t)\bigr) \\
&- \operatorname{co}\bigl[a_{ij}^-, a_{ij}^+\bigr]\bigl(\operatorname{sign}_{ij} f_j\bigl(\hat{y}_j(t)\bigr) - \hat{x}_i(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{co}\big[b_{ij}^-, b_{ij}^+\big]\big(\operatorname{sign}_{ij} f_j\big(y_j\big(t-\tau_j(t)\big)\big) - x_i(t)\big) \\
&- \operatorname{co}\big[b_{ij}^-, b_{ij}^+\big]\big(\operatorname{sign}_{ij} f_j\big(\hat{y}_j\big(t-\tau_j(t)\big)\big) - \hat{x}_i(t)\big)\big] - u_i(t)\Bigg\}\,\mathrm{d}t \\
&+ \sum_{j=1}^{m} \sigma\big(t, e_{jj}(t), e_{yj}\big(t-\tau_j(t)\big)\big)\,\mathrm{d}\Omega_j(t),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}e_{yj}(t) \in \Bigg\{ &-r_j^* e_{yj}(t) + \sum_{i=1}^{n}\big[\operatorname{co}\big[p_{ji}^-, p_{ji}^+\big]\big(\operatorname{sign}_{ji} g_i\big(x_i(t)\big) - y_j(t)\big) \\
&- \operatorname{co}\big[p_{ji}^-, p_{ji}^+\big]\big(\operatorname{sign}_{ji} g_i\big(\hat{x}_i(t)\big) - \hat{y}_j(t)\big) \\
&+ \operatorname{co}\big[q_{ji}^-, q_{ji}^+\big]\big(\operatorname{sign}_{ji} g_i\big(x_i\big(t-\tau_i(t)\big)\big) - y_j(t)\big) \\
&- \operatorname{co}\big[q_{ji}^-, q_{ji}^+\big]\big(\operatorname{sign}_{ji} g_i\big(\hat{x}_i\big(t-\tau_i(t)\big)\big) - \hat{y}_j(t)\big)\big] - v_j(t)\Bigg\}\,\mathrm{d}t \\
&+ \sum_{i=1}^{n} \sigma\big(t, e_{xi}(t), e_{xi}\big(t-\tau_i(t)\big)\big)\,\mathrm{d}\Omega_i(t).
\end{aligned}
$$

Definition 1. (See [9].) For any $\mathbf{e}(0) = (e_{x1}(0), e_{x2}(0), \dots, e_{xn}(0), e_{y1}(0), e_{y2}(0), \dots, e_{ym}(0))^{\mathrm{T}} \in \mathbb{R}^{n+m}$, if

$$\mathbf{P}\Big\{\lim_{t\to\infty} \big\|\mathbf{e}\big(t, \mathbf{e}(0)\big)\big\| = 0\Big\} = 1$$

holds, where $\mathbf{e}(t, \mathbf{e}(0)) = (e_{x1}(t), e_{x2}(t), \dots, e_{xn}(t), e_{y1}(t), e_{y2}(t), \dots, e_{ym}(t))^{\mathrm{T}}$, then system (4) is said to be globally asymptotically stable in probability.

Definition 2. (See [6, 18].) For system (4), define $T_0(\mathbf{e}_0, \Omega) = \inf\{T \geqslant 0\colon \mathbf{e}(t, \mathbf{e}(0)) = 0\ \forall t \geqslant T\}$ as the stochastic settling time function. If for any $\mathbf{e}(0) \in \mathbb{R}^{n+m}$,

$$\mathbf{P}\Big\{\lim_{t\to T_0} \big\|\mathbf{e}\big(t, \mathbf{e}(0)\big)\big\| = 0,\ \big\|\mathbf{e}\big(t, \mathbf{e}(0)\big)\big\| = 0,\ t \geqslant T_0\Big\} = 1$$

holds, then system (4) is said to be globally stochastically finite-time stable in probability, and T_0 is called the stochastic settling time.

Based on Definition 1, the finite-time stochastic synchronization of networks (2) and (3) is equivalent to the globally finite-time stochastic stability of the error system (4) at the origin.

In order to obtain our main theorems, we make the following assumptions.

Assumption 1. There exist constants $R_1 \geqslant 0$ and $R_2 \geqslant 0$ such that

$$\sigma^2(t, x, y) \leqslant R_1 x^2 + R_2 y^2.$$

Assumption 2. For $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and for all $x, y \in \mathbb{R}$, there exist constants h_{fj}, h_{gi} such that

$$\big|f_j(x)\big| \leqslant h_{fj}, \qquad \big|g_i(x)\big| \leqslant h_{gi}.$$

Let $\Omega(t) = (\Omega_1(t), \Omega_2(t), \dots, \Omega_m(t))^{\mathrm{T}}$ be a standard $\mathbb{R}^m$-valued Brownian motion defined on a complete probability space. For any given $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ associated with stochastic system

$$x(t) = x(0) + \int_0^t f\bigl(x(s)\bigr)\,\mathrm{d}s + \sum_{i=1}^m \int_0^t \sigma_i\bigl(x(s)\bigr)\,\mathrm{d}\Omega_i(s), \tag{5}$$

the infinitesimal generator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}V(x,t) = \sum_{i=1}^n f_i(x)\frac{\partial V(x,t)}{\partial x_i} + \frac{\partial V(x,t)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^n \beta_{ij}(x)\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j},$$

where $\beta_{ij}(x) = \sum_{k=1}^m \sigma_k(x_i(s))\sigma_k(x_j(s))$.

To prove our results, the following lemmas are necessary.

Lemma 1. (*See* [6].) *For stochastic differential system* (5)*, if there exist a positive definite, twice continuous differentiable and radially unbounded Lyapunov function* $V: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}_+$ *and a continuous differentiable function* $r: \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

(i) $\mathcal{L}V(\mathbf{x},t) \leqslant -r(V(\mathbf{x},t))$,
(ii) *for any* $0 \leqslant \epsilon < +\infty$, $\int_0^\epsilon 1/r(v)\,\mathrm{d}v < +\infty$,
(iii) *for* $v > 0$, $r'(v) > 0$,

then the origin of system (5) *is globally stochastically finite-time stable in probability. Moreover, the stochastic settling time* T_0 *satisfies* $\mathbf{E}[T_0(\mathbf{x}_0, \Omega)] \leqslant \int_0^{V(\mathbf{x}_0)} 1/r(v)\,\mathrm{d}v$.

Lemma 2. (*See* [13].) *If* $a_1, a_2, \dots, a_n$ *are positive numbers and* $0 < p < q$*, then* $(\sum_{i=1}^n a_i^q)^{1/q} \leqslant (\sum_{i=1}^n a_i^p)^{1/p}$.

3 Main results

In this section, the finite-time synchronization of MBAMNN (2) with stochastic disturbances under previously mentioned controller (4) is investigated. By a further discussion, a sufficient condition is given to ensure the synchronization achieve in a shorter time with an error-dependent switching controller. The main results are given as Theorems 1 and 2.

Theorem 1. *Under Assumptions* 1 *and* 2*, the error system* (4) *is globally stochastically finite-time stable in probability with controller* (4) *if the feedback gains* α, η, k_i^1, k_i^2, k_i, w_j^1, w_j^2, w_j *satisfy*

$$\begin{aligned}
&0 \leqslant \alpha < 1, \qquad \eta > 0, \qquad k_i \geqslant \frac{mR_2}{1-\lambda}, \qquad w_j \geqslant \frac{nR_2}{1-\lambda},\\
&k_i^1 \geqslant \frac{1}{2}\left[-2d_i^* - 2\sum_{j=1}^m \bigl(a_{ij}^- + b_{ij}^-\bigr) + mR_1 + k_i\right], \qquad k_i^2 \geqslant 2\sum_{j=1}^m \bigl(a_{ij}^+ + b_{ij}^+\bigr)h_{fj},\\
&w_j^1 \geqslant \frac{1}{2}\left[-2r_j^* - 2\sum_{i=1}^n \bigl(p_{ji}^- + q_{ji}^-\bigr) + nR_1 + w_j\right], \qquad w_j^2 \geqslant 2\sum_{i=1}^n \bigl(p_{ji}^+ + q_{ji}^+\bigr)h_{gi},
\end{aligned} \tag{6}$$

$i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. *And the stochastic settling time satisfies* $\mathbf{E}(T_0) \leqslant (V(0)^{(1-\alpha)/2}/\eta)(1-\alpha)$.

Proof. Employ the following Lyapunov–Krasovskii functional

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{i=1}^{n} \left[e_{xi}^2(t) + \int\limits_{t-\tau_i(t)}^{t} k_i e_{xi}^2(s)\,\mathrm{d}s \right],$$

$$V_2(t) = \sum_{j=1}^{m} \left[e_{yj}^2(t) + \int\limits_{t-\tau_j(t)}^{t} w_j e_{yj}^2(s)\,\mathrm{d}s \right].$$

By computing $\mathcal{L}V_1(t)$ along the trajectories of error system (4), we can derive that

$$\begin{aligned}
\mathcal{L}V_1(t) = {}& 2\sum_{i=1}^{n} e_{xi}(t) \Biggl\{ -d_i^* e_{xi}(t) + \sum_{j=1}^{m} \bigl[a_{ij}\bigl(x_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(y_j(t)\bigr) - x_i(t)\bigr) \\
& - a_{ij}\bigl(\hat{x}_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(\hat{y}_j(t)\bigr) - \hat{x}_i(t)\bigr) \\
& + b_{ij}\bigl(x_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(y_j\bigl(t-\tau_j(t)\bigr)\bigr) - x_i(t)\bigr) \\
& - b_{ij}\bigl(\hat{x}_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(\hat{y}_j\bigl(t-\tau_j(t)\bigr)\bigr) - \hat{x}_i(t)\bigr)\bigr] - u_i(t) \Biggr\} \\
& + \sum_{i=1}^{n} k_i \bigl[e_{xi}^2(t) - e_{xi}^2\bigl(t-\tau_i(t)\bigr)\bigl(1-\tau_i'(t)\bigr)\bigr] \\
& + n\sum_{j=1}^{m} \sigma^2\bigl(t, e_{yj}(t), e_{yj}\bigl(t-\tau_j(t)\bigr)\bigr).
\end{aligned} \tag{7}$$

For each t, we discuss six possible cases for the values of $x_i(t)$ and $\hat{x}_i(t)$:

Case 1: $x_i(t) > \mathrm{sign}_{ij}\, f_j(y_j(t))$, $\hat{x}_i(t) > \mathrm{sign}_{ij}\, f_j(\hat{y}_j(t))$, then

$$\begin{aligned}
&2e_{xi}(t)\bigl(a_{ij}\bigl(x_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(y_j(t)\bigr) - x_i(t)\bigr) - a_{ij}\bigl(\hat{x}_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(\hat{y}_j(t)\bigr) - \hat{x}_i(t)\bigr)\bigr) \\
&\quad \leqslant 4a_{ij}^- h_{fj}\bigl|e_{xi}(t)\bigr| - 2a_{ij}^-\bigl|e_{xi}(t)\bigr|^2.
\end{aligned}$$

Case 2: $x_i(t) \leqslant \mathrm{sign}_{ij}\, f_j(y_j(t))$, $\hat{x}_i(t) \leqslant \mathrm{sign}_{ij}\, f_j(\hat{y}_j(t))$, then

$$\begin{aligned}
&2e_{xi}(t)\bigl(a_{ij}\bigl(x_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(y_j(t)\bigr) - x_i(t)\bigr) - a_{ij}\bigl(\hat{x}_i(t)\bigr)\bigl(\mathrm{sign}_{ij}\, f_j\bigl(\hat{y}_j(t)\bigr) - \hat{x}_i(t)\bigr)\bigr) \\
&\quad \leqslant 4a_{ij}^+ h_{fj}\bigl|e_{xi}(t)\bigr| - 2a_{ij}^+\bigl|e_{xi}(t)\bigr|^2.
\end{aligned}$$

Case 3: $x_i(t) > \mathrm{sign}_{ij}\, f_j(y_j(t))$, $-h_{fj} \leqslant \hat{x}_i(t) \leqslant \mathrm{sign}_{ij}\, f_j(\hat{y}_j(t))$, then by Assumption 2 we have

$$\begin{aligned}
&2e_{xi}(t)\big(a_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big) - a_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - \hat{x}_i(t)\big)\big) \\
&\quad = 2e_{xi}(t)\big(a_{ij}^{-}\,\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - a_{ij}^{+}\,\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - a_{ij}^{-} e_{xi}(t) + \big(a_{ij}^{+} - a_{ij}^{-}\big)\hat{x}_i(t)\big) \\
&\quad \leqslant 4a_{ij}^{+} h_{fj}\big|e_{xi}(t)\big| - 2a_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned}$$

Case 4: $x_i(t) > \mathrm{sign}_{ij}\, f_j(y_j(t)) \geqslant -h_{fj}$, $\hat{x}_i(t) \leqslant -h_{fj}$, then

$$\begin{aligned}
&2e_{xi}(t)\big(a_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big) - a_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - \hat{x}_i(t)\big)\big) \\
&\quad = 2e_{xi}(t)\big(a_{ij}^{-}\,\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - a_{ij}^{+}\,\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - a_{ij}^{-} e_{xi}(t) + \big(a_{ij}^{+} - a_{ij}^{-}\big)\hat{x}_i(t)\big) \\
&\quad \leqslant 2(a_{ij}^{+} + a_{ij}^{-}) h_{fj}\big|e_{xi}(t)\big| - 2a_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned}$$

Case 5: $-h_{fj} \leqslant x_i(t) \leqslant \mathrm{sign}_{ij}\, f_j(y_j(t))$, $\hat{x}_i(t) > \mathrm{sign}_{ij}\, f_j(\hat{y}_j(t))$, then

$$\begin{aligned}
&2e_{xi}(t)\big(a_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big) - a_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - \hat{x}_i(t)\big)\big) \\
&\quad = 2e_{xi}(t)\big(a_{ij}^{+}\,\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - a_{ij}^{-}\,\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - a_{ij}^{-} e_{xi}(t) + \big(a_{ij}^{-} - a_{ij}^{+}\big)x_i(t)\big) \\
&\quad \leqslant 4a_{ij}^{+} h_{fj}\big|e_{xi}(t)\big| - 2a_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned}$$

Case 6: $x_i(t) \leqslant -h_{fj}$, $\hat{x}_i(t) > \mathrm{sign}_{ij}\, f_j(\hat{y}_j(t)) \geqslant -h_{fj}$, then

$$\begin{aligned}
&2e_{xi}(t)\big(a_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big) - a_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - \hat{x}_i(t)\big)\big) \\
&\quad = 2e_{xi}(t)\big(a_{ij}^{+}\,\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - a_{ij}^{-}\,\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - a_{ij}^{-} e_{xi}(t) + \big(a_{ij}^{-} - a_{ij}^{+}\big)x_i(t)\big) \\
&\quad \leqslant 2(a_{ij}^{+} + a_{ij}^{-}) h_{fj}\big|e_{xi}(t)\big| - 2a_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned}$$

So, we can conclude that

$$\begin{aligned}
&2e_{xi}(t)\big(a_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j(t)\big) - x_i(t)\big) - a_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j(t)\big) - \hat{x}_i(t)\big)\big) \\
&\quad \leqslant 4a_{ij}^{+} h_{fj}\big|e_{xi}(t)\big| - 2a_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned} \tag{8}$$

By a similar analysis to (8), we can get

$$\begin{aligned}
&2e_{xi}(t)\big(b_{ij}\big(x_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(y_j\big(t-\tau_j(t)\big)\big) - x_i(t)\big) \\
&\qquad - b_{ij}\big(\hat{x}_i(t)\big)\big(\mathrm{sign}_{ij}\, f_j\big(\hat{y}_j\big(t-\tau_j(t)\big)\big) - \hat{x}_i(t)\big)\big) \\
&\quad \leqslant 4b_{ij}^{+} h_{fj}\big|e_{xi}(t)\big| - 2b_{ij}^{-}\big|e_{xi}(t)\big|^2.
\end{aligned} \tag{9}$$

Since $|\tau_i'(t)| \leqslant \lambda < 1$, so that

$$k_i\big[e_{xi}^2(t) - e_{xi}^2\big(t-\tau_i(t)\big)\big(1-\tau_i'(t)\big)\big] \leqslant k_i\big[e_{xi}^2(t) - e_{xi}^2\big(t-\tau_i(t)\big)\big(1-\lambda\big)\big]. \tag{10}$$

According to Assumption 1, we can obtain that

$$\sigma^2\bigl(t, e_{yj}(t), e_{yj}\bigl(t-\tau_j(t)\bigr)\bigr) \leqslant R_1 {e_{yj}}^2(t) + R_2 {e_{yj}}^2\bigl(t-\tau_j(t)\bigr). \tag{11}$$

It follows form (7)–(11) that

$$\begin{aligned}\mathcal{L}V_1(t) \leqslant{}& \sum_{i=1}^{n}\Biggl[-2d_i^* - 2\sum_{j=1}^{m}\bigl(a_{ij}^- + b_{ij}^-\bigr)\Biggr]\bigl|e_{xi}(t)\bigr|^2 \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m}\bigl(4a_{ij}^+ + 4b_{ij}^+\bigr)h_{fj}\bigl|e_{xi}(t)\bigr| - \sum_{i=1}^{n}2e_{xi}(t)u_i(t) \\ &+ \sum_{i=1}^{n}k_i\bigl[e_{xi}^2(t) - e_{xi}^2\bigl(t-\tau_i(t)\bigr)(1-\lambda)\bigr] \\ &+ n\sum_{j=1}^{m}\bigl[R_1{e_{yj}}^2(t) + R_2{e_{yj}}^2\bigl(t-\tau_j(t)\bigr)\bigr]. \end{aligned}\tag{12}$$

With the similar process of $\mathcal{L}V_1(t)$, we also get

$$\begin{aligned}\mathcal{L}V_2(t) \leqslant{}& \sum_{j=1}^{m}\Biggl[-2r_j^* - 2\sum_{i=1}^{n}\bigl(p_{ji}^- + q_{ji}^-\bigr)\Biggr]\bigl|e_{yj}(t)\bigr|^2 \\ &+ \sum_{j=1}^{m}\sum_{i=1}^{n}\bigl(4p_{ji}^+ + 4q_{ji}^+\bigr)h_{gi}\bigl|e_{yj}(t)\bigr| - \sum_{j=1}^{m}2e_{yj}(t)v_j(t) \\ &+ \sum_{j=1}^{m}w_j\bigl[e_{yj}^2(t) - e_{yj}^2\bigl(t-\tau_j(t)\bigr)(1-\lambda)\bigr] \\ &+ m\sum_{i=1}^{n}\bigl[R_1e_{xi}^2(t) + R_2e_{xi}^2\bigl(t-\tau_i(t)\bigr)\bigr]. \end{aligned}\tag{13}$$

Add (12) and (13) up and substitute (4) into the sum, by condition (6) and Lemma 2, we have $\mathcal{L}V(t) \leqslant -2\eta V^{(\alpha+1)/2}(t)$. So, it follows from Lemma 1 that the origin of error system (4) is globally finite-time stochastically stable in probability. Accordingly, the finite-time synchronization of networks (2) and (3) can be obtained. Moreover, the stochastic settling time satisfies

$$\mathbf{E}(T_0) \leqslant \int_0^{V(0)} \frac{1}{2\eta v^{(1+\alpha)/2}}\,\mathrm{d}v = \frac{V(0)^{(1-\alpha)/2}}{\eta(1-\alpha)}.$$

This completes the proof. □

Next, based on Theorem 1, we consider a switching controller under which the error system (4) may achieve stable in probability with a shorter stochastic settling time.

Define $T(\alpha) = V(0)^{(1-\alpha)/2}/\eta(1-\alpha)$, $0 \leqslant \alpha < 1$. Computing the derivative of $T(\alpha)$, we can get

$$\frac{\mathrm{d}T(\alpha)}{\mathrm{d}\alpha} = \frac{(\alpha-1)\ln V(0)+2}{2\eta(1-\alpha)^2 V(0)^{(\alpha-1)/2}}.$$

Obviously, enlarging the gain constant η and reducing $V(0)$ can shorten the stochastic settling time, and when η and $V(0)$ are determined constants, there are two possible cases to be discussed.

Case 1: If $V(0) > \mathrm{e}^2$ (where e is the shorthand of constant exponent), $\mathrm{d}T(\alpha)/\mathrm{d}\alpha = 0$ if and only if $\alpha = 1 - 2/\ln V(0)$. Moreover,

$$\frac{\mathrm{d}T(\alpha)}{\mathrm{d}\alpha} < 0 \quad \text{when } 0 \leqslant \alpha < 1 - \frac{2}{\ln V(0)},$$
$$\frac{\mathrm{d}T(\alpha)}{\mathrm{d}\alpha} > 0 \quad \text{when } 1 - \frac{2}{\ln V(0)} < \alpha < 1.$$

So that $T(\alpha)$ take the minimal value at $\alpha = 1 - 2/\ln V(0)$.

Case 2: If $V(0) \leqslant \mathrm{e}^2$, then $\mathrm{d}T(\alpha)/\mathrm{d}\alpha \geqslant 0$ for all $\alpha \in [0,1)$, in this case, $T(\alpha)$ take the minimal value at $\alpha = 0$.

According to the discussion above, we design the following switching controller:

$$u_i(t) = k_i^1 e_{xi}(t) + k_i^2 \operatorname{sign}\bigl(e_{xi}(t)\bigr) + \eta\left(\int_{t-\tau_i(t)}^{t} k_i \mathrm{e}_{xi}^2(s)\,\mathrm{d}s\right)^{(\alpha(t)+1)/2} \frac{e_{xi}(t)}{|e_{xi}(t)|^2} + \eta \operatorname{sign}\bigl(e_{xi}(t)\bigr)\bigl|e_{xi}(t)\bigr|^{\alpha(t)},$$
$$v_j(t) = w_j^1 e_{yi}(t) + w_j^2 \operatorname{sign}\bigl(e_{yj}(t)\bigr) + \eta\left(\int_{t-\tau_j(t)}^{t} w_j \mathrm{e}_{yj}^2(s)\,\mathrm{d}s\right)^{(\alpha(t)+1)/2} \frac{e_{yj}(t)}{|e_{yj}(t)|^2} + \eta \operatorname{sign}\bigl(e_{yj}(t)\bigr)\bigl|e_{yj}(t)\bigr|^{\alpha(t)},$$

$\alpha(t)$ satisfies

$$\alpha(t) = \begin{cases} 1 - \frac{2}{\ln V(0)} & \text{if } V(t) > \mathrm{e}^2, \\ 0 & \text{if } V(t) \leqslant \mathrm{e}^2. \end{cases}$$

Based on Theorem 1, we can get the following theorem.

Theorem 2. *Under Assumptions* 1 *and* 2*, the error system* (4) *is globally stochastically finite-time stable in probability with controller* (14) *if the feedback gains* η, k_i^1, k_i^2, k_i,

w_j^1, w_j^2, w_j *satisfy*

$$\begin{aligned} &\eta > 0, \qquad k_i \geqslant \frac{mR_2}{1-\lambda}, \qquad w_j \geqslant \frac{nR_2}{1-\lambda}, \\ &k_i^1 \geqslant \frac{1}{2}\left[-2d_i^* - 2\sum_{j=1}^{m}\left(a_{ij}^- + b_{ij}^-\right) + mR_1 + k_i\right], \qquad k_i^2 \geqslant 2\sum_{j=1}^{m}\left(a_{ij}^+ + b_{ij}^+\right)h_{fj}, \\ &w_j^1 \geqslant \frac{1}{2}\left[-2r_j^* - 2\sum_{i=1}^{n}\left(p_{ji}^- + q_{ji}^-\right) + nR_1 + w_j\right], \qquad w_j^2 \geqslant 2\sum_{i=1}^{n}\left(p_{ji}^+ + q_{ji}^+\right)h_{gi}. \end{aligned} \tag{14}$$

And the stochastic settling time satisfies

$$\mathbf{E}(T_0) \leqslant \begin{cases} \frac{V(0)^{1/2}}{\eta} & \text{if } V(0) \leqslant \mathrm{e}^2, \\ \frac{(\mathrm{e} - \mathrm{e}^{2/\ln V(0)})\ln V(0) + 2\mathrm{e}}{2\eta} & \text{if } V(0) > \mathrm{e}^2. \end{cases}$$

Proof. We consider the same Lyapunov–Krasovskii functional as it is in the proof of Theorem 1:

$$V(t) = \sum_{i=1}^{n}\left[e_{xi}^2(t) + \int_{t-\tau_i(t)}^{t} k_i e_{xi}^2(s)\,\mathrm{d}s\right] + \sum_{j=1}^{m}\left[e_{yj}^2(t) + \int_{t-\tau_j(t)}^{t} w_j e_{yj}^2(s)\,\mathrm{d}s\right].$$

Similar to the process of former proof, if the assumptions and condition (14) are satisfied, one has

$$\mathcal{L}V(t) \leqslant -2\eta V^{(\alpha(t)+1)/2}(t).$$

Then it follows from Lemma 1 that the origin of error system (4) is globally stochastically finite-time stable in probability. And the stochastic settling time can be calculated as

$$\mathbf{E}(T_0) \leqslant \int_0^{V(0)} \frac{1}{2\eta v^{(1+\alpha(t))/2}}\,\mathrm{d}v.$$

If $V(0) \leqslant \mathrm{e}^2$, then

$$\mathbf{E}(T_0) \leqslant \int_0^{V(0)} \frac{1}{2\eta v^{1/2}}\,\mathrm{d}v = \frac{V(0)^{1/2}}{\eta}.$$

If $V(0) > \mathrm{e}^2$ then

$$\mathbf{E}(T_0) \leqslant \int_{\mathrm{e}^2}^{V(0)} \frac{1}{2\eta v^{1-1/\ln V(0)}}\,\mathrm{d}v + \int_0^{\mathrm{e}^2} \frac{1}{2\eta v^{1/2}}\,\mathrm{d}v = \frac{(\mathrm{e} - \mathrm{e}^{2/\ln V(0)})\ln V(0) + 2\mathrm{e}}{2\eta}.$$

This completes the proof. □

Obviously, in controller (4) the α is a constant. In controller (14), the value of $\alpha(t)$ switches between 0 and a constant. Although comparing with the controller (4) in Theorem 1, controller (14) requires more calculations in deciding the value of $\alpha(t)$, we can get a better estimation of settling time with it. We will show its advantages by numerical simulations in the next section.

Remark 1. In some existing papers (e.g., [4, 5, 19, 24, 32, 33]), the switching of memristors in neural networks depended on the absolute value of states of neural networks. However, based on the character of memristor, the state of memristor depends on how much charge has passed through it in a particular direction. And this kind of switching conditions can not reflect the character of memristors, which has been point out in [22]. The parameters' switching conditions in our model were derived from the circuit structure of the MBAMNNs and basic principle of memristors. The memristors' states depended on the sign of the potential difference between the two sides of devices. So, our assumption on the switching conditions is more reasonable than that in some existing papers.

Remark 2. To deal with the parameters mismatch problem caused by employing memristor, in some existing papers (e.g., [11, 34]), authors assumed the activation functions take the value 0 at switching points. This is a quite strong assumption that most activation functions do not satisfy. Comparing with it, the activation functions in this paper are only required to be bounded (see Assumption 2), this assumption can be satisfied by many usual activation functions. Meanwhile, with the improvement of the switching conditions in the model, the model turned to be more reasonable, the parameters mismatch problem became more difficult to be settled. The switching conditions in [4, 5, 19, 24, 32, 33] can decide both the upper and lower bound of the networks' states while the switching conditions in this paper can only limit the upper or the lower bound of the states. The similar switching condition can also be found in [12]. However, in [12], the influence of memritors to the self-inhibition of the states was neglected, the parameters mismatch problem only existed in the activation function terms. In this paper, the parameters mismatch problem in both activation function terms and self-inhibition terms were settled.

Remark 3. Theorems 1 and 2 are also feasible for certain MBAMNNs without stochastic disturbances by determining $R_1 = R_2 = 0$. So, our results extended some previous researches.

Remark 4. In [33], authors analysed the finite-time anti-synchronization of MBAMNNs with stochastic disturbances. However, their conditions to ensure the finite-time anti-synchronization may only be satisfied if the activation functions are monotonically decreasing. Comparing with the conditions in [33], our conditions can be satisfied by adjusting the parameters in controller, and do not rely on the parameters in neural networks. So, our results are more practical.

4 Numerical simulations

In this section, a MBAMNN is given to demonstrate the effectiveness of the theoretical results in Section 3.

We firstly consider a 2-dimensional drive network taking the form of (2), where

$$d_1 = \sum_{j=1}^{2}(a_{1j}+b_{1j}) + 0.82, \qquad d_2 = \sum_{j=1}^{2}(a_{2j}+b_{2j}) + 0.11,$$

$$a_{11} = \begin{cases} 0.78, & x_1(t) < -f_1(y_1(t)),\\ 0.53, & x_1(t) > -f_1(y_1(t)), \end{cases} \qquad b_{11} = \begin{cases} 0.33, & x_1(t) < -f_1(y_1(t-\tau_1(t))),\\ 0.29, & x_1(t) > -f_1(y_1(t-\tau_1(t))), \end{cases}$$

$$a_{12} = \begin{cases} 0.39, & x_1(t) < f_2(y_2(t)),\\ 0.17, & x_1(t) > f_2(y_2(t)), \end{cases} \qquad b_{12} = \begin{cases} 0.65, & x_1(t) < f_2(y_2(t-\tau_2(t))),\\ 0.46, & x_1(t) > f_2(y_2(t-\tau_2(t))), \end{cases}$$

$$a_{21} = \begin{cases} 0.13, & x_2(t) < f_1(y_1(t)),\\ 0.03, & x_2(t) > f_1(y_1(t)), \end{cases} \qquad b_{21} = \begin{cases} 0.84, & x_2(t) < f_1(y_1(t-\tau_1(t))),\\ 0.02, & x_2(t) > f_1(y_1(t-\tau_1(t))), \end{cases}$$

$$a_{22} = \begin{cases} 0.94, & x_2(t) < -f_2(y_2(t)),\\ 0.31, & x_2(t) > -f_2(y_2(t)), \end{cases} \qquad b_{22} = \begin{cases} 0.86, & x_2(t) < -f_2(y_2(t-\tau_2(t))),\\ 0.56, & x_2(t) > -f_2(y_2(t-\tau_2(t))), \end{cases}$$

$$r_1 = \sum_{i=1}^{2}(p_{1i}+q_{1i}) + 1.13, \qquad r_2 = \sum_{i=1}^{2}(p_{2i}+q_{2i}) + 0.90,$$

$$p_{11} = \begin{cases} 0.44, & y_1(t) < -g_1(x_1(t)),\\ 0.34, & y_1(t) > -g_1(x_1(t)), \end{cases} \qquad q_{11} = \begin{cases} 0.98, & y_1(t) < -g_1(x_1(t-\tau_1(t))),\\ 0.54, & y_1(t) > -g_1(x_1(t-\tau_1(t))), \end{cases}$$

$$p_{12} = \begin{cases} 0.17, & y_1(t) < g_2(x_2(t)),\\ 0.05, & y_1(t) > g_2(x_2(t)), \end{cases} \qquad q_{12} = \begin{cases} 0.99, & y_1(t) < g_2(x_2(t-\tau_2(t))),\\ 0.70, & y_1(t) > g_2(x_2(t-\tau_2(t))), \end{cases}$$

$$p_{21} = \begin{cases} 0.66, & y_2(t) < g_1(x_1(t)),\\ 0.33, & y_2(t) > g_1(x_1(t)), \end{cases} \qquad q_{21} = \begin{cases} 0.41, & y_2(t) < g_1(x_1(t-\tau_1(t))),\\ 0.28, & y_2(t) > g_1(x_1(t-\tau_1(t))), \end{cases}$$

$$p_{22} = \begin{cases} 0.89, & y_2(t) < -g_2(x_2(t)),\\ 0.12, & y_2(t) > -g_2(x_2(t)), \end{cases} \qquad q_{22} = \begin{cases} 0.76, & y_2(t) < -g_2(x_2(t-\tau_2(t))),\\ 0.46, & y_2(t) > -g_2(x_2(t-\tau_2(t))). \end{cases}$$

Select the delays and activation functions as $\tau_i(t) = \tau_j(t) = 0.8$, $f_j(x) = g_i(x) = 2\cos(4x)$ $(i,j = 1,2)$. Then we have $\lambda = 0$, $\tau = 0.5$, $h_{fj} = 2$, $h_{gi} = 2$ $(i,j = 1,2)$.

We choose two initial values, $\phi_1(t) = (-6,-6,-6,-6)^{\mathrm{T}}$, $\phi_2(t) = (6,6,6,6)^{\mathrm{T}}$, $t \in [-0.8, 0]$. The state trajectories of the drive network with initial values $\phi_1(t)$ and $\phi_2(t)$ are shown in Fig. 2.

Then we illustrate the finite-time synchronization of the drive network with Theorem 1. The response network is given in the form of (3) in which $\sigma(t,x,y) = 0.1x + 0.1y$. Then $R_1 = R_2 = 0.02$, conditions (6) can be calculated as $k_1 \geqslant 0.04$, $k_1^1 \geqslant -2.97$,

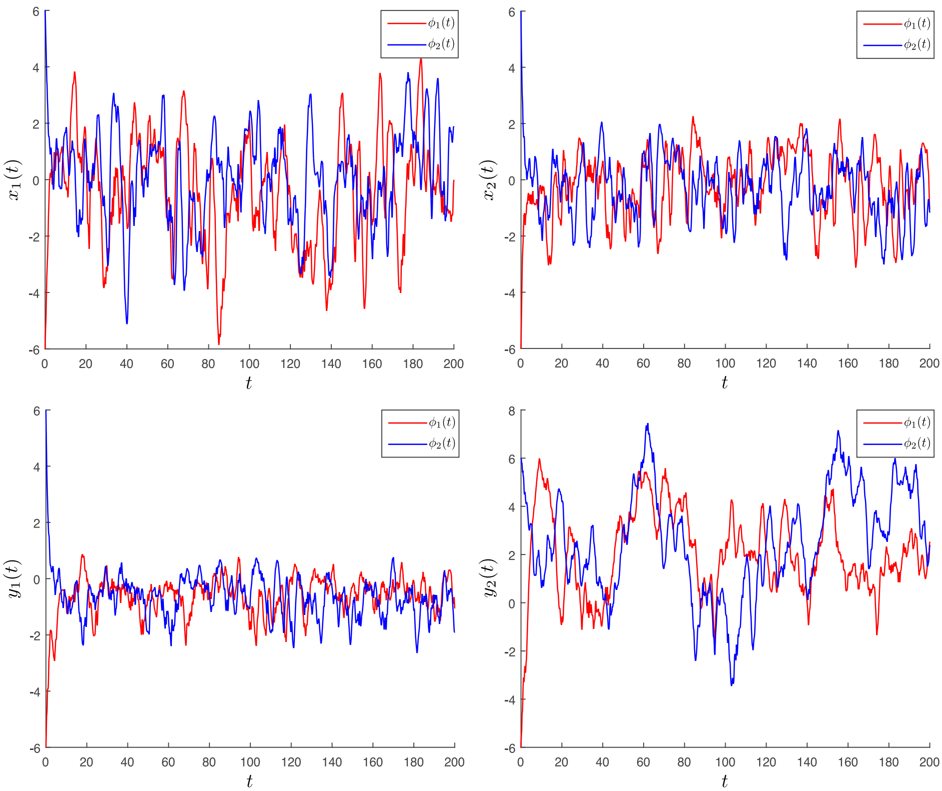

Figure 2. The state trajectories of the drive network with initial values $\phi_1(t)$ and $\phi_2(t)$.

$k_1^2 \geqslant 8.60$, $k_2 \geqslant 0.04$, $k_2^1 \geqslant -0.99$, $k_2^2 \geqslant 11.08$, $w_1 \geqslant 0.04$, $w_1^1 \geqslant -2.72$, $w_1^2 \geqslant 10.32$, $w_2 \geqslant 0.04$, $w_2^1 \geqslant -2.05$, $w_2^2 \geqslant 10.88$. Choose $\alpha = 0.3$, $\eta = 1$, and other parameters of controller (4) take the minimum of the feasible values. Take $\phi_1(t)$ and $\phi_2(t)$ as the initial values of the drive and response networks, respectively. By Theorem 1, the stochastic settling time can be estimated as $\mathbf{E}(T_0) \leqslant 13.3610$. To simulate the stochastic differential system, we used the method approved in [14]. The sate trajectories of the drive and response networks are shown in Fig. 3. The settling time in this simulation is 2.1151.

Next, we will demonstrate the effectiveness of Theorem 2. Still, we consider the drive and response networks above. Take $\phi_1(t)$ and $\phi_2(t)$ as initial values of the drive and response networks, respectively. By calculating the conditions (14) in Theorem 2, we choose $\eta = 1$, $k_1 = 0.04$, $k_1^1 = -2.97$, $k_1^2 = 8.60$, $k_2 = 0.04$, $k_2^1 = -0.99$, $k_2^2 = 11.08$, $w_1 = 0.04$, $w_1^1 = -2.72$, $w_1^2 = 10.32$, $w_2 = 0.04$, $w_2^1 = -2.05$, $w_2^2 = 10.88$, the switching parameter $\alpha(t)$ can be calculated as

$$\alpha(t) = \begin{cases} 0.6869 & \text{if } V(t) > \mathrm{e}^2, \\ 0 & \text{if } V(t) \leqslant \mathrm{e}^2. \end{cases}$$

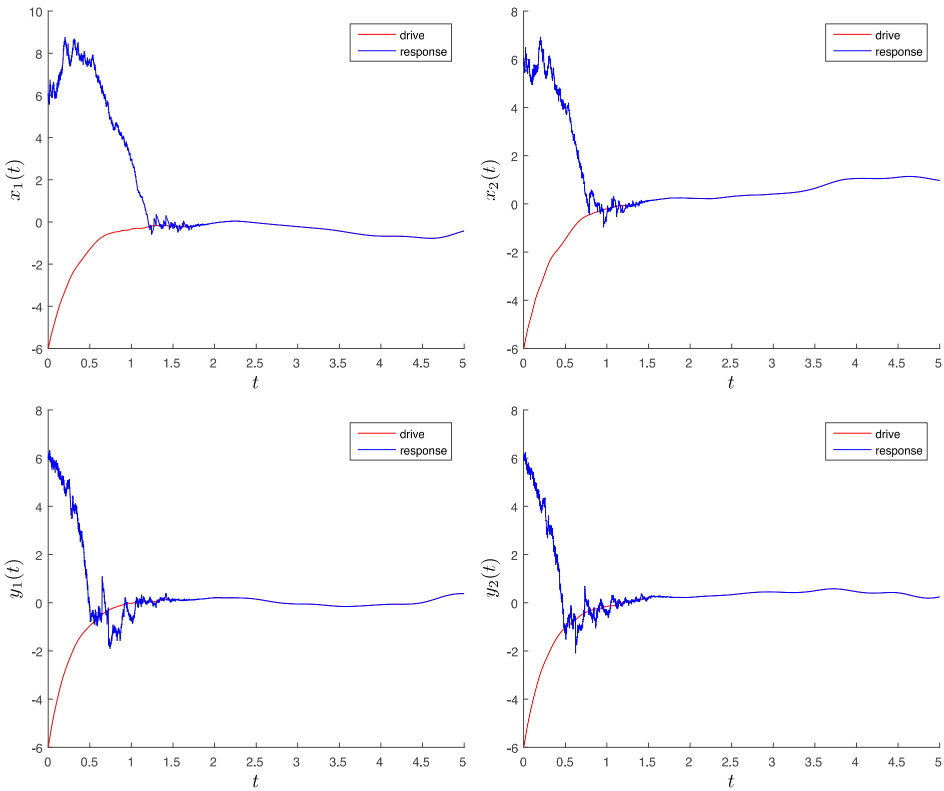

Figure 3. The state trajectories of the drive and response networks with $\alpha = 0.1$, $\eta = 1$. The initial values are $\phi_1(t)$ and $\phi_2(t)$, respectively.

Table 1. Average settling time of the drive and response networks with initial values $\phi_1(t)$ and $\phi_2(t)$ under different α

α	$\alpha(t)$	0	0.2	0.4	0.6	0.6869	0.8
Average time	1.6891	2.4803	2.1717	2.1057	1.8185	1.7581	1.7904

By Theorem 2, the stochastic settling time can be estimated as $\mathbf{E}(T_0) \leqslant 7.0319$. The simulation results are as follows. Figure 4 depicts the sate trajectories of the drive and response networks. The settling time in this simulation is 1.6521. Then we select α as $\alpha(t)$, 0, 0.2, 0.4, 0.6, 0.6869, 0.8, respectively. And for each α we simulate for 10 times. The average settling time for each α are listed in Table 1.

From the results we can see that the drive and response networks can achieve synchronization in finite time with controllers (4) and (14) under Theorems 1 and 2, respectively. The average settling time in simulation is shorter than the theoretical upper limit value. Moreover, the networks achieve synchronization in a shorter time with the error- dependent switching controller (14) in this example. However, we have to claim that the error-dependent switching controller can only provide a smaller upper limit estimation of

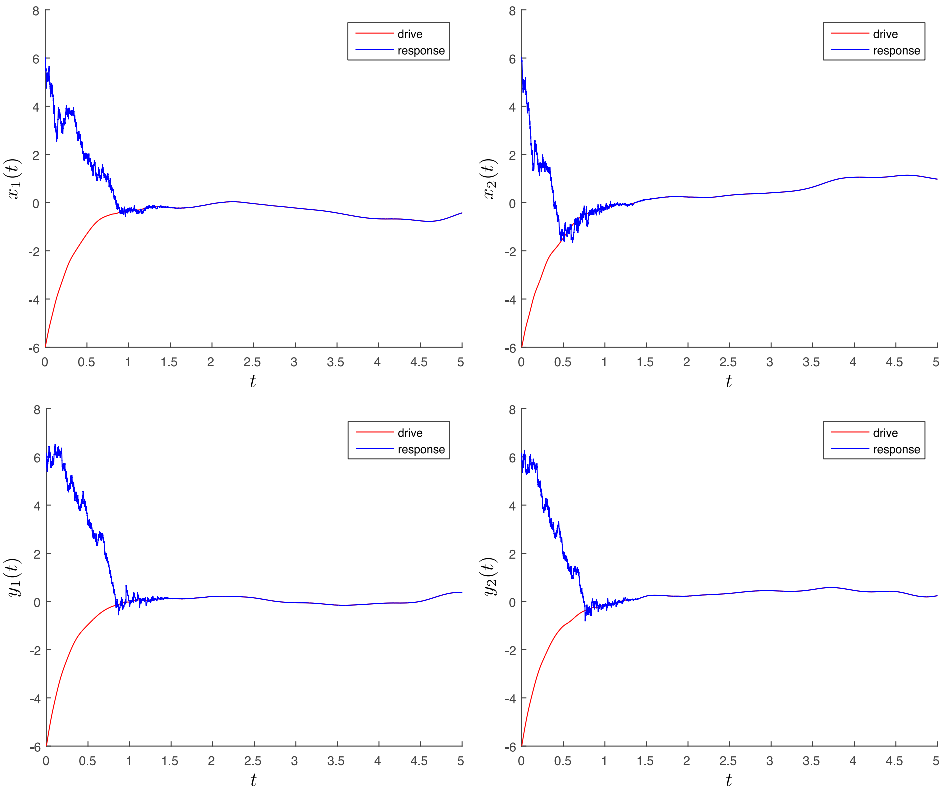

Figure 4. The state trajectories of the drive and response networks with switching controller (14). The initial values are $\phi_1(t)$ and $\phi_2(t)$, respectively.

settling time. It can not guarantee the networks achieve synchronization in a shorter time under all circumstances. We obtained the switching condition by analyzing the derivative of the expectation of settling time, but some enlarged operations in the process that we got the expectation of settling time make the switching point not optimal.

5 Summary

In this paper, the finite-time stochastic synchronization for a class of memristor-based BAM neural networks with time-varying delays and stochastic disturbances was investigated. Based on the structure of BAM neural networks and the basic principle of memristors, we came up with a new kind of switching conditions for the parameters. Based on the theory of Filippov's solution, by using Lyapunov–Krasovskii functionals and stochastic analysis technique, we presented sufficient conditions to ensure the finite-time stochastic synchronization of MBAMNNs with a certain controller and an error-dependent switching controller, respectively. Numerical examples were given to illustrate the effectiveness of our theorems. The drive and response networks achieved finite-time stochastic synchronization in simulations. Moreover, the error-dependent switching controller showed its advantage. Still there are some interesting yet challenging open problems for future study. For example, in some cases the signals provided by controllers we proposed are required to be quite large, however it is difficult to deliver arbitrarily large signals through real actuators. In [25, 26], actuator saturation was considered in the study of control analysis, which may be used to improve our results.

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