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Necessary optimality conditions for Lagrange problems involving ordinary control systems described by fractional Laplace operators

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Abstract. In this paper, optimal control problems containing ordinary nonlinear control systems described by fractional Dirichlet and Dirichlet–Neumann Laplace operators and a nonlinear integral performance index are studied. Using smooth-convex maximum principle, the necessary optimality conditions for such problems are derived.

Keywords: smooth-convex extremum principle, spectral representation of a self-adjoint operator, fractional Laplace operator, Dirichlet and Dirichlet–Neumann boundary conditions.

1 Introduction

In this paper, we consider the following two optimal control problems:

$$\begin{cases} \begin{cases} (-\Delta_k)^\beta x(t) = f\big(t, x(t), u(t)\big), & t \in (0,\pi) \text{ a.e.}, \qquad (\mathrm{E}_k) \\ u(t) \in M \subset \mathbb{R}^m, & t \in (0,\pi), \end{cases} \\ J(x,u) = \int_0^\pi f_0\big(t, x(t), u(t)\big)\,\mathrm{d}t \to \min, \end{cases} \qquad (\mathrm{OCP}_k)$$

where $k = 1, 2$, $\beta > 1/4$, $f : (0,\pi) \times \mathbb{R}^n \times M \to \mathbb{R}^n$ and $f_0 : (0,\pi) \times \mathbb{R}^n \times M \to \mathbb{R}$. The first of them, denoted by (OCP_1), contains the control system (E_1) involving the one-dimensional Dirichlet Laplace operator of order β $(-\Delta_1)^\beta$. The second one (OCP_2) includes the system (E_2), which is described by the Dirichlet–Neumann Laplace operator $(-\Delta_2)^\beta$. Operators $(-\Delta_1)^\beta$ and $(-\Delta_2)^\beta$ are defined through the spectral decomposition of the Laplace operator $-\Delta$ in $(0,\pi)$ with zero Dirichlet and Dirichlet–Neumann boundary conditions, respectively (cf. Section 2.2).

In the last years, fractional Laplacians are a topic of research of many scientists. There exist many definitions of such operators (e.g. based on Fourier transform [19, 25], hypersingular integral [25], Riesz potential operator [24], Bochner's definition [26], spectral decomposition (cf. [6, 18])). These different definitions typically lead to different operators (cf. [1, Sect. 2.3.]). Recent intensive investigations show that Laplacians can be applied in various areas; for example, in economics (cf. [5, 18]), probability (cf. [5, 9, 10, 17]), mechanics [8, 10], material science (cf. [7]), fluid mechanics and hydrodynamics (cf. [11, 14–16, 29–31]). Over the last years, they have been attracted interest of many mathematicians also in the field of optimal control theory. In [12, 13], some optimal control problems with a fractional Dirichlet–Laplace operator are investigated. The results concerning the existence, stability, continuous dependence of solutions on controls and existence of optimal solutions minimizing a some integral cost functional have been obtained there. In [28], a some optimal control problem (inspired by considerations in mathematical biology) with a general positive defined fractional operator (so-called diffusion operator) is studied. To be more specific, an evolution equation of a diffusion type with a some integral cost functional is considered. The necessary and sufficient optimality conditions for such a problem have been derived. Results of such a type have been also obtained in [3,4], where the linear–quadratic optimal control problems involving fractional powers of elliptic operators are investigated. Furthermore, a numerical scheme to solve the fractional optimal problems has been proposed there.

The aim of this paper is to derive the necessary optimality conditions for problems (OCP_k). Below, we formulate a result of such a type for the problem (OCP_1) obtained in [22] via Dubovitskii–Miljutin (D–M) approach (more details concerning (D–M) method can be found in [22, Sect. 2.1]):

Theorem 1. *Let us assume that*:

(i) $M \subset \mathbb{R}^m$ *is a closed convex set with nonempty interior,*

(ii) f, f_0 *are measurable on* $(0,\pi)$*, continuously differentiable on* $\mathbb{R}^n \times \mathbb{R}^m$ *and*

$$\bigl|f(t,x,u)\bigr|, \bigl|f_x(t,x,u)\bigr|, \bigl|f_u(t,x,u)\bigr| \leqslant a(t)\gamma\bigl(|x|\bigr) + b(t)\delta\bigl(|u|\bigr);$$
$$\bigl|f_0(t,x,u)\bigr|, \bigl|(f_0)_x(t,x,u)\bigr|, \bigl|(f_0)_u(t,x,u)\bigr| \leqslant d(t)c\bigl(|x|,|u|\bigr)$$

for $(t,x,u) \in (0,\pi) \times \mathbb{R}^n \times \mathbb{R}^m$*, where* $a, b \in L^2$*,* $d \in L^1$*,* $\gamma, \delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$*,* $c : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are continuous functions.*

If $(x_*, u_*) \in ((-\Delta_1)^\beta) \times L^\infty$ *is a local minimum point for problem* (OCP_1)*, one of the following conditions*:

- $\beta > 1/2$ *and* $\|f_x(\cdot, x_*(\cdot), u_*(\cdot))\|_{L^1} < \pi/(2\xi(2\beta))$;
- $\beta > 1/2$ *and* $f_x(t, x_*(t), u_*(t)) \leqslant 0$ *for a.e.* $t \in (0,\pi)$;
- $\beta > 1/4$*,* $f_x(\cdot, x_*(\cdot), u_*(\cdot)) \in L^\infty$ *and* $\|f_x(\cdot, x_*(\cdot), u_*(\cdot))\|_{L^\infty} < 1$;

is fulfilled and $(f_0)_x(\cdot, x_*(\cdot), u_*(\cdot))$*,* $(f_0)_u(\cdot, x_*(\cdot), u_*(\cdot))$ *are not all identically zero, then there exists a function* $\lambda \in D((-\Delta_1)^\beta)$ *such that*

$$(-\Delta_1)^\beta \lambda(t) = H_x\bigl(t, x_*(t), \lambda(t), u_*(t)\bigr), \quad t \in (0,\pi) \text{ a.e.},$$
$$H_u\bigl(t, x_*(t), \lambda(t), u_*(t)\bigr)\bigl(u - u_*(t)\bigr) \geqslant 0, \quad t \in (0,\pi) \text{ a.e.},$$

for any $u \in M$*, whereby* $H : (0, \pi) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$:

$$H(t, x, \lambda, u) = \lambda^{\mathrm{T}} f(t, x, u) + f_0(t, x, u).$$

However, there exist optimal control problems of type (OCP$_k$) where the (D–M) method cannot be applied. For example, if M is a finite set containing at least two elements (such a set is often used in optimal control theory), then M does not satisfy assumption (i) of Theorem 1 (the set M is not convex and its interior is empty). Moreover, in view of a specific structure of the set M, the assumption that f and f_0 are differentiable with respect to u does not make sense. So, it is necessary to use an alternative method to obtain optimality conditions. In our study, we use a smooth-convex extremum principle (cf. [23]). In this approach, the assumption of differentiability of f, f_0 with respect to u is replaced with some "convexity assumption" (consequently, maximum conditions obtained in both methods are different). On the other hand, in our approach, compactness of the set M is required (cf. Lemma 4). Consequently, in contrast to the (D–M) approach, it cannot be applied to optimal control problems where M is unbounded (in particular, $M = \mathbb{R}^m$). To sum up, from the above discussion it follows that both methods are useful in practical applications. One can also show that if f, f_0 are smooth and convex in u and M is convex, then both methods are equivalent (more precisely, the minimum conditions in both approaches are equivalent).

The paper is organized as follows. In Section 2, we give necessary notions and facts concerning ordinary Dirichlet and Dirichlet–Neumann Laplace operators of fractional order, as well as the extremum principle for a smooth-convex problem is formulated. In Section 3, we derive the main result of this paper, namely the necessary optimality conditions for problems (OCP$_k$), $k = 1, 2$ (Theorem 3). Two illustrative examples are presented in Section 4. We finish with Appendix A containing some basics from the spectral theory of self-adjoint operators in a real Hilbert space.

2 Preliminaries

In the first part of this section, we formulate the so-called smooth-convex optimal control problem and recall the extremum principle for it (cf. [23]). This tool will be used in the proof of the main result of this paper (Theorem 3). The second part concerns fractional ordinary Dirichlet and mixed Dirichlet–Neumann Laplace operators. The definitions of these operators come from the Stone-von Neumann operator calculus and are based on the spectral integral representation theorem for a self-adjoint operator in a Hilbert space (cf. [21, 22] and Appendix A).

2.1 Smooth-convex extremum principle

Let X, Y be the Banach spaces, and $\mathcal{U}$ denotes an arbitrary nonempty set. Let us consider the following problem:

$$f_0(x, u) \to \inf; \tag{1}$$

$$F(x, u) = 0, \tag{2}$$

$$f_1(x,u) \leqslant 0, \quad \dots, \quad f_n(x,u) \leqslant 0, \tag{3}$$

$$u \in \mathcal{U}, \tag{4}$$

where $f_0, \dots, f_n : X \times \mathcal{U} \to \mathbb{R}$ and $F : X \times \mathcal{U} \to Y$.

If the functions $f_0, \dots, f_n$ and the mapping F satisfy certain conditions of smoothness in x and "convexity" in u (cf. assumptions (a), (b) in Theorem 2), then the above problem is called the smooth-convex problem.

The Lagrange function for problem (1)–(4) is given by

$$\mathcal{L}(x, u, \lambda_0, \dots, \lambda_n, y^*) = \sum_{i=0}^{n} \lambda_i f_i(x,u) + \langle y^* F(x,u) \rangle,$$

where $\lambda_0, \dots, \lambda_n$ are real numbers, and $y^* \in Y^*$ (Y^* is the dual space to Y).

We say that a pair $(x_*, u_*) \in X \times \mathcal{U}$, satisfying constraints (2)–(4), is a local minimum point of problem (1)–(4) if there exists a neighborhood V of x_* such that

$$f_0(x_*, u_*) \leqslant f_0(x,u)$$

for any pair $(x,u) \in V \times \mathcal{U}$ satisfying constraints (2)–(4).

Theorem 2 [Smooth-convex extremum principle]. *Let the pair (x_*, u_*) satisfies conditions* (2)–(4)*, and assume that there exists a neighborhood $V \subset X$ of x_* such that*

(a) *for every $u \in \mathcal{U}$, the mapping $x \to F(x,u)$ and the functions $x \to f_i(x,u)$, $i = 0, \dots, n$, are continuously differentiable at the point x_*;*

(b) *for every $x \in V$, the mapping $u \to F(x,u)$ and the functions $u \to f_i(x,u)$, $i = 0, \dots, n$, satisfy the following convexity condition: for every $u_1, u_2 \in \mathcal{U}$ and $\beta \in [0,1]$, there exists an element $u \in \mathcal{U}$ such that*

$$F(x,u) = \beta F(x,u_1) + (1-\beta) F(x,u_2),$$
$$f_i(x,u) \leqslant \beta f_i(x,u_1) + (1-\beta) f_i(x,u_2), \quad i = 0, \dots, n;$$

(c) *the range $\operatorname{Im} F_x(x_*, u_*)$ of the linear operator $F_x(x_*, u_*) : X \to Y$ is closed and has a finite codimension in Y (i.e. a complementary subspace to $\operatorname{Im} F_x(x_*, u_*)$ has a finite dimension in Y).*

If (x_*, u_*) is a local minimum point of problem (1)–(4)*, then there exist the Lagrange multipliers $\lambda_0 \geqslant 0, \dots, \lambda_n \geqslant 0$, $y^* \in Y^*$ (not all zero) such that*

$$\mathcal{L}_x(x_*, u_*, \lambda_0, \dots, \lambda_n, y^*) = \sum_{i=0}^{n} \lambda_i (f_i)_x(x_*, u_*) + F_x^*(x_*, u_*) y^* = 0,$$
$$\mathcal{L}(x_*, u_*, \lambda_0, \dots, \lambda_n, y^*) = \min_{u \in U} \mathcal{L}(x_*, u, \lambda_0, \dots, \lambda_n, y^*),$$
$$\lambda_i f_i(x_*, u_*) = 0 \quad \textit{for } i = 1, \dots, n.$$

If, additionally,

(iv) *the image of the set $X \times U$ under the mapping*

$$(x,u) \to F_x(x_*, u_*) x + F(x_*, u)$$

contains a neighborhood of the origin of Y *and if there exists a point* (x,u) *such that*

$$F_x(x_*,u_*)x + F(x_*,u) = 0,$$
$$\big\langle (f_i)_x(x_*,u_*),x\big\rangle + f_i(x_*,u) < 0$$

for all $i = 1,\dots,n$ *for which* $f_i(x_*,u_*) = 0$*, then* $\lambda_0 \neq 0$ *and it can be assumed that* $\lambda_0 = 1$.

2.2 One-dimensional Dirichlet and Dirichlet–Neumann Laplace operators of fractional order

Let $-\Delta$ be the one-dimensional Laplace operator on the interval $(0,\pi)$ given by

$$-\Delta u = -u''. \tag{5}$$

Let us define the following spaces of functions:

$$H_D := H_0^1 \cap H^2 \quad \text{and} \quad H_{DN} := \big\{z \in H^2\colon\ z(0) = z'(\pi) = 0\big\}$$

(here $H_0^1 = H_0^1((0,\pi),\mathbb{R}^n)$ and $H^2 = H^2((0,\pi),\mathbb{R}^n)$ are classical Sobolev spaces).

Conditions $z(0) = z(\pi) = 0$ (hidden in the definition of H_D) and $z(0) = z'(\pi) = 0$ are called Dirichlet and Dirichlet–Neumann boundary conditions, respectively. Of course, H_D and H_{DN} are dense subspaces of $L^2 = L^2((0,\pi),\mathbb{R}^n)$.

By the one-dimensional Dirichlet Laplace operator $-\Delta_D : H_D \subset L^2 \to L^2$ we mean the operator $-\Delta$ given by (5) under Dirichlet boundary conditions. Similarly, by the one-dimensional Dirichlet–Neumann Laplace operator $-\Delta_{DN} : H_{DN} \subset L^2 \to L^2$ we mean the operator $-\Delta$ under Dirichlet–Neumann boundary conditions.

In an elementary way, one can show that operators $-\Delta_D$ and $-\Delta_{DN}$ are self-adjoint. Moreover, their spectra are given by

$$\sigma(-\Delta_D) = \sigma_p(-\Delta_D) = \big\{j^2,\ j = 1,2,\dots\big\},$$
$$\sigma(-\Delta_{DN}) = \sigma_p(-\Delta_{DN}) = \left\{\left(j - \frac{1}{2}\right)^2,\ j = 1,2,\dots\right\},$$

respectively, and the eigenspaces $\mathrm{Eig}_j(-\Delta_D)$ (associated with the eigenvalues $\lambda_j = j^2$), $\mathrm{Eig}_j(-\Delta_{DN})$ (associated with the eigenvalues $\lambda_j = (j-1/2)^2$) are sets

$$\mathrm{Eig}_j(-\Delta_D) = \big\{c\sin jt,\ c \in \mathbb{R}^n\big\},$$
$$\mathrm{Eig}_j(-\Delta_{DN}) = \left\{d\sin\left(j - \frac{1}{2}\right)t,\ d \in \mathbb{R}^n\right\}.$$

In what follows, we shall use the fact that systems of functions

$$c_{i,j} = \Bigg(0,\dots,0,\underbrace{\sqrt{\frac{2}{\pi}}\sin jt}_{i},0,\dots,0\Bigg), \quad i = 1,\dots,n,\ j = 1,2,\dots,$$

$$d_{i,j} = \Bigl(0,\dots,0,\underbrace{\sqrt{\frac{2}{\pi}}\sin\Bigl(j-\frac{1}{2}\Bigr)t}_{i},0,\dots,0\Bigr),\quad i=1,\dots,n,\ j=1,2,\dots,$$

are complete orthonormal systems in L^2.

Now, assume $\beta > 0$ and consider the operator

$$(-\Delta_D)^\beta\colon\ D\bigl((-\Delta_D)^\beta\bigr) \subset L^2 \to L^2$$

given by (cf. Appendix A)

$$(-\Delta_D)^\beta x(t) = \Bigl(\int\limits_{\sigma(-\Delta_D)} \lambda^\beta E(\mathrm{d}\lambda)x\Bigr)(t) = \sum_{j=1}^{\infty} j^{2\beta} a_j \sqrt{\frac{2}{\pi}}\sin jt$$

for $x \in D((-\Delta_D)^\beta)$, where

$$D\bigl((-\Delta_D)^\beta\bigr) = \Biggl\{x(t) = \Bigl(\int\limits_{\sigma(-\Delta_D)} 1E(\mathrm{d}\lambda)x\Bigr)(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}}\sin jt \in L^2;\ \int\limits_{\sigma(-\Delta_D)} \bigl|\lambda^\beta\bigr|^2 \bigl\|E(\mathrm{d}\lambda)x\bigr\|_{L^2}^2 = \sum_{j=1}^{\infty} j^{4\beta}|a_j|^2 < \infty\Biggr\}$$

(here E is the spectral measure (cf. Appendix A.2) for the operator $-\Delta_D$, and $a_j\sqrt{2/\pi}\times \sin jt$ is the projection of x on the n-dimensional eigenspace $\mathrm{Eig}_j(-\Delta_D)$).

The operator $(-\Delta_D)^\beta$ is called the fractional Dirichlet Laplace operator of order β, and the function $(-\Delta_D)^\beta x$ – the fractional Dirichlet Laplacian of order β of x.

Similarly, we define the fractional Dirichlet–Neumann Laplace operator of order β. This is the operator

$$(-\Delta_{DN})^\beta\colon\ D\bigl((-\Delta_{DN})^\beta\bigr) \subset L^2 \to L^2$$

given by

$$(-\Delta_{DN})^\beta x(t) = \Bigl(\int\limits_{\sigma(-\Delta_{DN})} \lambda^\beta F(\mathrm{d}\lambda)x\Bigr)(t) = \sum_{j=1}^{\infty}\Bigl(j-\frac{1}{2}\Bigr)^{2\beta} b_j \sqrt{\frac{2}{\pi}}\sin\Bigl(j-\frac{1}{2}\Bigr)t$$

for $x \in D((-\Delta_{DN})^\beta)$, where

$$D\bigl((-\Delta_{DN})^\beta\bigr) = \Biggl\{x(t) = \Bigl(\int\limits_{\sigma(-\Delta_{DN})} 1F(\mathrm{d}\lambda)x\Bigr)(t) = \sum_{j=1}^{\infty} b_j \sqrt{\frac{2}{\pi}}\sin\Bigl(j-\frac{1}{2}\Bigr)t \in L^2;\ \int\limits_{\sigma(-\Delta_{DN})} \bigl|\lambda^\beta\bigr|^2 \bigl\|F(\mathrm{d}\lambda)x\bigr\|_{L^2}^2 = \sum_{j=1}^{\infty}\Bigl(j-\frac{1}{2}\Bigr)^{4\beta}|b_j|^2 < \infty\Biggr\}$$

(here F is the spectral measure for the operator $-\Delta_{DN}$, and $b_j\sqrt{2/\pi}\sin(j-1/2)t$ is the projection of x on the n-dimensional eigenspace $\mathrm{Eig}_j(-\Delta_{DN})$).

Remark 1. To shorten the notation, in the next sections, the fractional Dirichlet Laplace operator of order β is denoted by $(-\Delta_1)^\beta$. Similarly, by $(-\Delta_2)^\beta$ we mean the fractional Dirichlet–Neumann Laplace operator.

We have (cf. [21, 22])

Lemma 1. *$D((-\Delta_D)^\beta)$ is complete with the scalar products*

$$\langle x, y\rangle_{D_\beta} = \langle x, y\rangle_{L^2} + \big\langle(-\Delta_D)^\beta x, (-\Delta_D)^\beta y\big\rangle_{L^2}$$

and

$$\langle x, y\rangle_{D_{\sim\beta}} = \big\langle(-\Delta_D)^\beta x, (-\Delta_D)^\beta y\big\rangle_{L^2}. \tag{6}$$

Moreover, norms generated by these products are equivalent.

Remark 2. Completeness of the domain $D((-\Delta_D)^\beta)$ follows from the fact that the operator $(-\Delta_D)^\beta$ is self-adjoint (cf. Appendix A), so it is closed. Equivalence of norms $\|\cdot\|_{D_\beta}$ and $\|\cdot\|_{D_{\sim\beta}}$ guarantees the following Poincaré inequality on $D((-\Delta_D)^\beta)$ (cf. [21]):

$$\|x\|^2_{L^2} \leqslant \|x\|^2_{D_{\sim\beta}}, \quad x \in D\big((-\Delta_D)^\beta\big).$$

Using the similar argumentation (cf. Remark 2), we also obtain

Lemma 2. *$D((-\Delta_{DN})^\beta)$ is complete with the scalar products*

$$\langle x, y\rangle_{DN_\beta} = \langle x, y\rangle_{L^2} + \big\langle(-\Delta_{DN})^\beta x, (-\Delta_{DN})^\beta y\big\rangle_{L^2}$$

and

$$\langle x, y\rangle_{DN_{\sim\beta}} = \big\langle(-\Delta_{DN})^\beta x, (-\Delta_{DN})^\beta y\big\rangle_{L^2}. \tag{7}$$

Moreover, norms generated by these products are equivalent.

In particular, equivalence of norms $\|\cdot\|_{DN_\beta}$ and $\|\cdot\|_{DN_{\sim\beta}}$ is provided due to the following Poincaré inequality on $D((-\Delta_{DN})^\beta)$:

$$\|x\|^2_{L^2} = \sum_{j=1}^{\infty} b_j^2 \leqslant \sum_{j=1}^{\infty} (2j-1)^{4\beta} b_j^2 = 16^\beta \sum_{j=1}^{\infty} \left(j - \frac{1}{2}\right)^{4\beta} b_j^2 = 16^\beta \|x\|^2_{DN_{\sim\beta}}$$

for $x \in D((-\Delta_{DN})^\beta)$.

In the proof of the main result of this paper, we shall use the following

Lemma 3. *If $\beta > 1/4$, then*

$$\|x\|_{L^\infty} \leqslant \sqrt{\frac{2}{\pi}\zeta(4\beta)}\,\|x\|_{D_{\sim\beta}}, \quad x \in D\big((-\Delta_D)^\beta\big), \tag{8}$$

$$\|x\|_{L^\infty} \leqslant 4^\beta \sqrt{\frac{2}{\pi}\zeta(4\beta)}\,\|x\|_{DN_{\sim\beta}}, \quad x \in D\big((-\Delta_{DN})^\beta\big), \tag{9}$$

and therefore embeddings

$$D\big((-\Delta_D)^\beta\big) \subset L^\infty, \qquad D\big((-\Delta_{DN})^\beta\big) \subset L^\infty$$

are continuous (here ζ is the Riemann zeta function given by $\zeta(\gamma) = \sum_{k=1}^{\infty} 1/k^\gamma$).

Proof. The proof of (8) can be found in [21].

Now, let $x \in D((-\Delta_{DN})^\beta)$. Then

$$\begin{aligned} |x(t)|^2 &= \left|\sum_{j=1}^{\infty} b_j \sqrt{\frac{2}{\pi}} \sin\left(j - \frac{1}{2}\right)t\right|^2 \leqslant \frac{2}{\pi}\left(\sum_{j=1}^{\infty} |b_j|\right)^2 = \frac{2}{\pi}\left(\sum_{j=1}^{\infty} \frac{(j-\frac{1}{2})^{2\beta}|b_j|}{(j-\frac{1}{2})^{2\beta}}\right)^2 \\ &\leqslant \frac{2}{\pi}\left(\sum_{j=1}^{\infty}\left(j-\frac{1}{2}\right)^{4\beta} b_j^2\right)\sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{2})^{4\beta}} \leqslant \frac{2}{\pi}\|x\|^2_{DN_{\sim\beta}}\sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{2})^{4\beta}} \\ &= \frac{2}{\pi}\|x\|^2_{DN_{\sim\beta}}\sum_{j=1}^{\infty}\frac{1}{(\frac{1}{2}j)^{4\beta}} = 16^\beta\frac{2}{\pi}\|x\|^2_{DN_{\sim\beta}}\zeta(4\beta) < \infty, \quad t \in (0,\pi) \text{ a.e.} \end{aligned}$$

Hence, we obtain inequality (9).

The proof is completed. □

3 Necessary optimality conditions

In this part of the paper, we derive the necessary optimality conditions for optimal control problems (OCP_k), $k = 1, 2$.

We define the set of controls

$$\mathcal{U}_M := \bigl\{u \in L^1((0,\pi),\mathbb{R}^m)\colon\ u(t) \in M \subset \mathbb{R}^m,\ t \in (0,\pi)\bigr\}.$$

Let us fix $k = 1, 2$. We say that a pair $(x_*, u_*) \in D((-\Delta_k)^\beta) \times \mathcal{U}_M$ is a locally optimal solution of the problem (OCP_k) if x_* is a solution of (E_k) corresponding to the control u_* and there exists a neighborhood W_k of the point x_* in $D((-\Delta_k)^\beta)$ such that

$$J(x_*, u_*) \leqslant J(x, u)$$

for all pairs $(x, u) \in W_k \times \mathcal{U}_M$ satisfying (E_k).

In the proof of the main result, we shall use the following two lemmas.

Lemma 4. *Assume that*

(i) $M \subset \mathbb{R}^m$ *is compact*;
(ii) $h(\cdot, x, u)$ *is measurable on* $[a, b]$ *for all* $x \in \mathbb{R}^n$, $u \in M$;
(iii) $h(t, \cdot, u)$ *is continuous on* $\mathbb{R}^n$ *for a.e.* $t \in [a, b]$ *and all* $u \in M$;
(iv) $h(t, x, \cdot)$ *is continuous on* M *for a.e.* $t \in [a, b]$ *and all* $x \in \mathbb{R}^n$;
(v) *for a.e.* $t \in [a, b]$ *and all* $x \in \mathbb{R}^n$, *the set*

$$h(t, x, M) := \bigl\{h(t, x, u) \in \mathbb{R}^n,\ u \in M\bigr\}$$

is convex.

Then for all $u_1, u_2 \in \mathcal{U}_M$, $x \in D((-\Delta_k)^\beta)$, $k = 1, 2$, *and* $\gamma \in [0, 1]$, *there exists* $\widetilde{u} \in \mathcal{U}_M$ *such that*

$$h\bigl(t, x(t), \widetilde{u}(t)\bigr) = \gamma h\bigl(t, x(t), u_1(t)\bigr) + (1-\gamma) h\bigl(t, x(t), u_2(t)\bigr), \quad t \in [a, b] \text{ a.e.}$$

Lemma 5. *Let $u_* \in \mathcal{U}_M$ and $\varphi : [a,b] \times M \to \mathbb{R}$ be such that $\varphi(\cdot, u)$ is measurable on $[a,b]$ for all $u \in M$, and let $\varphi(t,\cdot)$ is continuous on M for a.e. $t \in [a,b]$. If*

$$-\infty < \int_a^b \varphi\bigl(t, u_*(t)\bigr)\,\mathrm{d}t \leqslant \int_a^b \varphi\bigl(t, u(t)\bigr)\,\mathrm{d}t < +\infty, \quad u \in \mathcal{U}_M,$$

then

$$\varphi\bigl(t, u_*(t)\bigr) \leqslant \varphi(t, u)$$

for a.e. $t \in [a,b]$ and all $u \in M$.

Remark 3. Lemma 4 can be proved just as Lemma 5 in [20]. Lemma 5 differs from the appropriate result proved in [20, Lemma 6] in the "control domain" – we replace $\mathbb{R}^r$ by M. It can be proved in the same way as in [20] with the aid of the theorem on the measurability of the superposition $[a,b] \ni t \to g(t, u(t)) \in \mathbb{R}^n$ of a Carathéodory function $g : [a,b] \times M \to \mathbb{R}^n$ with a measurable one $u : [a,b] \to M$ (cf. [23, p. 330]).

Now, we derive the maximum principle for problems (OCP_1) and (OCP_2). We have

Theorem 3. *Let us fix $k = 1, 2$. We assume that M is compact, $\beta > 1/4$ and*

(A) *$f_0(\cdot, x, u)$ is measurable on $(0,\pi)$ for all $x \in \mathbb{R}^n$, $u \in M$, $f_0(t, x, \cdot)$ is continuous on M for a.e. $t \in (0,\pi)$ and all $x \in \mathbb{R}^n$, $f_0 \in C^1$ with respect to $x \in \mathbb{R}^n$ and*

$$\bigl|f_0(t,x,u)\bigr|, \bigl|(f_0)_x(t,x,u)\bigr| \leqslant a(t)\eta\bigl(|x|\bigr)$$

for a.e. $t \in (0,\pi)$ and all $(x,u) \in \mathbb{R}^n \times M$, where $a \in L^2((0,\pi), \mathbb{R}_0^+)$, $\eta \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$;

(B) *$f(\cdot, x, u)$ is measurable on $(0,\pi)$ for all $x \in \mathbb{R}^n$, $u \in M$, $f(t, x, \cdot)$ is continuous on M for a.e. $t \in (0,\pi)$ and all $x \in \mathbb{R}^n$, $f \in C^1$ with respect to $x \in \mathbb{R}^n$ and*

$$\bigl|f(t,x,u)\bigr| \leqslant b(t)\delta\bigl(|x|\bigr)$$

for a.e. $t \in (0,\pi)$ and all $(x,u) \in \mathbb{R}^n \times M$, where $b \in L^2((0,\pi), \mathbb{R}_0^+)$, $\delta \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$;

(C) *for any $(x,u) \in D((-\Delta_k)^\beta) \times \mathcal{U}_M$, $f_x(\cdot, x(\cdot), u(\cdot)) \in L^\infty((0,\pi), \mathbb{R}^{n\times n})$ and*

$$\bigl\|f_x\bigl(\cdot, x(\cdot), u(\cdot)\bigr)\bigr\|_{L^\infty} < 1 \qquad \text{if } k = 1, \tag{10}$$

$$\bigl\|f_x\bigl(\cdot, x(\cdot), u(\cdot)\bigr)\bigr\|_{L^\infty} < \frac{1}{4^\beta} \quad \text{if } k = 2; \tag{11}$$

(D) *for a.e. $t \in (0,\pi)$ and all $x \in \mathbb{R}^n$, the set*

$$\bigl\{\bigl(f_0(t,x,u), f(t,x,u)\bigr) \in \mathbb{R}^{n+1},\ u \in M\bigr\}$$

is convex.

If the pair $(x_*, u_*) \in D((-\Delta_k)^\beta) \times \mathcal{U}_M$ *is a locally optimal solution of problem* (OCP$_k$)*, then there exists a function* $\lambda \in D((-\Delta_k)^\beta)$ *such that*

$$\begin{aligned}(-\Delta_k)^\beta \lambda(t) \\ &= f_x^{\mathrm{T}}\big(t, x_*(t), u_*(t)\big)\lambda(t) - (f_0)_x\big(t, x_*(t), u_*(t)\big), \quad t \in (0,\pi) \text{ a.e.},\end{aligned} \tag{12}$$

and

$$\begin{aligned}&f_0\big(t, x_*(t), u_*(t)\big) - \lambda(t) f\big(t, x_*(t), u_*(t)\big) \\ &\quad = \min_{u \in M}\big\{f_0\big(t, x_*(t), u\big) - \lambda(t) f\big(t, x_*(t), u\big)\big\} \quad t \in (0,\pi) \text{ a.e.}\end{aligned} \tag{13}$$

Proof. Let us fix $k = 1, 2$ and define the operators:

$$F_k : D\big((-\Delta_k)^\beta\big) \times \mathcal{U}_M \to L^2\colon\ F_k\big(x(\cdot), u(\cdot)\big) = (-\Delta_k)^\beta x(\cdot) - f\big(\cdot, x(\cdot), u(\cdot)\big),$$

$$F_k^0 : D\big((-\Delta_k)^\beta\big) \times \mathcal{U}_M \to \mathbb{R}\colon\ F_k^0\big(x(\cdot), u(\cdot)\big) = \int_0^\pi f_0\big(t, x(t), u(t)\big)\,\mathrm{d}t.$$

Then problem (OCP$_k$) can be formulated as

$$\begin{aligned}&F_k^0\big(x(\cdot), u(\cdot)\big) \to \min, \\ &F_k\big(x(\cdot), u(\cdot)\big) = 0, \\ &u(\cdot) \in \mathcal{U}_M.\end{aligned}$$

We shall show that F_k and F_k^0 satisfy all assumptions of Theorem 2.

First, let us note that from Lemma 4 applied to the function $h = (f_0, f)$ it follows that for any $x \in D((-\Delta_k)^\beta)$, $u_1, u_2 \in \mathcal{U}_M$ and $\gamma \in [0,1]$, there exists a function $\widetilde{u} \in \mathcal{U}_M$ such that

$$\begin{aligned}F_k^0(x, \widetilde{u}) &= \gamma F_k^0(x, u_1) + (1-\gamma) F_k^0(x, u_2), \\ F_k(x, \widetilde{u}) &= \gamma F_k(x, u_1) + (1-\gamma) F_k(x, u_2).\end{aligned}$$

This means that assumption (b) of Theorem 2 is satisfied.

Using assumptions (B), (C) and analogous arguments as in [21, Prop. 5.1], we check that the mapping F_k is continuously differentiable with respect to $x \in D((-\Delta_k)^\beta)$ and the differential $(F_k)_x(x, u) : D((-\Delta_k)^\beta) \to L^2$ of F_k at the point (x, u) is given by

$$(F_k)_x(x, u)h = (-\Delta_k)^\beta h(t) - f_x\big(t, x(t), u(t)\big)h(t)$$

for any fixed $u \in \mathcal{U}_M$.

Similarly (using assumption (A) and analogous arguments as in [22, Prop. 3.2]), we obtain a differentiability property of the mapping F_k^0 [1], whereby the differential

[1] In order to prove a differentiability property of mappings F_2 and F_2^0 (then $(-\Delta_2)^\beta = (-\Delta_{DN})^\beta$ denotes the Dirichlet–Neumann Laplace operator of order β), we use the norm generated by the scalar product (7) and the estimation (9) instead of (6) and (8), respectively.

$(F_k^0)_x(x,u) : D((-\Delta_k)^\beta) \to \mathbb{R}$ of F_k^0 at the point (x,u) is given by

$$(F_k)_x(x,u)h = \int_0^\pi (f_0)_x\big(t,x(t),u(t)\big)h(t)\,\mathrm{d}t$$

for any fixed $u \in \mathcal{U}_M$.

The fact that the range $\operatorname{Im}(F_k)_x(x_*,u_*)$ of the mapping $(F_k)_x(x_*,u_*)$ satisfies assumption (c) of Theorem 2 follows from

- condition (10) and [21, Prop. 5.2] in the case of $k=1$;
- condition (11) and the bijectivity of $(F_2)_x(x_*,u_*)$[2] in the case of $k=2$

(more precisely, in both cases, the range $\operatorname{Im}(F_k)_x(x_*,u_*)$ is a whole space L^2, so it is closed and its codimension is equal to zero).

So, all assumptions of the smooth-convex extremum principle are satisfied. Consequently, there exist (not all equal to zero) $\lambda_0 \geqslant 0$ and $\lambda \in L^2$ such that

$$\lambda_0 \int_0^\pi (f_0)_x\big(t,x_*(t),u_*(t)\big)h(t)\,\mathrm{d}t + \int_0^\pi \lambda(t)\big((-\Delta_k)^\beta h(t) - f_x\big(t,x_*(t),u_*(t)\big)h(t)\big)\,\mathrm{d}t = 0 \tag{14}$$

for any $h \in D((-\Delta_k)^\beta)$ and

$$\begin{aligned}&\lambda_0 \int_0^\pi f_0\big(t,x_*(t),u_*(t)\big)\,\mathrm{d}t + \int_0^\pi \lambda(t)\big((-\Delta_k)^\beta x_*(t) - f\big(t,x_*(t),u_*(t)\big)\big)\,\mathrm{d}t \\ &= \min_{u\in\mathcal{U}_M}\bigg\{\lambda_0 \int_0^\pi f_0\big(t,x_*(t),u(t)\big)\,\mathrm{d}t \\ &\quad + \int_0^\pi \lambda(t)\big((-\Delta_k)^\beta x_*(t) - f\big(t,x_*(t),u(t)\big)\big)\,\mathrm{d}t\bigg\}.\end{aligned} \tag{15}$$

Since $\operatorname{Im}(F_k)_x(x_*,u_*) = L^2$, therefore $\lambda_0 \neq 0$ and, without loss of generality, we can put $\lambda_0 = 1$. Then equality (14) can be rewritten as

$$\int_0^\pi \lambda(t)(-\Delta_k)^\beta h(t)\,\mathrm{d}t = \int_0^\pi -V(t)h(t)\,\mathrm{d}t, \quad h \in D\big((-\Delta_k)^\beta\big),$$

[2] Using analogous arguments as in the proof of Proposition 5.2(c) (cf. [21]) one can obtain the bijectivity of the mapping $(F_2)_x(x_*,u_*)$.

where

$$V(t) = (f_0)_x\bigl(t, x_*(t), u_*(t)\bigr) - \lambda(t) f_x\bigl(t, x_*(t), u_*(t)\bigr), \quad t \in (0, \pi) \text{ a.e.}$$

From assumption (C) it follows that $V \in L^2$. Hence and from the fact that the operator $(-\Delta_k)^\beta$ is self-adjoint it follows that $\lambda \in D((-\Delta_k)^\beta)$ and

$$(-\Delta_k)^\beta \lambda(t) = -V(t), \quad t \in (0, \pi) \text{ a.e.}$$

Consequently, condition (12) holds.

In order to prove condition (13), let us observe that condition (15) is equivalent to the following one:

$$\int_0^\pi \bigl(f_0\bigl(t, x_*(t), u_*(t)\bigr) - \lambda(t) f\bigl(t, x_*(t), u_*(t)\bigr)\bigr)\,\mathrm{d}t$$
$$= \min_{u \in \mathcal{U}_M} \left\{ \int_0^\pi \bigl(f_0\bigl(t, x_*(t), u(t)\bigr) - \lambda(t) f\bigl(t, x_*(t), u(t)\bigr)\bigr)\,\mathrm{d}t \right\}.$$

So, (13) follows from Lemma 5.

The proof is completed. □

4 Examples

In this section, we present two theoretical examples, which illustrate obtained maximum principle.

Example 1. Let us consider the following optimal control problem:

$$(-\Delta_1)^\beta x(t) = \frac{1}{2}\sin^2 x(t) + \bigl|u(t)\bigr|, \quad t \in (0, \pi) \text{ a.e.,} \tag{16}$$

$$J(x, u) = \int_0^\pi \bigl(\cos x(t) + 2\bigl|u(t)\bigr|\bigr)\,\mathrm{d}t \to \min, \tag{17}$$

whereby $\beta > 1/4$, $m = n = 1$, $u \in \mathcal{U}_{[-1,1]}$ and

$$f(t, x, u) = \frac{1}{2}\sin^2 x + |u|, \qquad f_0(t, x, u) = \cos x + 2|u|.$$

It is easy to check that all assumptions of Theorem 3 are satisfied. Consequently, if the pair $(x_*, u_*) \in D((-\Delta_1)^\beta) \times \mathcal{U}_{[-1,1]}$ is a locally optimal solution of problem (16)–(17), then there exists a function $\lambda \in D((-\Delta_1)^\beta)$ such that

$$(-\Delta_1)^\beta \lambda(t) = \frac{1}{2}\sin\bigl(2x_*(t)\bigr)\lambda(t) + \sin\bigl(x_*(t)\bigr), \quad t \in (0, \pi) \text{ a.e.,} \tag{18}$$

and

$$2\bigl|u_*(t)\bigr| - \lambda(t)\bigl|u_*(t)\bigr| = \min_{u \in [-1,1]} \bigl\{2|u| - \lambda(t)|u|\bigr\}, \quad t \in (0, \pi) \text{ a.e.} \tag{19}$$

It is clear that the pair $(x_*, u_*) = (0,0)$ can be an optimal solution of (16)–(17) (it satisfies (16), (18) and (19) with $\lambda(t) \equiv 0$). Furthermore, we see that Theorem 1 cannot be applied in this problem (the functions f and f_0 are not differentiable at $u = 0$).

Example 2. Let us consider the following optimal control problem:

$$(-\Delta_2)^{1/2}x(t) = \frac{1}{4}x(t) + u(t), \quad t \in (0,\pi) \text{ a.e.}, \tag{20}$$

$$J(x,u) = \int_0^\pi \left(-\sin\left(\frac{t}{2}\right)x(t) + 4u(t)\right) \mathrm{d}t \to \min, \tag{21}$$

whereby $m = n = 1$ and $u \in \mathcal{U}_{[-1,1]}$. It is easy to check that all assumptions of Theorem 3 are satisfied. Consequently, if the pair $(x_*, u_*) \in D((-\Delta_2)^{1/2}) \times \mathcal{U}_{[-1,1]}$ is a locally optimal solution of problem (20)–(21), then there exists a function $\lambda \in D((-\Delta_2)^{1/2})$ such that

$$(-\Delta_2)^{1/2}\lambda(t) = \frac{1}{4}\lambda(t) + \sin\frac{t}{2}, \quad t \in (0,\pi) \text{ a.e.}, \tag{22}$$

and

$$4u_*(t) - \lambda(t)u_*(t) = \min_{u\in[-1,1]} \bigl\{4u - \lambda(t)u\bigr\}, \quad t \in (0,\pi) \text{ a.e.} \tag{23}$$

Let $\lambda(t) = \sum_{j=1}^\infty b_j\sqrt{2/\pi}\sin(j-1/2)t$. Then (22) can be written as follows:

$$\sum_{j=1}^\infty \left(j-\frac{1}{2}\right)b_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t$$
$$= \frac{1}{4}\sum_{j=1}^\infty b_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t + \sum_{j=1}^\infty c_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t, \quad t \in (0,\pi) \text{ a.e.},$$

whereby $c_j = \int_0^\pi \sin(t/2)\sqrt{2/\pi}\sin(j-1/2)t\,\mathrm{d}t$, $j \in \mathbb{N}$.

Consequently,

$$b_j = \frac{c_j}{j-\frac{3}{4}} = \begin{cases} 4\sqrt{\frac{\pi}{2}}, & j = 1, \\ 0, & j > 1. \end{cases}$$

This means that

$$\lambda(t) = 4\sin\frac{t}{2}, \quad t \in (0,\pi) \text{ a.e.}$$

is a solution of (22). Hence and from (23) we conclude that

$$u_*(t) = -1, \quad t \in (0,\pi) \text{ a.e.}, \tag{24}$$

and

$$(-\Delta_2)^{1/2}x_*(t) = \frac{1}{4}x_*(t) - 1, \quad t \in (0,\pi) \text{ a.e.} \tag{25}$$

Let $x_*(t) = \sum_{j=1}^{\infty} d_j \sqrt{2/\pi} \sin(j-1/2)t$. Then (25) can be written as follows:

$$\sum_{j=1}^{\infty}\left(j-\frac{1}{2}\right)d_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t$$
$$=\frac{1}{4}\sum_{j=1}^{\infty}d_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t-\sum_{j=1}^{\infty}e_j\sqrt{\frac{2}{\pi}}\sin\left(j-\frac{1}{2}\right)t, \quad t\in(0,\pi), \text{ a.e.,}$$

whereby $e_j = \int_0^{\pi} \sqrt{2/\pi} \sin(j-1/2)t \,\mathrm{d}t$, $j \in \mathbb{N}$. Consequently,

$$d_j = \frac{-e_j}{j-\frac{3}{4}} = -\sqrt{\frac{2}{\pi}}\frac{1}{(j-\frac{1}{2})(j-\frac{3}{4})}, \quad j\in\mathbb{N},$$

and therefore

$$x_*(t) = -\frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\sin(j-\frac{1}{2})t}{(j-\frac{1}{2})(j-\frac{3}{4})}, \quad t\in(0,\pi) \text{ a.e.} \tag{26}$$

It means that the pair (x_*, u_*) given by (26) and (24) is the only pair, which can be a locally optimal solution of problem (20)–(21). Moreover, the minimal value of the cost functional J is equal to

$$J(x_*,u_*) = \int_0^{\pi}\left(-\sin\left(\frac{t}{2}\right)x_*(t)+4u_*(t)\right)\mathrm{d}t$$
$$=\frac{2}{\pi}\int_0^{\pi}\sin\left(\frac{t}{2}\right)\sum_{j=1}^{\infty}\frac{\sin(j-\frac{1}{2})t}{(j-\frac{1}{2})(j-\frac{3}{4})}\,\mathrm{d}t-4\pi = 8-4\pi.$$

5 Conclusions

In the paper, we investigated the Lagrange problems containing nonlinear control systems with Dirichlet and Dirichlet–Neumann Laplace operators of fractional orders. The main result, obtained in this work, is the Pontryagin maximum principle for such problems (Theorem 3). Obtained optimality conditions consist of the adjoint system (12) with Dirichlet and Dirichlet–Neumann boundary conditions, respectively, and the minimum condition (13). We derived our result using the smooth-convex extremum principle due to Ioffe and Tikchomirov. The result of such a type for the problem with Dirichlet boundary conditions was proved in [22, Thm. 4] via Dubovitskii–Milyutin method. Two illustrative examples were presented. In particular, we observed that [22, Thm. 4] cannot be used to problem (16)–(17).

The aim of a forthcoming work is studying of the sufficient optimality conditions for problems (OCP_1) and (OCP_2).

Appendix: Basics of self-adjoint operators in a Hilbert space

In this section, we give the necessary notions and facts from the theory of unbounded self-adjoint operators in a real Hilbert space (cf. [21,22]). More details can be found in [2,27], where all results are obtained in the case of a complex Hilbert space. Nevertheless, their proofs can be reproduced (if required, with small changes) in the case of a real Hilbert space.

So, in this section, we shall assume that H is a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_H$.

A.1 Self-adjoint operator

Let $T : D(T) \subset H \to H$ be a densely defined linear operator ($\overline{D(T)} = H$) with the domain $D(T)$. We define

$$D(T^*) := \bigl\{x \in H\colon \exists_{z\in H}\ \langle x, Ty\rangle_H = \langle z, y\rangle_H \text{ for all } y \in D(T)\bigr\}.$$

For $x \in D(T^*)$, we denote $T^*x = z$ (this element is uniquely determined due to the density of $D(T)$). The operator $T^* : D(T^*) \subset H \to H$ is called the adjoint operator to T. If $T = T^*$, then T is called self-adjoint. We note that whenever T is self-adjoint operator one has

$$\langle Tx, y\rangle_H = \langle x, Ty\rangle_H, \quad x, y \in D(T).$$

A.2 Spectral integral and decomposition theorem

Let $\mathcal{B}$ be a σ-algebra of Borel subsets of $\mathbb{R}$, and $\mathcal{P}(H)$ denotes the set of all orthogonal projection operators onto closed linear subspaces of H. A set function $E : \mathcal{B} \to \mathcal{P}(H)$ is called a spectral measure (or a decomposition of the identity) if

(i) for any $x \in H$, the set function $\mathcal{B} \ni P \to E(P)x$ is σ-additive;
(ii) $E(\mathbb{R}) = I$ (here I denotes the identity operator on H);
(iii) $E(P \cap Q) = E(P) \circ E(Q)$ for $P, Q \in \mathcal{B}$.

Let W be the union of all open sets $V \subset \mathbb{R}$ such that $E(V) = 0$. Then the complement $\mathbb{R} \setminus W$ is called a support of a spectral measure E and denoted by $\operatorname{supp}(E)$.

Let us assume that $u : \mathbb{R} \to \mathbb{R}$ defined E-a.e. is a bounded Borel measurable function. Then, in the usual way (via a sequence of simple functions), one can show that for any $x \in H$, there exists the integral (with respect to the vector measure $E(\cdot)x$) $\int_{-\infty}^{+\infty} u(\lambda) \times E(\mathrm{d}\lambda)x$. We define the integral with respect to the spectral measure E $\int_{-\infty}^{+\infty} u(\lambda)\, E(\mathrm{d}\lambda) : H \to H$ in the following way:

$$\left(\int_{-\infty}^{+\infty} u(\lambda)\, E(\mathrm{d}\lambda)\right)x = \int_{-\infty}^{+\infty} u(\lambda)\, E(\mathrm{d}\lambda)x.$$

One proves that the above operator is linear, continuous and symmetric.

Now, let $u:\mathbb{R}\to\mathbb{R}$ defined E-a.e. be an unbounded Borel measurable function. Let us define the sequence of functions u_n:

$$u_n(\lambda)=\begin{cases}u(\lambda) & \text{if } |u(\lambda)|\leqslant n,\\ 0 & \text{if } |u(\lambda)|>n.\end{cases}$$

Functions u_n are Borel measurable and bounded. Consequently, there exist integrals $\int_{-\infty}^{+\infty} f_n(\lambda)\,E(\mathrm{d}\lambda)$, $n\in\mathbb{N}$. Let us consider the set

$$D=\left\{x\in H\colon \int_{-\infty}^{+\infty}\left|u(\lambda)\right|^2\left\|E(\mathrm{d}\lambda)x\right\|_H^2<\infty\right\}. \tag{A.1}$$

One can show that D is a dense linear subspace of H and for $x\in D$, there exists the limit $\lim_{n\to\infty}\int_{-\infty}^{+\infty}u_n(\lambda)\,E(\mathrm{d}\lambda)x$. So, we can define the operator $\int_{-\infty}^{+\infty}u(\lambda)\,E(\mathrm{d}\lambda)\colon D\subset H\to H$ in the following way:

$$\left(\int_{-\infty}^{+\infty}u(\lambda)\,E(\mathrm{d}\lambda)\right)x=\lim_{n\to\infty}\int_{-\infty}^{+\infty}u_n(\lambda)\,E(\mathrm{d}\lambda)x.$$

One can prove that

$$\left\|\int_{-\infty}^{+\infty}u(\lambda)\,E(\mathrm{d}\lambda)x\right\|_H^2=\int_{-\infty}^{+\infty}\left|u(\lambda)\right|^2\left\|E(\mathrm{d}\lambda)x\right\|_H^2 \tag{A.2}$$

and the operator $\int_{-\infty}^{+\infty}u(\lambda)E(\mathrm{d}\lambda)$ is self-adjoint.

If $u:\mathbb{R}\to\mathbb{R}$ is a Borel measurable function and $\omega\in\mathcal{B}$, then

$$\int_\omega u(\lambda)\,E(\mathrm{d}\lambda):=\int_{-\infty}^{+\infty}\chi_\omega(\lambda)u(\lambda)\,E(\mathrm{d}\lambda),$$

where χ_ω is a characteristic function of the set ω.

In order to define the spectral integral in the case of a Borel measurable function $u:P\to\mathbb{R}$, where $P\in\mathcal{B}$ contains the support $\operatorname{supp}(E)$, it is sufficient to extend u on $\mathbb{R}$ to any Borel measurable function.

Now, we formulate a spectral decomposition theorem, which plays a crucial role in the spectral theory of self-adjoint operators.

Theorem A1 [Spectral decomposition theorem for self-adjoint operators]. *Let $T: D(T)\subset H\to H$ be a self-adjoint operator such that the resolvent set $\rho(T)$ is nonempty. Then there exists a unique spectral measure E with the closed support* $\operatorname{supp}(E)=\sigma(T)$ *such that*

$$T=\int_{-\infty}^{+\infty}\lambda\,E(\mathrm{d}\lambda)=\int_{\sigma(T)}\lambda\,E(\mathrm{d}\lambda). \tag{A.3}$$

In conclusion of this section, we shall define a function of a self-adjoint operator. Let $T : D(T) \subset H \to H$ be a self-adjoint operator with $\rho(T) \neq \emptyset$. From Theorem A1 it follows that T has the integral representation given by (A.3). For a Borel measurable function $u : \mathbb{R} \to \mathbb{R}$ defined E-a.e., we define the operator $u(T)$ as follows:

$$u(T) = \int_{-\infty}^{+\infty} u(\lambda)\, E(\mathrm{d}\lambda) = \int_{\sigma(T)} u(\lambda)\, E(\mathrm{d}\lambda).$$

According to general properties of the spectral integrals presented above, the domain $D(u(T))$ is given by (A.1), equality (A.2) holds and $u(T)$ is self-adjoint. Moreover, its spectrum is given by

$$\sigma\bigl(u(T)\bigr) = \overline{u\bigl(\sigma(T)\bigr)},$$

provided that u is continuous on $\sigma(T)$.

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