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Uniqueness of iterative positive solutions for the singular infinite-point p-Laplacian fractional differential system via sequential technique*

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Abstract. By sequential techniques and mixed monotone operator, the uniqueness of positive solution for singular p-Laplacian fractional differential system with infinite-point boundary conditions is obtained. Green's function is derived, and some useful properties of Green' function are obtained. Based on these new properties, the existence of unique positive solutions is established, moreover, an iterative sequence and a convergence rate are given, which are important for practical application, and an example is given to demonstrate the validity of our main results.

Keywords: fractional differential system, iterative positive solution, sequential techniques, mixed monotone operator, singular problem.

1 Introduction

Boundary value problems for nonlinear fractional differential equations arise from the studies of complex problems in many disciplinary areas such as aerodynamics, fluid flows, electrodynamics of complex medium, electrical networks, rheology, polymer rheology, economics, biology chemical physics, control theory, signal and image processing, blood flow phenomena, and so on. There has been a significant development in the study of fractional differential equations in recent years. For an extensive collection of such literature, readers can refer to [1,3,4,7,11–17] and the references therein. Multi-point boundary

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value problems are a significant development for fractional differential equation, and the system in this paper is infinite-points boundary value problem, and about values at infinite-points are involved in the boundary conditions that we refer the reader to [3, 4, 11] and the references therein. For p-Laplacian fractional differential equation, we refer the reader to [9, 12]. In this paper, we consider the following singular infinite-point p-Laplacian nonlinear fractional differential equation system:

$$\begin{aligned}
&D_{0+}^{\alpha}\big(\varphi_{p_1}\big(D_{0+}^{\gamma}u\big)\big)(t)+\lambda^{1/(q_1-1)}f\big(t,u(t),D_{0+}^{\mu_1}u(t),D_{0+}^{\mu_2}u(t),\dots,\\
&\quad D_{0+}^{\mu_{n-2}}u(t),v(t)\big)=0,\quad 0<t<1,\\
&D_{0+}^{\beta}\big(\varphi_{p_2}\big(D_{0+}^{\delta}v\big)\big)(t)+\mu^{1/(q_2-1)}g\big(t,u(t),D_{0+}^{\eta_1}u(t),D_{0+}^{\eta_2}u(t),\dots,\\
&\quad D_{0+}^{\eta_{m-2}}u(t)\big)=0,\quad 0<t<1,\\
&u(0)=D_{0+}^{\mu_i}u(0)=0,\quad D_{0+}^{\gamma}u(0)=0,\quad i=1,2,\dots,n-2,\\
&D_{0+}^{\mu_{n-2}+r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0+}^{\mu_{n-2}+r_2}u(\xi_j),\\
&v(0)=D_{0+}^{\eta_i}v(0)=0,\quad D_{0+}^{\delta}v(0)=0,\quad i=1,2,\dots,m-2,\\
&D_{0+}^{\eta_{m-2}+\overline{r}_1}v(1)=\sum_{j=1}^{\infty}\overline{\eta}_j D_{0+}^{\eta_{m-2}+\overline{r}_2}v(\overline{\xi}_j),
\end{aligned}\tag{1}$$

where $\alpha,\beta,\gamma,\delta,\mu_\kappa,\eta_\varrho\in\mathbb{R}_+^1$ ($\kappa=1,2,\dots,n-2$; $\varrho=1,2,\dots,m-2$), $n,m\in\mathbb{N}$ (natural number set), $n,m\geqslant 2$, and $1/2<\alpha,\beta\leqslant 1$, $n-1<\gamma\leqslant n$, $m-1<\delta\leqslant m$, $p_1,\overline{p}_1\in[2,n-2]$. p-Laplacian operator φ_{p_i} is defined as $\varphi_{p_i}(s)=|s|^{p_i-2}s$, $p_i,q_i>1$, $1/p_i+1/q_i=1$ ($i=1,2$), $n-1-\kappa<\gamma-\mu_\kappa\leqslant n-\kappa$, $m-1-\varrho<\delta-\eta_\varrho\leqslant m-\varrho$ ($\kappa=1,2,\dots,n-2$; $\varrho=1,2,\dots,m-2$), $\eta_i\leqslant\mu_{n-2}$ ($i=1,2,\dots,m-2$), and $0<\eta,\vartheta\leqslant 1$. $\lambda,\mu,\chi,\iota>0$ are parameters, $f\in C((0,1)\times(0,+\infty)^n,\mathbb{R}_+^1)$ and $f(t,x_1,x_2,\dots,x_n)$ has singularity at $x_i=0$ ($i=1,2,\dots,n$) and $t=0,1$, $g\in C((0,1)\times\mathbb{R}_+^{m-1},\mathbb{R}_+^1)$, $h,a\in C(0,1)$ with $\int_0^\eta \chi t^{\gamma-\mu_{n-2}-1}h(t)\,\mathrm{d}A(t)<1$, $\int_0^\vartheta \iota t^{\delta-\eta_{m-2}-1}a(t)\,\mathrm{d}B(t)<1$. A, B are functions of bounded variation, $\int_0^\eta h(t)D_{0+}^{\mu_{n-2}}u(t)\,\mathrm{d}A(t)$, $\int_0^\vartheta a(t)D_{0+}^{\eta_{m-2}}v(t)\,\mathrm{d}B(t)$ denote the Riemann–Stieltjes integral with respect to A and B. $D_{0+}^{\alpha}u$, $D_{0+}^{\beta}v$, $D_{0+}^{\gamma}u$, $D_{0+}^{\delta}v$, $D_{0+}^{\mu_\kappa}u$, $D_{0+}^{\eta_\varrho}u$ are the standard Riemann–Liouville derivative.

Motivated by the results above, we utilize fixed point theorem to investigate the existence results of positive solution of BVP (1). Compared with our paper [6], $p_1\neq p_2$ in p-Laplacian system of (1), but $p_1=p_2$ in p-Laplacian system in [6]. Compared with our papers [5, 6], values at infinite points are involved in the boundary conditions of the boundary value problem (1), and the method that we used in this paper is sequential techniques. Compared with [3,4,11], the method, which we used is sequential techniques, and the positive solution, which we obtained, is iterative solution. Compared with [16,17], fractional derivatives are involved in the nonlinear terms, and the solution we obtained is iterative solution, the result is accurate.

Next we list the following assumptions for convenience.

(S1) $f(t,x_1,x_2,\dots,x_n)=\phi(t,x_1,x_2,\dots,x_n)+\psi(t,x_1,x_2,\dots,x_n)$, where $\phi:(0,1)\times[0,+\infty)^n\to\mathbb{R}_+^1$ and $\psi:(0,1)\times(0,+\infty)^n\to\mathbb{R}_+^1$ are continuous, and for any fixed $t \in [0,1]$, $\phi(t, x_1, x_2, \dots, x_n)$ is nondecreasing and $\psi(t, x_1, x_2, \dots, x_n)$ is nonincreasing on $x_i > 0$ $(i = 1, 2, \dots, n)$, respectively.

(S2) There exists $\sigma \in (0,1)$ such that, for $x_i > 0$ $(i = 1, 2, \dots, n)$ and for any $l \in (0,1)$, $t \in (0,1)$,

$$\phi(t, lx_1, lx_2, \dots, lx_n) \geqslant l^{\sigma^{1/(q_1-1)}} \phi(t, x_1, x_2, \dots, x_n),$$
$$\psi\bigl(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_n\bigr) \geqslant l^{\sigma^{1/(q_1-1)}} \psi(t, x_1, x_2, \dots, x_n).$$

(S3) $g \in C([0,1] \times \mathbb{R}_+^{m-1}, \mathbb{R}_+^1)$ is nondecreasing on $x_i > 0$ $(i = 1, 2, \dots, m-1)$, and $g(t, 1, \dots, 1) \neq 0$. Moreover, for $t \in (0,1)$, there exists $\varsigma \in [0,1]$ such that

$$g(t, lx_1, lx_2, \dots, lx_{m-1}) \geqslant l^{\varsigma^{1/(q_2-1)}} g(t, x_1, x_2, \dots, x_{m-1}), \quad l \in (0,1).$$

(S4) $0 < \int_0^1 \phi^2(\tau, 1, 1, \dots, 1)\,\mathrm{d}\tau < +\infty$, $0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q_1-1)}} \psi^2(\tau, 1, 1, \dots, 1)\,\mathrm{d}\tau < +\infty$, $0 < \int_0^1 g^2\bigl(\tau, 1, 1, \dots, 1\bigr)\,\mathrm{d}\tau < +\infty$.

Remark 1. According to (S2) and (S3), for all $x_i > 0$ $(i = 1, 2, \dots, n)$, $t \in (0,1)$, $l \geqslant 1$, we have

$$g(t, lx_1, lx_2, \dots, lx_{m-1}) \leqslant l^{\varsigma^{1/(q_2-1)}} g(t, x_1, x_2, \dots, x_{m-1}),$$
$$\phi(t, lx_1, lx_2, \dots, lx_n) \leqslant l^{\sigma^{1/(q_1-1)}} \phi(t, x_1, x_2, \dots, x_n),$$
$$\psi\bigl(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_n\bigr) \leqslant l^{\sigma^{1/(q_1-1)}} \psi(t, x_1, x_2, \dots, x_n).$$

2 Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas, which are useful for the following research and can be found in the recent literature such as [8, 10].

In what follows, we will give the expression of the linear problems.

Lemma 1. *Let $y, \overline{y} \in L^1(0,1) \cap C(0,1)$, then the equation of the BVPs*

$$\begin{aligned} &-D_{0+}^{\gamma-\mu_{n-2}} u(t) = y(t), \quad 0 < t < 1, \\ &u(0) = 0, \qquad D_{0+}^{r_1} u(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2} u(\xi_j), \\ &-D_{0+}^{\delta-\eta_{m-2}} v(t) = \overline{y}(t), \quad 0 < t < 1, \\ &v(0) = 0, \qquad D_{0+}^{\overline{r}_1} v(1) = \sum_{j=1}^{\infty} \overline{\eta}_j D_{0+}^{\overline{r}_2} v(\overline{\xi}_j) \end{aligned} \tag{2}$$

has integral representation

$$u(t) = \int_0^1 G(t,s) y(s)\,\mathrm{d}s, \qquad v(t) = \int_0^1 H(t,s) \overline{y}(s)\,\mathrm{d}s, \tag{3}$$

respectively, where

$$G(t,s) = \frac{1}{\varDelta\Gamma(\gamma-\mu_{n-2})} \times \begin{cases} \Gamma(\gamma-\mu_{n-2})t^{\gamma-\mu_{n-2}-1}P(s)(1-s)^{\gamma-\mu_{n-2}-r_1-1} \\ \quad -\varDelta(t-s)^{\gamma-\mu_{n-2}-1}, & 0\leqslant s\leqslant t\leqslant 1, \\ \Gamma(\gamma-\mu_{n-2})t^{\gamma-\mu_{n-2}-1}P(s)(1-s)^{\gamma-\mu_{n-2}-r_1-1}, & 0\leqslant t\leqslant s\leqslant 1, \end{cases} \tag{4}$$

$$H(t,s) = \frac{1}{\overline{\varDelta}\Gamma(\delta-\eta_{m-2})} \times \begin{cases} \Gamma(\delta-\eta_{m-2})t^{\delta-\eta_{m-2}-1}\overline{P}(s)(1-s)^{\delta-\eta_{m-2}-\overline{r}_1-1} \\ \quad -\overline{\varDelta}(t-s)^{\delta-\eta_{m-2}-1}, & 0\leqslant s\leqslant t\leqslant 1, \\ \Gamma(\delta-\eta_{m-2})t^{\delta-\eta_{m-2}-1}\overline{P}(s)(1-s)^{\delta-\eta_{m-2}-\overline{r}_1-1}, & 0\leqslant t\leqslant s\leqslant 1, \end{cases} \tag{5}$$

in which

$$\begin{aligned} P(s) &= \frac{1}{\Gamma(\gamma-\mu_{n-2}-r_1)} \\ &\quad - \frac{1}{\Gamma(\gamma-\mu_{n-2}-r_2)} \sum_{s\leqslant\xi_j} \eta_j \left(\frac{\xi_j-s}{1-s}\right)^{\gamma-\mu_{n-2}-r_2-1} (1-s)^{r_1-r_2}, \\ \overline{P}(s) &= \frac{1}{\Gamma(\delta-\eta_{m-2}-\overline{r}_1)} \\ &\quad - \frac{1}{\Gamma(\delta-\eta_{m-2}-\overline{r}_2)} \sum_{s\leqslant\overline{\xi}_j} \overline{\eta}_j \left(\frac{\overline{\xi}_j-s}{1-s}\right)^{\delta-\eta_{m-2}-\overline{r}_1-1} (1-s)^{\overline{r}_1-\overline{r}_2}, \\ \varDelta &= \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_1)} - \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_2)} \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-\mu_{n-2}-r_2-1}, \\ \overline{\varDelta} &= \frac{\Gamma(\delta-\eta_{m-2})}{\Gamma(\delta-\eta_{m-2}-\overline{r}_1)} - \frac{\Gamma(\delta-\eta_{m-2})}{\Gamma(\delta-\eta_{m-2}-\overline{r}_2)} \sum_{j=1}^{\infty} \overline{\eta}_j \overline{\xi}_j^{\delta-\eta_{m-2}-\overline{r}_2-1}. \end{aligned}$$

Proof. We only need to prove (4), the proof of (5) is similar with the proof of (4). By means of the definition of fractional differential integral, we can reduce (2) to an equivalent integral equation

$$u(t) = -I_{0^+}^{\gamma-\mu_{n-2}} y(t) + C_1 t^{\gamma-\mu_{n-2}-1} + C_2 t^{\gamma-\mu_{n-2}-2}$$

for $C_1 C_2 \in \mathbb{R}$. From $u(0) = 0$ we have $C_2 = 0$. Consequently, we get

$$u(t) = C_1 t^{\gamma-\mu_{n-2}-1} - I_{0^+}^{\gamma-\mu_{n-2}} y(t).$$

By some properties of the fractional integrals and fractional derivatives, we have

$$
\begin{aligned}
D_{0^+}^{r_1}u(t) &= C_1\frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_1)}t^{\gamma-\mu_{n-2}-r_1-1} - I_{0^+}^{\gamma-\mu_{n-2}-r_1}y(t),\\
D_{0^+}^{r_2}u(t) &= C_1\frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_2)}t^{\gamma-\mu_{n-2}-r_2-1} - I_{0^+}^{\gamma-\mu_{n-2}-r_2}y(t).
\end{aligned} \tag{6}
$$

On the other hand, combining $D_{0^+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0^+}^{r_2}u(\xi_j)$ with (6), we get

$$
\begin{aligned}
C_1 &= \int_0^1 \frac{(1-s)^{\gamma-\mu_{n-2}-r_1-1}}{\Gamma(\gamma-\mu_{n-2}-r_1)\Delta}y(s)\,\mathrm{d}s - \sum_{j=1}^{\infty}\eta_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\gamma-\mu_{n-2}-r_2-1}}{\Gamma(\gamma-\mu_{n-2}-r_2)\Delta}y(s)\,\mathrm{d}s\\
&= \int_0^1 \frac{(1-s)^{\gamma-\mu_{n-2}-r_1-1}P(s)}{\Delta}y(s)\,\mathrm{d}s,
\end{aligned}
$$

where $P(s), \Delta$ are as (4). Hence,

$$
\begin{aligned}
u(t) &= C_1 t^{\gamma-\mu_{n-2}-1} - I_{0^+}^{\gamma-\mu_{n-2}}y(t)\\
&= -\int_0^t \frac{\Delta(t-s)^{\gamma-\mu_{n-2}-1}}{\Gamma(\gamma-\mu_{n-2})\Delta}y(s)\,\mathrm{d}s + \int_0^1 \frac{(1-s)^{\gamma-\mu_{n-2}-r_1-1}t^{\gamma-\mu_{n-2}-1}P(s)}{\Delta}y(s)\,\mathrm{d}s\\
&= \int_0^1 G(t,s)y(s)\,\mathrm{d}s.
\end{aligned}
$$

Therefore, (4) holds, similarly, (5) holds. □

Lemma 2. *Let* $\Delta, \overline{\Delta} > 0$ *for* $s\in[0,1]$, *then the Green functions defined by* (3) *satisfies*:

(i) $G, H : [0,1]\times[0,1]\to\mathbb{R}_+$ *are continuous, and* $G(t,s), H(t,s) > 0$ *for all* $t,s\in(0,1)$;

(ii)
$$
\frac{1}{\Gamma(\gamma-\mu_{n-2})}t^{\gamma-\mu_{n-2}-1}j(s) \leqslant G(t,s) \leqslant a^* t^{\gamma-\mu_{n-2}-1}, \tag{7}
$$
$$
\frac{1}{\Gamma(\delta-\eta_{m-2})}t^{\delta-\eta_{m-2}-1}\overline{j}(s) \leqslant H(t,s) \leqslant \overline{a}^* t^{\delta-\eta_{m-2}-1}, \tag{8}
$$

where

$$
\begin{aligned}
j(s) &= (1-s)^{\gamma-\mu_{n-2}-r_1-1}\bigl[1-(1-s)^{r_1}\bigr],\\
\overline{j}(s) &= (1-s)^{\delta-\eta_{m-2}-\overline{r}_1-1}\bigl[1-(1-s)^{\overline{r}_1}\bigr],\\
a^* &= \frac{1}{\Delta\Gamma(\gamma-\mu_{n-2}-r_1)},\qquad \overline{a}^* = \frac{1}{\overline{\Delta}\Gamma(\delta-\eta_{m-2}-\overline{r}_1)},
\end{aligned}
$$

Δ, $\overline{\Delta}$ *are defined as in Lemma* 1.

Proof. Let

$$G_0(t,s) = \frac{1}{\Gamma(\gamma-\mu_{n-2})} \times \begin{cases} t^{\gamma-\mu_{n-2}-1}(1-s)^{\gamma-\mu_{n-2}-r_1-1} - (t-s)^{\gamma-\mu_{n-2}-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\gamma-\mu_{n-2}-1}(1-s)^{\gamma-\mu_{n-2}-r_1-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

From [7], for $r_1 \in [2, n-2]$, we have

$$\begin{aligned} 0 &\leqslant t^{\gamma-\mu_{n-2}-1}(1-s)^{\gamma-\mu_{n-2}-r_1-1}\bigl[1-(1-s)^{r_1}\bigr] \\ &\leqslant \Gamma(\gamma-\mu_{n-2})G_0(t,s) \leqslant t^{\gamma-\mu_{n-2}-1}(1-s)^{\gamma-\mu_{n-2}-r_1-1}. \end{aligned} \tag{9}$$

By direct calculation, we get $P'(s) \geqslant 0$, $s \in [0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $r_2 \leqslant r_1$, $r_1, r_2 \in [2, n-2]$, $s \in [0,1]$, we get

$$\begin{aligned} \Gamma(\gamma-\mu_{n-2})P(s) &= \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_1)} \\ &\quad - \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_2)} \sum_{s\leqslant\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-\mu_{n-2}-r_2-1} (1-s)^{r_1-r_2} \\ &\geqslant \Gamma(\gamma-\mu_{n-2})P(0) \\ &= \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_1)} - \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-2}-r_2)} \sum \eta_j \xi_j^{\gamma-\mu_{n-2}-r_2-1} \\ &= \Delta. \end{aligned}$$

By (4) and (9), we have

$$\begin{aligned} \Delta\Gamma(\gamma-\mu_{n-2})G(t,s) &\geqslant \Delta\Gamma(\gamma-\mu_{n-2})G_0(t,s) \\ &\geqslant \Delta t^{\gamma-\mu_{n-2}-1}(1-s)^{\gamma-\mu_{n-2}-r_1-1}\bigl[1-(1-s)^{r_1}\bigr]. \end{aligned}$$

Clearly, $\Delta\Gamma(\gamma-\mu_{n-2})G(t,s) \leqslant \Gamma(\gamma-\mu_{n-2})t^{\gamma-\mu_{n-2}-1}P(s)(1-s)^{\gamma-\mu_{n-2}-r_1-1}$. So the proof of (7) is completed. Similarly, (8) also holds. □

To study the PFDE (1), in what follows, we consider the associated linear PFDE

$$\begin{aligned} &D_{0+}^{\alpha}\bigl(\varphi_{p_1}\bigl(D_{0+}^{\gamma-\mu_{n-2}}x\bigr)\bigr)(t) + \rho(t) = 0, \quad 0 < t < 1, \\ &x(0) = 0, \qquad D_{0+}^{\gamma}x(0) = 0, \qquad D_{0+}^{r_1}x(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2}x(\xi_j), \end{aligned} \tag{10}$$

and

$$\begin{aligned} &D_{0+}^{\beta}\bigl(\varphi_{p_2}\bigl(D_{0+}^{\delta-\eta_{m-2}}y\bigr)\bigr)(t) + \overline{\rho}(t) = 0, \quad 0 < t < 1, \\ &y(0) = 0, \qquad D_{0+}^{\delta}y(0) = 0, \qquad D_{0+}^{\overline{r}_1}y(1) = \sum_{j=1}^{\infty} \overline{\eta}_j D_{0+}^{\overline{r}_2}y(\overline{\xi}_j). \end{aligned} \tag{11}$$

Lemma 3. *The PFDE* (10), (11) *has the following unique positive solution*:

$$x(t)=\int_0^1 G(t,s)\left(\int_0^s \overline{a}(s-\tau)^{\alpha-1}\rho(\tau)\,\mathrm{d}\tau\right)^{q_1-1}\mathrm{d}s,\quad t\in[0,1], \tag{12}$$

$$y(t)=\int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-\tau)^{\beta-1}\overline{\rho}(\tau)\,\mathrm{d}\tau\right)^{q_2-1}\mathrm{d}s,\quad t\in[0,1], \tag{13}$$

respectively, in which $\overline{a}=1/\Gamma(\alpha)$, $\overline{b}=1/\Gamma(\beta)$.

Proof. Let $h=D_{0+}^{\gamma-\mu_{n-2}}x, k=\varphi_{p_1}(h)$, then the solution of the initial value problem

$$D_{0+}^{\alpha}k(t)+\rho(t)=0,\quad 0<t<1,\qquad k(0)=0$$

is given by $k(t)=C_1t^{\alpha-1}-I_{0+}^{\alpha}\rho(t)$, $t\in[0,1]$. By the relations $k(0)=0$, we have $C_1=0$, and hence

$$k(t)=-I_{0+}^{\alpha}\rho(t),\quad t\in[0,1]. \tag{14}$$

By $D_{0+}^{\gamma}x=h, h=\varphi_{p_1}^{-1}(k)$, we have from (14) that the solution of (10) satisfies

$$\begin{aligned}&D_{0+}^{\gamma-\mu_{n-2}}x(t)=\varphi_{p_1}^{-1}\bigl(-I_{0+}^{\alpha}\rho(t)\bigr),\quad 0<t<1,\\&x(0)=0,\qquad D_{0+}^{r_1}x(1)=\sum_{j=1}^{\infty}\eta_jD_{0+}^{r_2}x(\xi_j).\end{aligned} \tag{15}$$

By (3), the solution of Eq. (15) can be written as

$$x(t)=-\int_0^1 G(t,s)\varphi_{p_1}^{-1}\bigl(-I_{0+}^{\alpha}\rho(s)\bigr)\,\mathrm{d}s,\quad t\in[0,1].$$

Since $\rho(s)\geqslant 0, s\in[0,1]$, we have $\varphi_{p_1}^{-1}(-I_{0+}^{\alpha}\rho(s))=-(I_{0+}^{\alpha}\rho(s))^{q_1-1}$, $s\in[0,1]$, which implies that the solution of Eq.(10) is (12). Similarly, the solution of Eq.(11) is (13). □

Let $u(t)=I_{0+}^{\mu_{n-2}}x(t)$, $v(t)=I_{0+}^{\eta_{m-2}}y(t)$, problem (1) can turn into the following modified problem of the PFDE (16):

$$\begin{aligned}&D_{0+}^{\alpha}\bigl(\varphi_{p_1}\bigl(D_{0+}^{\gamma-\mu_{n-2}}x\bigr)\bigr)(t)+\lambda^{1/(q_1-1)}f\bigl(t,I_{0+}^{\mu_{n-2}}x(t),I_{0+}^{\mu_{n-2}-\mu_1}x(t),\dots,\\&\quad I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(t),x(t),I_{0+}^{\eta_{m-2}}y(t)\bigr)=0,\quad 0<t<1,\\&D_{0+}^{\beta}\bigl(\varphi_{p_2}\bigl(D_{0+}^{\delta-\eta_{m-2}}y\bigr)\bigr)(t)+\mu^{1/(q_2-1)}g\bigl(t,I_{0+}^{\mu_{n-2}}x(t),I_{0+}^{\mu_{n-2}-\eta_1}x(t),\dots,\\&\quad I_{0+}^{\mu_{n-2}-\eta_{m-2}}x(t)\bigr)=0,\quad 0<t<1,\\&x(0)=0,\quad D_{0+}^{\gamma}x(0)=0,\qquad D_{0+}^{r_1}x(1)=\sum_{j=1}^{\infty}\eta_jD_{0+}^{r_2}x(\xi_j),\\&y(0)=0,\quad D_{0+}^{\delta}y(0)=0,\qquad D_{0+}^{\overline{r}_1}y(1)=\sum_{j=1}^{\infty}\overline{\eta}_jD_{0+}^{\overline{r}_2}y(\overline{\xi}_j).\end{aligned} \tag{16}$$

Lemma 4. *Let* $u(t) = I_{0^+}^{\mu_{n-2}} x(t), v(t) = I_{0^+}^{\eta_{m-2}} y(t), x(t), y(t) \in C[0,1]$. *Then* (1) *can be transformed into* (16)*. Moreover, if* $(x,y) \in C[0,1] \times C[0,1]$ *is a positive solution of problem* (16)*, then* $(I_{0^+}^{\mu_{n-2}} x, I_{0^+}^{\eta_{m-2}} y)$ *is a positive solution of problem* (1)*.*

Proof. The proof is similar with Lemma 2.5 in [5], we omit it here. □

In order to establish the existence of positive solution for system (1), we shall consider the following problem:

$$\begin{aligned}
&D_{0+}^{\alpha}\big(\varphi_{p_1}\big(D_{0+}^{\gamma-\mu_{n-2}}x\big)\big)(t)+\lambda^{1/(q_1-1)}f\bigg(t, I_{0^+}^{\mu_{n-2}}x(t)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(t)+\frac{1}{k},\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(t)+\frac{1}{k}, x(t)+\frac{1}{k}, I_{0^+}^{\eta_{m-2}}y(t)+\frac{1}{k}\bigg)=0, \quad 0<t<1,\\
&D_{0+}^{\beta}\big(\varphi_{p_2}\big(D_{0+}^{\delta-\eta_{m-2}}y\big)\big)(t)+\mu^{1/(q_2-1)}g\big(t, I_{0^+}^{\mu_{n-2}}x(t), I_{0^+}^{\mu_{n-2}-\eta_1}x(t),\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(t)\big)=0, \quad 0<t<1,\\
&x(0)=0, \quad D_{0+}^{\gamma}x(0)=0, \qquad D_{0+}^{r_1}x(1)=\sum_{j=1}^{\infty}\eta_j D_{0+}^{r_2}x(\xi_j),\\
&y(0)=0, \quad D_{0+}^{\delta}y(0)=0, \qquad D_{0+}^{\overline{r}_1}y(1)=\sum_{j=1}^{\infty}\overline{\eta}_j D_{0+}^{\overline{r}_2}y(\overline{\xi}_j),
\end{aligned} \tag{17}$$

where $t \in (0,1), k \in \{2,3,\dots\}$. Assume that $f : [0,1] \times (\mathbb{R}^1 \setminus \{0\})^n \to \mathbb{R}_+^1$ is continuous, then (x,y) is a solution of system (17) if and only if $(x,y) \in C[0,1] \times C[0,1]$ is a solution of the following nonlinear integral equation system (18):

$$\begin{aligned}
x(t)&=\lambda\int_0^1 G(t,s)\Bigg(\int_0^s \overline{a}(s-\tau)^{\alpha-1}f\bigg(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots,\\
&\qquad I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k}, x(\tau)+\frac{1}{k}, I_{0^+}^{\eta_{m-2}}y(\tau)+\frac{1}{k}\bigg)\,\mathrm{d}\tau\Bigg)^{q_1-1}\mathrm{d}s, \quad t\in[0,1],\\
y(\tau)&=\mu\int_0^1 H(\tau,s)\Bigg(\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w, I_{0^+}^{\mu_{n-2}}x(w), I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,\\
&\qquad I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\big)\,\mathrm{d}w\Bigg)^{q_2-1}\mathrm{d}s, \quad t\in[0,1].
\end{aligned} \tag{18}$$

Easily, we get the following integral equation:

$$\begin{aligned}
x(t)=\lambda\int_0^1 G(t,s)\Bigg(\int_0^s \overline{a}(s-\tau)^{\alpha-1}f\bigg(&\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots,\\
&I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k}, x(\tau)+\frac{1}{k},
\end{aligned}$$

$$I_{0^+}^{\eta_{m-2}}\left[\mu\int_0^1 H(\tau,s)\left(\int_0^s \bar{b}(s-w)^{\beta-1} g\big(w,\dots,I_{0^+}^{\mu_{n-2}}x(w),I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\right.\right.$$

$$\left.\left.I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\big)\,\mathrm{d}w\right)^{q_2-1}\mathrm{d}s\right]+\frac{1}{k}\Bigg)\,\mathrm{d}\tau\Bigg)^{q_1-1}\mathrm{d}s,\quad t\in[0,1]. \tag{19}$$

Let P be a normal cone of a Banach space E, and $e\in P$ with $\|e\|\leqslant 1$, $e\neq\theta$ (θ is a zero element of E). Define $Q_e=\{u\in P$: there exist constants $c,C>0$ such that $ce\leqslant u\leqslant Ce\}$. Assume $A:Q_e\times Q_e\to Q_e$. A is said to be mixed monotone if $A(u,y)$ is nondecreasing in u and nonincreasing in y, i.e., $u_1\leqslant u_2$ ($u_1,u_2\in Q_e$) implies $A(u_1,y)\leqslant A(u_2,y)$ for any $y\in Q_e$, and $y_1\leqslant y_2$ ($y_1,y_2\in Q_e$) implies $A(u,y_1)\geqslant A(u,y_2)$ for any $u\in Q_e$. The element $u^*\in Q_e$ is called a fixed point of A if $A(u^*,u^*)=u^*$. Now we give the following lemma.

Lemma 5. (*See* [2].) *Suppose that $A:Q_e\times Q_e\to Q_e$ is a mixed monotone operator and there exists a constant σ satisfying $0<\sigma<1$ such that*

$$A\left(lx,\frac{1}{l}y\right)\geqslant l^\sigma A(x,y),\quad x,y\in Q_e,\ 0<l<1, \tag{20}$$

then A has a unique fixed point $x^*\in Q_e$, and for any $x_0,y_0\in Q_e$, we have

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=x^*,$$

where

$$x_n=A(x_{n-1},y_{n-1}),\quad y_n=A(y_{n-1},x_{n-1}),\quad n=1,2,\dots,$$

and convergence rate

$$\|x_n-x^*\|=o\big(1-r^{\sigma^n}\big),\qquad \|y_n-x^*\|=o\big(1-r^{\sigma^n}\big),$$

r is a constant, $0<r<1$, and dependent on x_0, y_0.

Let $e(t)=t^{\gamma-\mu_{n-1}-1}$, we define a normal cone of $C[0,1]$ by

$$P=\big\{x\in C[0,1]\colon x(t)\geqslant 0,\ 0\leqslant t\leqslant 1\big\},$$

also define

$$Q_e=\left\{x\in P\colon \text{there exists } D\geqslant 1,\ \frac{1}{D}e(t)\leqslant x(t)\leqslant De(t),\ t\in[0,1]\right\}.$$

Remark 2. Let $s=\tau t$, by simple calculation, we have

$$I_{0^+}^{\mu_{n-2}}e(t)=\frac{1}{\Gamma(\mu_{n-2})}\int_0^t (t-s)^{\mu_{n-2}-1}s^{\gamma-\mu_{n-2}-1}\,\mathrm{d}s=\frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma)}t^{\gamma-1}. \tag{21}$$

Similarly, we have

$$I_{0^+}^{\eta_{m-2}} t^{\delta-\eta_{m-2}} = \frac{\Gamma(\delta-\eta_{m-2})}{\Gamma(\delta)} t^{\delta-1},$$

$$I_{0^+}^{\mu_{n-2}-\mu_\kappa} e(t) = \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_\kappa)} t^{\gamma-\mu_\kappa-2}, \quad \kappa = 1,2,\dots,n-3, \tag{22}$$

$$I_{0^+}^{\mu_{m-2}-\eta_\varrho} e(t) = \frac{\Gamma(\gamma-\mu_{m-2})}{\Gamma(\gamma-\eta_\varrho)} t^{\gamma-\eta_\varrho-1}, \quad \varrho = 1,2,\dots,m-2. \tag{23}$$

3 Main results

Theorem 1. *Suppose that* (S1)–(S4) *hold. Then the PFDE* (1) *has a unique positive solution* (u^*, v^*) *for all* $t \in [0,1]$, *which satisfies*

$$\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma)} t^{\gamma-1} \leqslant u^*(t) \leqslant \frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1},$$
$$\frac{\mu t^{\delta-1}\overline{b}^{q_2-1}}{\Gamma(\delta)} K_1 \leqslant v^*(t) \leqslant \frac{\Gamma(\delta-\eta_{m-2})\overline{a}^*\mu t^{\delta-1}\overline{b}^{q_2-1}}{\Gamma(\delta)(2\beta-1)^{(q_2-1)/2}} K_2, \tag{$*$}$$

where

$$K_1 = \left(\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma)}\right)^{\varsigma} \int_0^1 \overline{j}(s) \left[\int_0^s w^{(\gamma-1)\varsigma^{1/(q_2-1)}} (s-w)^{\beta-1} g(w,1,\dots,1)\,\mathrm{d}w\right]^{q_2-1} \mathrm{d}s,$$

$$K_2 = \left(\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_{m-2})} + 1\right)^{\varsigma} \int_0^1 \left[s^{(2\beta-1)/2}\left(\int_0^s g^2(w,1,1,\dots,1)\,\mathrm{d}w\right)^{1/2}\right]^{q_2-1} \mathrm{d}s.$$

Moreover, for any $u_0 \in Q_e$, *constructing successively sequences*

$$\begin{aligned} u_{k+1}(t) = I_{0^+}^{\mu_{n-2}} \Biggl\{ \lambda \int_0^1 G(t,s) \Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \bigl(\phi\bigl(\tau, u_k(\tau), D_{0^+}^{\mu_1} u_k(\tau), \dots, \\ D_{0^+}^{\mu_{n-2}} u_k(\tau), u_k(\tau), A u_k^{(\mu_{n-2})}(\tau)\bigr) + \psi\bigl(\tau, u_k(\tau), D_{0^+}^{\mu_1} u_k(\tau), \dots, \\ D_{0^+}^{\mu_{n-2}} u_k(\tau), u_k(\tau), A u_k^{(\mu_{n-2})}(\tau)\bigr)\bigr)\,\mathrm{d}\tau \Biggr]^{q_1-1} \mathrm{d}s \Biggr\}, \quad t \in [0,1], \end{aligned}$$

$$\begin{aligned} v_{k+1}(t) = I_{0^+}^{\eta_{m-2}} \Biggl\{ \mu \int_0^1 H(\tau,s) \Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w, u_k(w), \\ D_{0^+}^{\eta_1} u_k(w), \dots, D_{0^+}^{\eta_{m-2}} u_k(w)\bigr)\,\mathrm{d}w \Biggr]^{q_2-1} \mathrm{d}s \Biggr\}, \quad t \in [0,1], \end{aligned}$$

$k = 0, 1, 2, \dots$, and we have $\|u_k - u^*\| \to 0$, $\|v_k - v^*\| \to 0$ as $k \to \infty$, the convergence rate is $\|u_k - u^*\| = o(1 - r^{\sigma^k})$, where r is a constant, $0 < r < 1$, and dependent on u_0.

Proof. We first consider the existence of a positive solution to problem (17). From the discussion in Section 2 we only need to consider the existence of a positive solution to BVP (19). In order to realize this purpose, let

$$Ax(\tau) = I_{0^+}^{\eta_{m-2}}\left\{\mu\int_0^1 H(\tau,s)\left[\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w, I_{0^+}^{\mu_{n-2}}x(w),\right.\right.$$
$$\left.\left. I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\big)\,\mathrm{d}w\right]^{q_2-1}\mathrm{d}s\right\},\quad \tau\in[0,1], \tag{24}$$

and define the operator $T_k : Q_e \times Q_e \to P$ by

$$T_k(x,z)(t) = \lambda\int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k},\dots,\right.\right.\right.$$
$$\left. I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k}, x(\tau)+\frac{1}{k}, Ax(\tau)+\frac{1}{k}\right)$$
$$+\psi\left(\tau, I_{0^+}^{\mu_{n-2}}z(\tau)+\frac{1}{k},\dots,I_{0^+}^{\mu_{n-2}-\mu_{n-3}}z(\tau)+\frac{1}{k},\right.$$
$$\left.\left.\left. z(\tau)+\frac{1}{k}, Az(\tau)+\frac{1}{k}\right)\right)\mathrm{d}\tau\right]^{q_1-1}\mathrm{d}s,\quad t\in[0,1].$$

Now we prove that $T_k : Q_e \times Q_e \to P$ is well defined. For any $x, z \in Q_e$, by (24), (S3), (21), (23) and Remark 1, for all $\tau \in [0,1]$, we have

$$\mu\int_0^1 H(\tau,s)\left[\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w, I_{0^+}^{\mu_{n-2}}x(w), I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,\right.$$
$$\left. I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\big)\,\mathrm{d}w\right]^{q_2-1}\mathrm{d}s$$
$$\leqslant \mu\overline{a}^*\tau^{\delta-\eta_{m-2}-1}\int_0^1\left[\int_0^s \overline{b}(s-w)^{\beta-1}g\left(w, \frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma)}w^{\gamma-1},\right.\right.$$
$$\left.\left.\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_1)}w^{\gamma-\eta_1-1},\dots,\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_{m-2})}w^{\gamma-\eta_{m-2}-1}\right)\mathrm{d}w\right]^{q_2-1}\mathrm{d}s$$
$$\leqslant \mu\overline{a}^*\tau^{\delta-\eta_{m-2}-1}\overline{b}^{q-1}$$
$$\times\left(\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_{m-2})}+1\right)^{\varsigma}\int_0^1\left[\int_0^s (s-w)^{\beta-1}g(w,1,1,\dots,1)\,\mathrm{d}w\right]^{q_2-1}\mathrm{d}s, \tag{25}$$

$$\mu\int_0^1 H(\tau,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\bigl(w,I_{0^+}^{\mu_{n-2}}x(w),I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,$$
$$I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\bigr)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s$$
$$\geqslant \frac{\mu}{\Gamma(\delta-\eta_{m-2})}\tau^{\delta-\eta_{m-2}-1}\int_0^1 \overline{j}(s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\biggl(w,\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma)}w^{\gamma-1},$$
$$\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma-\eta_1)}w^{\gamma-\eta_1-1},\dots,\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma-\eta_{m-2})}w^{\gamma-\eta_{m-2}-1}\biggr)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s$$
$$=\frac{\mu}{\Gamma(\delta-\eta_{m-2})}\tau^{\delta-\eta_{m-2}-1}\overline{b}^{q_2-1}\biggl(\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma)}\biggr)^{\varsigma}$$
$$\times\int_0^1 \overline{j}(s)\Biggl[\int_0^s w^{(\gamma-1)\varsigma^{1/(q_2-1)}}(s-w)^{\beta-1}g(w,1,1,\dots,1)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s. \qquad (26)$$

Hence, by (24), (25), (26) and Hölder inequality, we have

$$Ax(\tau)=I_{0^+}^{\eta_{m-2}}\Biggl(\mu\int_0^1 H(\tau,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\bigl(w,I_{0^+}^{\mu_{n-2}}x(w),$$
$$I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\bigr)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s\Biggr)$$
$$\leqslant \frac{\Gamma(\delta-\eta_{m-2})\overline{a}^*\mu\tau^{\delta-1}\overline{b}^{q_2-1}}{\Gamma(\delta)(2\beta-1)^{(q_2-1)/2}}\biggl(\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_{m-2})}+1\biggr)^{\varsigma}$$
$$\times\int_0^1\Biggl[s^{(2\beta-1)/2}\biggl(\int_0^s g^2(w,1,1,\dots,1)\,\mathrm{d}w\biggr)^{1/2}\Biggr]^{q_2-1}\mathrm{d}s,\quad \tau\in[0,1], \qquad (27)$$

$$Ax(\tau)=I_{0^+}^{\eta_{m-2}}\Biggl(\mu\int_0^1 H(\tau,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\bigl(w,I_{0^+}^{\mu_{n-2}}x(w),$$
$$I_{0^+}^{\mu_{n-2}-\eta_1}x(w),\dots,I_{0^+}^{\mu_{n-2}-\eta_{m-2}}x(w)\bigr)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s\Biggr)$$
$$\geqslant \frac{\mu\tau^{\delta-1}\overline{b}^{q_2-1}}{\Gamma(\delta)}\biggl(\frac{\Gamma(\gamma-\mu_{n-2})}{D\Gamma(\gamma)}\biggr)^{\varsigma}\int_0^1 \overline{j}(s)\Biggl[\int_0^s w^{(\gamma-1)\varsigma^{1/(q_2-1)}}(s-w)^{\beta-1}$$
$$\times g(w,1,1,\dots,1)\,\mathrm{d}w\Biggr]^{q_2-1}\mathrm{d}s,\quad \tau\in[0,1].$$

By (S4), we get that $Ax(\tau)$ is well defined. From (21), (22), (27), (S1), and Remark 1 we have

$$\begin{aligned}
&\phi\biggl(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots, I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k},\\
&\qquad x(\tau)+\frac{1}{k}, Ax(\tau)+\frac{1}{k}\biggr)\\
&\quad\leqslant \phi\bigl(\tau, Db+1, Db+1,\dots, D^{\varsigma}b+1\bigr)\\
&\quad\leqslant 2^{\sigma^{1/(q_1-1)}}b^{\sigma^{1/(q_1-1)}}D^{\sigma^{1/(q_1-1)}}\phi(\tau,1,1,\dots,1), \quad \tau\in(0,1),
\end{aligned}\tag{28}$$

where $D\geqslant 1, b$ are two positive constants. By (21), (22), (27), (S1), (S2), we also have

$$\begin{aligned}
&\psi\biggl(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots, I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k},\\
&\qquad x(\tau)+\frac{1}{k}, Ax(\tau)+\frac{1}{k}\biggr)\\
&\quad\leqslant \psi\biggl(\tau, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k}, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k},\dots, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k}\biggr)\\
&\quad\leqslant c^{-\sigma^{1/(q_1-1)}}D^{\sigma^{1/(q_1-1)}}\tau^{(\gamma-1)\sigma^{1/(q_1-1)}}\psi(\tau,1,1,\dots,1), \quad \tau\in(0,1),
\end{aligned}\tag{29}$$

where c is a positive constant. Noting $(c/D)\tau^{\gamma-1}<1$ and by (21), (22), (27), (S1), (S2), we have

$$\begin{aligned}
&\phi\biggl(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots, I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k},\\
&\qquad x(\tau)+\frac{1}{k}, Ax(\tau)+\frac{1}{k}\biggr)\\
&\quad\geqslant \phi\biggl(\tau, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k}, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k},\dots, \frac{c}{D}\tau^{\gamma-1}+\frac{1}{k}\biggr)\\
&\quad\geqslant \biggl(\frac{c}{D}\tau^{\gamma-1}+\frac{1}{k}\biggr)^{-\sigma}\phi(\tau,1,1,\dots,1)\\
&\quad= c^{\sigma^{1/(q_1-1)}}D^{-\sigma^{1/(q_1-1)}}\tau^{(\gamma-1)\sigma^{1/(q_1-1)}}\phi(\tau,1,1,\dots,1), \quad \tau\in(0,1).
\end{aligned}\tag{30}$$

By (21), (22), (27), (S1), and Remark 1, we also get

$$\begin{aligned}
&\psi\biggl(\tau, I_{0^+}^{\mu_{n-2}}x(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}x(\tau)+\frac{1}{k},\dots, I_{0^+}^{\mu_{n-2}-\mu_{n-3}}x(\tau)+\frac{1}{k},\\
&\qquad x(\tau)+\frac{1}{k}, Ax(\tau)+\frac{1}{k}\biggr)\\
&\quad\geqslant \psi\bigl(\tau, Db\tau^{\gamma-1}+1, Db\tau^{\gamma-\mu_1-1}+1,\dots, Db\tau^{\gamma-\mu_{n-2}-1}+1, Db\tau^{\delta-1}+1\bigr)\\
&\quad\geqslant 2^{-\sigma^{1/(q_1-1)}}b^{-\sigma^{1/(q_1-1)}}D^{-\sigma^{1/(q_1-1)}}\psi(\tau,1,1,\dots,1), \quad \tau\in(0,1).
\end{aligned}\tag{31}$$

For any $x, z \in Q_e$, it follows from (28), (29) that

$$
\begin{aligned}
&T_k(x,z)(t) \\
&\quad \leqslant \lambda a^* t^{\gamma-\mu_{n-2}-1} D^{\sigma} \overline{a}^{q_1-1} \frac{1}{(2\alpha-1)^{(q_1-1)/2}} \\
&\qquad \times \int_0^1 \Biggl[2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} \Biggl(\int_0^s \phi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \Biggr)^{1/2} \\
&\qquad + c^{-\sigma^{1/(q_1-1)}} \Biggl(\int_0^s \tau^{-2(\gamma-1)\sigma^{1/(q_1-1)}} \psi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \Biggr)^{1/2} \Biggr]^{q_1-1} \mathrm{d}s \\
&\quad < +\infty, \quad t \in [0,1].
\end{aligned}
\tag{32}
$$

By (S4), (32), we have that $T_k : Q_e \times Q_e \to P$ is well defined. Next, we will prove $T_k : Q_e \times Q_e \to Q_e$. Formula (32) implies that

$$
T_k(x,z)(t) \leqslant D t^{\gamma-\mu_{n-2}-1} = D e(t), \quad t \in [0,1].
$$

At the same time, by (30) and (31), we have

$$
\begin{aligned}
&T_k(x,z)(t) \\
&\quad = \lambda \int_0^1 G(t,s) \Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \Biggl(\phi\Biggl(\tau, I_{0+}^{\mu_{n-2}} x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1} x(\tau) + \frac{1}{k}, \dots, \\
&\qquad I_{0+}^{\mu_{n-2}-\mu_{n-3}} x(\tau) + \frac{1}{k}, x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k} \Biggr) + \psi\Biggl(\tau, I_{0+}^{\mu_{n-2}} z(\tau) + \frac{1}{k}, \\
&\qquad I_{0+}^{\mu_{n-2}-\mu_1} z(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}} z(\tau) + \frac{1}{k}, z(\tau) + \frac{1}{k}, Az(\tau) + \frac{1}{k} \Biggr) \Biggr) \mathrm{d}\tau \Biggr]^{q_1-1} \mathrm{d}s \\
&\quad \geqslant \frac{\lambda}{\Gamma(\gamma-\mu_{n-2})} t^{\gamma-\mu_{n-2}-1} D^{-\sigma} \overline{a}^{q_1-1} \\
&\qquad \times \int_0^1 j(s) \Biggl[\int_0^s (s-\tau)^{\alpha-1} \bigl(c^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \phi(\tau,1,1,\dots,1) \\
&\qquad + 2^{-\sigma^{1/(q_1-1)}} b^{-\sigma^{1/(q_1-1)}} \psi(\tau,1,1,\dots,1) \bigr)\,\mathrm{d}\tau \Biggr]^{q_1-1} \mathrm{d}s, \quad t \in [0,1].
\end{aligned}
\tag{33}
$$

Formula (33) implies that

$$
T_k(x,z)(t) \geqslant \frac{1}{D} t^{\gamma-\mu_{n-2}-1} = \frac{1}{D} e(t), \quad t \in [0,1].
$$

Hence, $T_k : Q_e \times Q_e \to Q_e$. It is easy to prove that $T_k : Q_e \times Q_e \to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T_k satisfies (20). For any $x, z \in Q_e$ and $l \in (0,1)$, by (S2) and Remark 1, for all $t \in [0,1]$, we have

$$\begin{aligned}
&\lambda \int_0^1 G(t,s)\Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\Bigg(\phi\Bigg(\tau, I_{0^+}^{\mu_{n-2}} lx(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1} lx(\tau)+\frac{1}{k},\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}} lx(\tau)+\frac{1}{k}, lx(\tau)+\frac{1}{k}, Alx(\tau)+\frac{1}{k}\Bigg)\\
&\quad + \psi\Bigg(\tau, I_{0^+}^{\mu_{n-2}}\frac{1}{l}z(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1}\frac{1}{l}z(\tau)+\frac{1}{k},\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}}\frac{1}{l}z(\tau)+\frac{1}{k}, \frac{1}{l}z(\tau)+\frac{1}{k}, A\frac{1}{l}z(\tau)+\frac{1}{k}\Bigg)\Bigg)\,\mathrm{d}\tau\Bigg]^{q_1-1}\mathrm{d}s\\
&\geqslant l^\sigma \lambda \int_0^1 G(t,s)\Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau, I_{0^+}^{\mu_{n-2}} x(\tau), I_{0^+}^{\mu_{n-2}-\mu_1} x(\tau),\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}} x(\tau), x(\tau), Ax(\tau)\big) + \psi\big(\tau, I_{0^+}^{\mu_{n-2}} z(\tau), I_{0^+}^{\mu_{n-2}-\mu_1} z(\tau),\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}} z(\tau), z(\tau), Az(\tau)\big)\big)\,\mathrm{d}\tau\Bigg]^{q_1-1}\mathrm{d}s.
\end{aligned} \tag{34}$$

Formula (34) implies that

$$T_k\left(lx, \frac{1}{l}z\right) \geqslant l^\sigma T_k(x,z), \quad x, z \in Q_e. \tag{35}$$

Hence, Lemma 5 assumes that there exists a unique positive solution $x_k^* \in Q_e$ such that $T_k(x_k^*, x_k^*) = x_k^*$. Consequently, x_k^* is a unique positive solution of (17) for every $k \in \{2,3,\dots\}$. Since $x_k^* \in Q_e$, so x_k^* has uniform lower and upper bounds. Thus, in order to pass the solution x_k^* of (17) to that of (16), we need that the fact that $\{x_k^*\}_{k\geqslant 2}$ is an equiconuous family on $[0,1]$. In fact, by (28), (29), $x_k^* \in Q_e$, we have

$$\begin{aligned}
&f\Bigg(\tau, I_{0^+}^{\mu_{n-2}} x_k^*(\tau)+\frac{1}{k}, I_{0^+}^{\mu_{n-2}-\mu_1} x_k^*(\tau)+\frac{1}{k},\dots,\\
&\quad I_{0^+}^{\mu_{n-2}-\mu_{n-3}} x_k^*(\tau)+\frac{1}{k}, Ax_k^*(\tau)+\frac{1}{k}\Bigg)\\
&\quad \leqslant 2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \phi(\tau,1,1,\dots,1)\\
&\qquad + c^{-\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \psi(\tau,1,1,\dots,1), \quad \tau \in (0,1),
\end{aligned}$$

and let

$$\begin{aligned}
\varphi(\tau) &= 2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \phi(\tau,1,1,\dots,1)\\
&\quad + c^{-\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \psi(\tau,1,1,\dots,1), \quad \tau \in (0,1),
\end{aligned}$$

by (S4), we easily get that $\varphi(s) \in L^1[0,1]$. Hence, for $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, we have

$$\begin{aligned}
&\left|(x_k^*)(t_2) - (x_k^*)(t_1)\right| \\
&\quad = \left| \int_0^1 \big(G(t_2,s) - G(t_1,s)\big) \left(\int_0^s \overline{a}(s-\tau)^{\alpha-1} f\left(\tau, I_{0+}^{\mu_{n-2}} x^*(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1} x_k^*(\tau) \right.\right.\right. \\
&\qquad \left.\left. + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}} x_k^*(\tau) + \frac{1}{k}, x_k^*(\tau) + \frac{1}{k}, A x_k^*(\tau) + \frac{1}{k}\right) \mathrm{d}\tau \right)^{q_1-1} \mathrm{d}s \\
&\quad \leqslant \left| \int_0^1 \big(G_1(t_2,s) + G_2(t_2,s) - G_1(t_1,s) - G_2(t_1,s)\big) \left(\int_0^s \overline{a}(s-\tau)^{\alpha-1} \varphi(\tau)\,\mathrm{d}\tau \right)^{q_1-1} \mathrm{d}s \right| \\
&\quad \leqslant \frac{\overline{a}^{q_1-1}\|\varphi\|_L}{\Delta(2\alpha-1)^{(q_1-1)/2}\Gamma(\gamma-\mu_{n-2}-r_1)} \big(t_2^{\gamma-\mu_{n-2}-1} - t_1^{\gamma-\mu_{n-2}-1}\big) \\
&\qquad + \frac{\overline{a}^{q_1-1}\|\varphi\|_L}{\Delta(2\alpha-1)^{(q_1-1)/2}} \Bigg[(t_2-t_1)^{\gamma-\mu_{n-2}-1} \\
&\qquad + \int_0^{t_1} \big((t_2-s)^{\gamma-\mu_{n-2}-1} - (t_1-s)^{\gamma-\mu_{n-2}-1}\big)\,\mathrm{d}s \Bigg].
\end{aligned} \tag{36}$$

Since $(t-s)^{\gamma-\mu_{n-2}-1}$ is uniformly continuous on $[0,1]\times[0,1]$ and $t^{\gamma-\mu_{n-2}-1}$ is uniformly continuous on $[0,1]$, so any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, $t_2 - t_1 < \delta$, $0 < s \leqslant t_1$,

$$t_2^{\gamma-\mu_{n-2}-1} - t_1^{\gamma-\mu_{n-2}-1} < \varepsilon,$$
$$(t_2-s)^{\gamma-\mu_{n-2}-1} - (t_1-s)^{\gamma-\mu_{n-2}-1} < \varepsilon.$$

Consequently, for all $x \in D$, $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ and $t_2 - t_1 < \min\{\delta, \sqrt[\gamma-\mu_{n-2}-1]{\varepsilon}\}$, the inequality

$$\left|(x_k^*)(t_2) - (x_k^*)(t_1)\right| \leqslant \frac{\overline{a}^{q_1-1}\|\varphi\|_L}{\Delta(2\alpha-1)^{(q_1-1)/2}} \big(2 + \Gamma(\gamma-\mu_{n-2}-r_1)\big)\varepsilon$$

holds. Hence, by Arzela–Ascoli theorem, we get that $\{x_k^*\}_{k\geqslant 2}$ is an equiconuous family on $[0,1]$. Hence, $\{x_k^*\}_{k\geqslant 2}$ is relatively compact in P, then the sequence $\{x_k^*\}$ has a subsequence converge to $x^* \subset P$. Without loss of generality, we still assume that $\{x_k^*\}$ itself uniformly converges to x^*, that is $\lim_{k\to\infty} x_k^* \to x^*$, then (x^*, y^*) is the solution of (19), which can be easily get by the Lebesgue dominated convergence theorem.

Moreover, for any $u_0(t) = I_{0+}^{\mu_{n-2}} x_0 \in Q_e$, by Lemma 5, constructing a successively sequence

$$\begin{aligned}
x_{m+1}(t) = \lambda \int_0^1 G(t,s) \Bigg[\int_0^s & \overline{a}(s-\tau)^{\alpha-1} \big(\phi(\tau, I_{0+}^{\mu_{n-2}} x_m(\tau), \dots, \\
& I_{0+}^{\mu_{n-2}-\mu_{n-3}} x_m(\tau), x_m(\tau), A x_m(\tau)\big) + \psi\big(\tau, I_{0+}^{\mu_{n-2}} x_m(\tau), \dots, \\
& I_{0+}^{\mu_{n-2}-\mu_{n-3}} x_m(\tau), x_m(\tau), A x_m(\tau)\big)\big)\,\mathrm{d}\tau \Bigg]^{q_1-1} \mathrm{d}s, \quad t \in [0,1],\ m = 1,2,\dots,
\end{aligned}$$

by $u_{m+1}(t) = I_{0+}^{\mu_{n-2}} x_{m+1}(t)$, then

$$u_{m+1}(t) = I^{\mu_{n-2}}\left\{\lambda \int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau, u_m(\tau), D_{0+}^{\mu_1}u_m(\tau), D_{0+}^{\mu_2}u_m(\tau)\ldots,\right.\right.$$
$$D_{0+}^{\mu_{n-2}}u_m(\tau), AD_{0+}^{\mu_{n-2}}u_m(\tau)\big) + \psi\big(\tau, u_m(\tau), D_{0+}^{\mu_1}u_m(\tau), D_{0+}^{\mu_2}u_m(\tau)\ldots,$$
$$\left.\left. D_{0+}^{\mu_{n-2}}u_m(\tau), AD_{0+}^{\mu_{n-2}}u_m(\tau)\big)\big)\,\mathrm{d}\tau\right]^{q_1-1}\mathrm{d}s\right\}, \quad t \in [0,1],\ m = 1,2,\ldots,$$

and we have $\|u_m - u^*\| = \|I_{0+}^{\mu_{n-2}}x_m - I_{0+}^{\mu_{n-2}}x^*\| \to 0$ as $m \to \infty$, convergence rate

$$\|u_m - u^*\| = \big\|I_{0+}^{\mu_{n-2}}x_m - I_{0+}^{\mu_{n-2}}x^*\big\| = o\big(1 - r^{\sigma^m}\big),$$

r is a constant, $0 < r < 1$, and dependent on u_0. Hence, for $t \in [0,1]$, by Lemma 4, $u^*(t) = I_{0+}^{\mu_{n-2}}x^*(t)$, $v^*(t) = I_{0+}^{\eta_{m-2}}y^*(t)$ is the unique positive solution of system (1), where

$$y^*(t) = \int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-w)^{\beta-1} g\big(w, I_{0+}^{\mu_{n-2}}x^*(w), I_{0+}^{\mu_{n-2}-\eta_1}x^*(w),\ldots,\right.$$
$$\left. I_{0+}^{\mu_{n-2}-\eta_{m-2}}x^*(w)\big)\,\mathrm{d}w\right)^{q_2-1}\mathrm{d}s. \tag{37}$$

By (25), (26), (35), (36), and $x^* \in Q_e$, we get (u^*, v^*), which satisfies $(*)$.

Moreover, by (37), we have

$$v^*(t) = I_{0+}^{\eta_{m-2}}\left\{\mu \int_0^1 H(\tau,s)\left[\int_0^s \overline{b}(s-w)^{\beta-1} g\big(w, u^*(w), D_{0+}^{\eta_1}u^*(w),\ldots,\right.\right.$$
$$\left.\left. D_{0+}^{\eta_{m-2}}(w)\big)\,\mathrm{d}w\right]^{q_2-1}\mathrm{d}s\right\}$$

and

$$v_{k+1} = I_{0+}^{\eta_{m-2}}\left(\mu \int_0^1 H(\tau,s)\left[\int_0^s \overline{b}(s-w)^{\beta-1} g\big(w, u_k(\omega),\right.\right.$$
$$\left.\left. D_{0+}^{\eta_1},\ldots, D_{0+}^{\eta_{m-2}}(w)\big)\,\mathrm{d}w\right]^{q_2-1}\mathrm{d}s\right)$$
$$\leqslant \frac{\Gamma(\delta-\eta_{m-2})\overline{a}^*\mu\tau^{\delta-1}\overline{b}^{q_2-1}}{\Gamma(\delta)(2\beta-1)^{(q_2-1)/2}}\left(\frac{D\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\eta_{m-2})}+1\right)^{\varsigma}$$
$$\times \int_0^1 \left[s^{(2\beta-1)/2}\left(\int_0^s g^2(w,1,1,\ldots,1)\,\mathrm{d}w\right)^{1/2}\right]^{q_2-1}\mathrm{d}s < +\infty.$$

By Lebesgue control convergence theorem, we have $\|v_k - v^*\| \to 0 (k \to +\infty)$. Therefore, the proof of Theorem 1 is completed. □

4 An example

Consider the following boundary value problem:

$$\begin{aligned}
&D_{0+}^{3/4}\big(\varphi_3\big(D_{0+}^{5/2-}u\big)\big)(t)+\lambda^2 f\big(t,u(t),D_{0+}^{1/2}u(t),v(t)\big)=0,\quad 0<t<1,\\
&D_{0+}^{3/4}\big(\varphi_2\big(D_{0+}^{3/2}v\big)\big)(t)+\mu^2 g\big(t,u(t)\big)=0,\quad 0<t<1,\\
&u(0)=0,\quad D_{0+}^{\gamma}u(0)=0,\qquad D_{0+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0+}^{r_2}u(\xi_j),\\
&v(0)=0,\quad D_{0+}^{\delta}v(0)=0,\qquad D_{0+}^{\overline{r}_1}v(1)=\sum_{j=1}^{\infty}\overline{\eta}_j D_{0+}^{\overline{r}_2}u(\overline{\xi}_j),
\end{aligned} \tag{38}$$

where $\gamma=5/2$, $\delta=3/2$, $\alpha=\beta=3/4$, $r_1=r_2=1/2$, $\overline{r}_1=\overline{r}_2=1/2$, $\eta_j=\overline{\eta}_j=1/(2j^5)$, $\xi_j=\overline{\xi}_j=1/j^2$, $p_1=3$, $q_1=3/2$, $p_2=2$, $q_2=1/2$, and

$$\begin{aligned}
\phi(t,x_1,x_2,x_3)&=\big(t^{-1/4}+\cos t\big)x_1^{1/9}+2tx_2^{1/8}+2x_3^{1/16},\\
\psi(t,x_1,x_2,x_3)&=t^{-1/16}x_1^{-1/8}+x_2^{-1/16}+(2-t)x_3^{-1/15},\\
g(t,u)&=\big(3t+t^2\big)u^{3/5}+(t\sin t+t)u^{2/3}.
\end{aligned}$$

Hence, by simple calcution, we have $(\Gamma(\gamma)/\Gamma(\gamma-p_2))\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-p_2-1}=0.5412\leqslant\Gamma(\gamma)/\Gamma(\gamma-p_1)=\Gamma(5/2)/\Gamma(3/2)$, $(\Gamma(\delta)/\Gamma(\delta-\overline{p}_2))\sum_{j=1}^{\infty}\overline{\eta}_j\overline{\xi}_j^{\alpha-\overline{p}_2-1}=0.5412\leqslant\Gamma(\delta)/\Gamma(\delta-\overline{p}_1)=\Gamma(3/2)/\Gamma(1/2)$. Moreover, for any $(t,x_1,x_2,x_3)\in(0,1)\times(0,\infty)^3$ and $0<l<1$, we have

$$\begin{aligned}
\phi(t,lx_1,lx_2,lx_3)&=\big(t^{-1/4}+\cos t\big)(lx_1)^{1/9}+2t(lx_2)^{1/8}+2(lx_3)^{1/16}\\
&\geqslant l^{1/8}\big(\big(t^{-1/4}+\cos t\big)x_1^{1/9}+2tx_2^{1/8}+2x_3^{1/16}\big)\\
&=l^{1/8}\phi(t,x_1,x_2,x_3)=l^{\sigma^{1/(q-1)}}\phi(t,x_1,x_2,x_3),\\
\psi\big(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3\big)&=t^{-1/16}\big(l^{-1}x_1\big)^{-1/8}+\big(l^{-1}x_2\big)^{-1/16}+(2-t)\big(l^{-1}x_3\big)^{-1/15}\\
&\geqslant l^{1/8}\big(t^{-1/16}x_1^{-1/8}+x_2^{-1/16}+(2-t)x_3^{-1/15}\big)\\
&=l^{1/8}\psi(t,x_1,x_2,x_3)=l^{\sigma^{1/(q_1-1)}}\psi(t,x_1,x_2,x_3),\\
g(t,lu)&=\big(3t+t^2\big)(lu)^{3/5}+(t\sin t+t)(lu)^{2/3}\\
&\geqslant l^{2/3}\big(\big(3t+t^2\big)u^{3/5}+(t\sin t+t)u^{2/3}\big)\\
&=l^{2/3}g(t,u)=l^{\varsigma^{1/(q_2-1)}}g(t,u).
\end{aligned}$$

Noting $\sigma = 1/(2\sqrt{2}) < 1$, $\varsigma = 2/3$, $\psi(\tau,1,1,1) = \tau^{-1/16} + 3 - \tau$, $\phi(\tau,1,1,1) = \tau^{-1/4} + \cos\tau + 2\tau + 2$, $g(\tau,1) = 3\tau + \tau^2 + \tau\sin\tau + \tau$, we have

$$\begin{aligned}
0 &< \int_0^1 \phi^2(\tau,1,1,1)\,\mathrm{d}\tau = \int_0^1 \left(\tau^{-1/4} + \cos\tau + 2\tau + \tau\right)^2 \mathrm{d}\tau \\
&\leqslant 19 + 8 + \frac{16}{7} < +\infty, \\
0 &< \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q-1)}} \psi^2(\tau,1,1,1)\,\mathrm{d}\tau = \int_0^1 \tau^{-3/8}\left(\tau^{-1/16} + 3 - \tau\right)^2 \mathrm{d}\tau \\
&= 2 + \frac{96}{9} + 9 < +\infty, \\
0 &< \int_0^1 g^2(\tau,1,1,1)\,\mathrm{d}\tau \leqslant \int_0^1 \left(3\tau + \tau^2 + \tau\sin\tau + \tau\right)^2 \mathrm{d}\tau \\
&\leqslant 17 \times \frac{1}{3} + 6\frac{1}{4} + \frac{1}{5} + 12 + 16 < +\infty.
\end{aligned}$$

Thus, assumptions (S1)–(S4) of Theorem 1 hold. Then Theorem 1 implies that problem (38) has a unique solution.

In addition, for any initial $u_0 \in Q_e$, we construct a successively sequence

$$\begin{aligned}
u_{k+1}(t) = \int_0^1 \lambda G(t,s) \Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \Big(\phi\big(t, u_k(t), D_{0^+}^{1/2} u_k(t), Au_k'(t)\big) \\
+ \psi\big(t, u_k(t), D_{0^+}^{1/2} u_k(t), Au_k'(t)\big)\Big)\,\mathrm{d}\tau \Bigg]^{q_1-1} \mathrm{d}s, \quad t \in [0,1],\ k = 1,2,\dots,
\end{aligned}$$

and we have $\|u_k - u^*\| \to 0$ as $k \to \infty$, the convergence rate is $\|u_k - u^*\| = o(1 - r^{\sigma^k})$, where r is a constant, $0 < r < 1$, and dependent on u_0.

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