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Study on a class of Schrödinger elliptic system involving a nonlinear operator*

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Abstract. This paper considers a class of Schrödinger elliptic system involving a nonlinear operator. Firstly, under the simple condition on ψ and φ, we prove the existence of the entire positive bounded radial solutions. Secondly, by using the iterative technique and the method of contradiction, we prove the existence and nonexistence of the entire positive blow-up radial solutions. Our results extend the previous existence and nonexistence results for both the single equation and systems. In the end, we give two examples to illustrate our results.

Keywords: Schrödinger system, nonlinear operator, blow-up solution, iterative method.

1 Introduction and preliminary

In this paper, our main objective is to show the positive radial solutions of the following nonlinear Schrödinger elliptic system involving a nonlinear operator:

$$\begin{aligned} &\operatorname{div}(\mathcal{G}\big(|\nabla y|^{p-2}\big)\nabla y) = b\big(|x|\big)\psi(z), \quad x \in \mathbb{R}^n,\\ &\operatorname{div}(\mathcal{G}\big(|\nabla z|^{p-2}\big)\nabla z) = h\big(|x|\big)\varphi(y), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $n \geqslant 3$, $b, h, \psi, \varphi \in C([0,+\infty),[0,+\infty))$, and $\mathcal{G}$ is a nonlinear operator on $\Theta = \{\mathcal{G} \in C^2([0,+\infty),(0,+\infty)) \mid \exists p = \mathrm{const} > 2\colon\ \mathcal{G}(ls) \leqslant l^{p-2}\mathcal{G}(s),\ 0<l<1\}$.

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The Schrödinger elliptic system arises in various areas of applied mathematics and physics and has been widely studied by many authors in many contexts. In particular, many rich results on the Schrödinger elliptic system have been obtained by using nonlinear functional analysis methods such as the variational method [4–7, 15, 18, 33, 35, 38], the fixed point theorem [1, 12, 14, 16, 20, 21], the upper and lower solution method [13, 39] and the method of moving planes [27, 31].

As a special case of system (1), when $G(X) = X$, $p = 2$, Li, Zhang and Zhang [11] investigated the existence of entire positive bounded and blow-up radial solutions to the following semilinear elliptic system:

$$\Delta y = b(|x|)\psi(z), \quad \Delta z = h(|x|)\varphi(y), \quad x \in \mathbb{R}^n.$$

On the other hand, if $G(X) = X$, $\psi(z) = z^\alpha$, $\varphi(y) = y^\beta$, $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, $p = 2$, then our system (1) takes the following form:

$$\Delta y = b(|x|)z^\alpha, \quad \Delta z = h(|x|)y^\beta, \quad x \in \mathbb{R}^n. \tag{2}$$

Lair [9] has considered the necessary and sufficient conditions for the existence of the nonnegative entire large radial solution of system (2). In addition, Lair and Wood [10] studied the existence of entire positive large radial solutions of system (2).

In a recent paper [35], by using the iterative method and the dual method, Zhang and Liu studied the existence and nonexistence of entire blow-up radial solutions for the following quasilinear p-Laplacian Schrödinger elliptic equation with a nonsquare diffusion term:

$$\begin{aligned} &-\Delta_p y - \Delta_p(|y|^{2\gamma})|y|^{2\gamma-2}y = h(x)\varphi(y), \quad y > 0, \\ &\lim_{|x|\to\infty} y(x) = \infty, \quad x \in \mathbb{R}^n, \end{aligned}$$

where $n \geqslant 1$, $\Delta_p y = \operatorname{div}(|\nabla y|^{p-2}\nabla y)$ with $p \geqslant 2\gamma$, $\gamma > 1/2$, h is a nonnegative continuous radial function on $\mathbb{R}^n$, $\varphi \in C([0, +\infty), [0, +\infty))$ is nondecreasing.

In 2018, by using the iterative technique and introducing a growth condition, Zhang and Wu [37] focused on the existence and nonexistence of the entire blow-up radial solutions for the following nonlinear Schrödinger elliptic equation:

$$\operatorname{div}(\mathcal{G}(|\nabla z|)\nabla z) = b(|x|)\psi(z), \quad x \in \mathbb{R}^n, \tag{3}$$

where $n \geqslant 2$, $\mathcal{G}$ is a nonlinear operator on $\Theta = \{\mathcal{G} \in C^2([0, +\infty), (0, +\infty)) \mid \exists\alpha = \text{const} > 0\text{: } \mathcal{G}(ls) \leqslant l^\alpha \mathcal{G}(s) 0 < l < 1\}$.

Inspired by the above excellent works, in this paper, by employing the monotone iterative method under some appropriate conditions on b, h, ψ, φ and $\mathcal{G}$. We first establish the existence of entire positive bounded radial solutions of the nonlinear Schrödinger elliptic system (1). Then the existence and nonexistence of entire positive blow-up radial solutions of the Schrödinger elliptic system (1) are also given. The monotone iterative method, as an effective tool, plays a crucial role in the study of nonlinear problem, see [2, 3, 8, 10, 11, 17, 19, 22–26, 28–30, 32, 34, 36, 37] and the references therein. To the best of our knowledge, although many papers have been found on the Schrödinger system, there is no work on the existence and nonexistence of blow-up radial solutions for the Schrödinger system (1) involving a nonlinear operator by using the monotone iterative method. It is worth mentioning that our nonlinear Schrödinger system (1) cover the problems studied in [3, 8–11, 19, 32, 37] as special cases.

Before we begin to give our concrete results, we first present some symbols, assumptions and lemmas, which will be used immediately in subsequent proof of our theorems.

Denote

$$B(r) = \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s)\,\mathrm{d}s\right) \mathrm{d}t, \quad r \geqslant 0, \qquad B(\infty) := \lim_{r\to\infty} B(r),$$

$$H(r) = \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s)\,\mathrm{d}s\right) \mathrm{d}t, \quad r \geqslant 0, \qquad H(\infty) := \lim_{r\to\infty} H(r),$$

and

$$A(r) = \int_a^r \frac{\mathrm{d}x}{\psi(x) + \varphi(x) + 1}, \quad r \geqslant a > 0, \qquad A(\infty) := \lim_{r\to\infty} A(r).$$

We can know that $A'(r) = 1/(\psi(r) + \varphi(r) + 1) > 0$ for all $r > a$ and A has an increasing inverse function A^{-1} on $[0, \infty)$.

Our assumptions are as follows:

(A1) $A(\infty) = \infty$;
(A2) $A(\infty) < \infty$;
(A3) $B(\infty) = \infty$, $H(\infty) = \infty$;
(A4) $B(\infty) < \infty$, $H(\infty) < \infty$;
(A5) $\psi, \varphi \in C([0, +\infty), [0, +\infty))$ are nondecreasing;
(A6) $0 \leqslant \psi(x) \leqslant \lambda_1 x^\alpha + \mu_1$, $0 \leqslant \varphi(x) \leqslant \lambda_2 x^\beta + \mu_2$, where $0 < \lambda_1, \lambda_2 \leqslant 1$, $\mu_1, \mu_2 > 0$ and $0 < \alpha, \beta < p - 1$.

In order to complete our paper better, we also need to introduce the following lemmas.

Lemma 1. (*See* [37].) *If* $\mathcal{G} \in \Theta$, *let* $\mathcal{R}(s) = s\mathcal{G}(s^{p-2})$. *Then*

(i) $\mathcal{R}(s)$ *has a nonnegative increasing inverse mapping* $\mathcal{R}^{-1}(s)$;
(ii) *when* $0 < l < 1$, *one has*
$$\mathcal{R}^{-1}(ls) \geqslant l^{1/(p-1)} \mathcal{R}^{-1}(s);$$
(iii) *when* $l \geqslant 1$, *one has*
$$\mathcal{R}^{-1}(ls) \leqslant l^{1/(p-1)} \mathcal{R}^{-1}(s).$$

With aid of a standard deduction, one can obtain the following conclusion. So, we omit its proof.

Lemma 2. *The Schrödinger elliptic system* (1) *has a radial solution* $(y, z) \in C^2[0, \infty) \times C^2[0, \infty)$ *if and only if it solves the following ordinary differential system*:

$$\begin{aligned} &\big(\mathcal{G}\big(\big|y'\big|^{p-2}\big)y'\big)' + \frac{n-1}{r}\mathcal{G}\big(\big|y'\big|^{p-2}\big)y' = b(r)\psi(z), \quad r > 0, \\ &\big(\mathcal{G}\big(\big|z'\big|^{p-2}\big)z'\big)' + \frac{n-1}{r}\mathcal{G}\big(\big|z'\big|^{p-2}\big)z' = h(r)\varphi(y), \quad r > 0. \end{aligned} \tag{4}$$

2 Existence of the positive bounded radial solutions

In this section, we pay close attention to the existence of the positive bounded radial solutions of the Schrödinger elliptic system (1) involving a nonlinear operator.

Theorem 1. *Assume that* (A1), (A4) *and* (A5) *hold. Then the Schrödinger system* (1) *has infinitely many positive bounded radial solutions* $(y, z) \in C^2[0, \infty) \times C^2[0, \infty)$.

Proof. By Lemma 1, system (4) is equivalent to the following system:

$$\begin{aligned} &\big(\mathcal{R}(y')\big)' + \frac{n-1}{r}\mathcal{R}\big(y'\big) = b(r)\psi\big(z(r)\big), \quad r > 0, \\ &\big(\mathcal{R}(z')\big)' + \frac{n-1}{r}\mathcal{R}\big(z'\big) = h(r)\varphi\big(y(r)\big), \quad r > 0. \end{aligned} \tag{5}$$

It is well known that the solutions of the above system (5) are the solutions of the following integral system:

$$\begin{aligned} y(r) &= y(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi\big(z(s)\big)\,\mathrm{d}s\right)\mathrm{d}t, \quad r \geqslant 0, \\ z(r) &= z(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi\big(y(s)\big)\,\mathrm{d}s\right)\mathrm{d}t, \quad r \geqslant 0. \end{aligned}$$

We choose the initial values $y(0) = z(0) = \gamma > 0$, then define $\{y_m\}_{m\geqslant 1}$ and $\{z_m\}_{m\geqslant 0}$ on $[0, \infty)$ by

$$\begin{aligned} z_0(r) &= \gamma, \\ y_m(r) &= \gamma + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi\big(z_{m-1}(s)\big)\,\mathrm{d}s\right)\mathrm{d}t, \quad r \geqslant 0, \\ z_m(r) &= \gamma + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi\big(y_m(s)\big)\,\mathrm{d}s\right)\mathrm{d}t, \quad r \geqslant 0. \end{aligned} \tag{6}$$

Clearly, for all $r \geqslant 0$ and $m \in \mathbb{N}$, $y_m(r) \geqslant \gamma$, $z_m(r) \geqslant \gamma$ and $z_0 \leqslant z_1$. By Lemma 1 and (A5) we yield $y_1(r) \leqslant y_2(r)$ for all $r \geqslant 0$, then $z_1(r) \leqslant z_2(r)$ for all $r \geqslant 0$. Continuing this process, we get that $\{y_m\}$ and $\{z_m\}$ are the increasing sequences.

It is inescapably clear that $y_m'(t) \geqslant 0$ and $z_m'(t) \geqslant 0$. Moreover, we obtain by Lemma 1 and monotonicity of ψ, φ, $\{y_m\}$ and $\{z_m\}$ that, for each $r \geqslant 0$,

$$\begin{aligned}
y_m'(r) &= \mathcal{R}^{-1}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b(s)\psi\bigl(z_{m-1}(s)\bigr)\,\mathrm{d}s\right)\\
&\leqslant \mathcal{R}^{-1}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b(s)\psi\bigl(z_m(s)\bigr)\,\mathrm{d}s\right)\\
&\leqslant \mathcal{R}^{-1}\left(\psi\bigl(z_m(r)\bigr)\frac{1}{r^{n-1}}\int_0^r s^{n-1}b(s)\,\mathrm{d}s\right)\\
&\leqslant \mathcal{R}^{-1}\left(\bigl(\psi\bigl(z_m(r)\bigr)+1\bigr)\frac{1}{r^{n-1}}\int_0^r s^{n-1}b(s)\,\mathrm{d}s\right)\\
&\leqslant \bigl(\psi\bigl(z_m(r)\bigr)+1\bigr)^{1/(p-1)}\mathcal{R}^{-1}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b(s)\,\mathrm{d}s\right)\\
&\leqslant \bigl(\psi\bigl(z_m(r)\bigr)+1\bigr)B'(r)\\
&\leqslant \bigl[\psi\bigl(z_m(r)+y_m(r)\bigr)+\varphi\bigl(z_m(r)+y_m(r)\bigr)+1\bigr]B'(r),
\end{aligned}$$

$$\begin{aligned}
z_m'(r) &= \mathcal{R}^{-1}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}h(s)\varphi(y_m(s))\,\mathrm{d}s\right)\\
&\leqslant \mathcal{R}^{-1}\left(\varphi\bigl(y_m(r)\bigr)\frac{1}{r^{n-1}}\int_0^r s^{n-1}h(s)\,\mathrm{d}s\right)\\
&\leqslant \mathcal{R}^{-1}\left(\bigl(\varphi\bigl(y_m(r)\bigr)+1\bigr)\frac{1}{r^{n-1}}\int_0^r s^{n-1}h(s)\,\mathrm{d}s\right)\\
&\leqslant \bigl(\varphi\bigl(y_m(r)\bigr)+1\bigr)^{1/(p-1)}\mathcal{R}^{-1}\left(\frac{1}{r^{n-1}}\int_0^r s^{n-1}h(s)\,\mathrm{d}s\right)\\
&\leqslant \bigl(\varphi\bigl(y_m(r)\bigr)+1\bigr)H'(r)\\
&\leqslant \bigl[\psi\bigl(z_m(r)+y_m(r)\bigr)+\varphi\bigl(z_m(r)+y_m(r)\bigr)+1\bigr]H'(r),
\end{aligned}$$

and

$$\int_0^r \frac{y_m'(t)+z_m'(t)}{\psi(y_m(t)+z_m(t))+\varphi(y_m(t)+z_m(t))+1}\,\mathrm{d}t \leqslant H(r)+B(r).$$

Consequently,

$$A\big(y_m(r)+z_m(r)\big)-A(2\gamma)\leqslant B(r)+H(r)\quad \forall r\geqslant 0. \tag{7}$$

Because of the monotonicity of A^{-1}, we get

$$y_m(r)+z_m(r)\leqslant A^{-1}\big(A(2\gamma)+B(r)+H(r)\big)\quad \forall r\geqslant 0. \tag{8}$$

Since $A(\infty)=\infty$, we can know that $A^{-1}(\infty)=\infty$.

It follows that $\{y_m\}$ and $\{z_m\}$ are the bounded and equicontinuous sequences on $[0,c_0]$ for arbitrary $c_0>0$. Using the Arzela–Ascoli theorem, we can get that the subsequences of $\{y_m\}$ and $\{z_m\}$ converge uniformly to y and z on $[0,c_0]$, respectively. It follows from the arbitrariness of $c_0>0$ that (y,z) is a positive radial solution of system (1). Then, according to the arbitrariness of the initial value $\gamma\in(0,\infty)$, we can know that system (1) has infinitely many positive radial solutions. Moreover, it follows from $B(\infty)<\infty$, $H(\infty)<\infty$ and (8) that

$$\begin{aligned} y(r)+z(r)&\leqslant A^{-1}\big(A(2\gamma)+B(r)+H(r)\big)\\ &\leqslant A^{-1}\big(A(2\gamma)+B(\infty)+H(\infty)\big)\\ &<\infty\quad \forall r\geqslant 0. \end{aligned}$$

Thus, the Schrödinger system (1) has infinitely many positive bounded radial solutions (y,z).

The proof is completed. □

Theorem 2. *Assume that* (A2), (A4) *and* (A5) *hold and there exists a constant* $\gamma>a/2$ *such that*

$$B(\infty)+H(\infty)<A(\infty)-A(2\gamma).$$

Then the Schrödinger system (1) *has infinitely many positive radial solutions* $(y,z)\in C^2[0,\infty)\times C^2[0,\infty)$ *satisfying*

$$\begin{aligned} &\gamma+\left(\min\left\{\frac{1}{2},\psi(\gamma)\right\}\right)^{1/(p-1)}B(r)\\ &\quad\leqslant y(r)\leqslant A^{-1}\big(A(2\gamma)+B(r)+H(r)\big)\quad \forall r\geqslant 0 \end{aligned}$$

and

$$\begin{aligned} &\gamma+\left(\min\left\{\frac{1}{2},\varphi(\gamma)\right\}\right)^{1/(p-1)}H(r)\\ &\quad\leqslant z(r)\leqslant A^{-1}\big(A(2\gamma)+B(r)+H(r)\big)\quad \forall r\geqslant 0. \end{aligned}$$

Proof. By the proof of Theorem 1, we can know that the Schrödinger system (1) has infinitely many positive radial solutions. Then, since $A(\infty)<\infty$, $B(\infty)<\infty$, $H(\infty)<\infty$ and there exists $\gamma>a/2$ such that $B(\infty)+H(\infty)<A(\infty)-A(2\gamma)$, we can know by (7) that

$$A\big(y_m(r)+z_m(r)\big)\leqslant A(2\gamma)+B(r)+H(r)<A(\infty)<\infty.$$

Because of the monotonicity of A^{-1}, we get

$$y_m(r) + z_m(r) \leqslant A^{-1}\big(A(2\gamma) + B(r) + H(r)\big) < \infty \quad \forall r \geqslant 0. \tag{9}$$

Taking the limit in (9), we see that

$$y(r) + z(r) \leqslant A^{-1}\big(A(2\gamma) + B(r) + H(r)\big) < \infty \quad \forall r \geqslant 0.$$

It follows from Lemma 1 and the monotonicity of ψ and φ that

$$\begin{aligned} y(r) &= y(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi\big(z(s)\big)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \int_0^r \mathcal{R}^{-1}\left(\psi(\gamma)\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \left(\min\left\{\frac{1}{2}, \psi(\gamma)\right\}\right)^{1/(p-1)} \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \left(\min\left\{\frac{1}{2}, \psi(\gamma)\right\}\right)^{1/(p-1)} B(r) \end{aligned}$$

and

$$\begin{aligned} z(r) &= z(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi\big(y(s)\big)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \int_0^r \mathcal{R}^{-1}\left(\varphi(\gamma)\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \left(\min\left\{\frac{1}{2}, \varphi(\gamma)\right\}\right)^{1/(p-1)} \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &\geqslant \gamma + \left(\min\left\{\frac{1}{2}, \varphi(\gamma)\right\}\right)^{1/(p-1)} H(r). \end{aligned}$$

The proof is completed. □

3 Existence and nonexistence of the positive blow-up radical solutions

Next, we are concerned with the existence and nonexistence of the positive blow-up radical solutions to the Schrödinger elliptic system (1) involving a nonlinear operator.

Theorem 3. *Assume that* (A1), (A3) *and* (A5) *hold. Then the Schrödinger system* (1) *has infinitely many positive blow-up radial solutions* $(y, z) \in C^2[0,\infty) \times C^2[0,\infty)$.

Proof. It is easy to know that the Schrödinger system (1) has infinitely many positive radial solutions. Moreover,

$$y(r) \geqslant \gamma + \left(\min\left\{\frac{1}{2}, \psi(\gamma)\right\}\right)^{1/(p-1)} B(r)$$

and

$$z(r) \geqslant \gamma + \left(\min\left\{\frac{1}{2}, \varphi(\gamma)\right\}\right)^{1/(p-1)} H(r),$$

which imply $\lim_{r\to\infty} y(r) = \infty$ and $\lim_{r\to\infty} z(r) = \infty$. Thus, the Schrödinger system (1) has infinitely many positive blow-up radial solutions (y, z).

The proof is completed. □

Theorem 4. *Assume that* (A3), (A5) *and* (A6) *hold. Then the Schrödinger system* (1) *has infinitely many positive entire blow-up radial solutions* $(y, z) \in C^2[0,\infty) \times C^2[0,\infty)$.

Proof. We firstly show that (1) has a positive radial solution. We research the integral form of system (1)

$$\begin{aligned} y(r) &= y(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s) \psi\big(z(s)\big)\,\mathrm{d}s\right) \mathrm{d}t, \quad r \geqslant 0, \\ z(r) &= z(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \varphi\big(y(s)\big)\,\mathrm{d}s\right) \mathrm{d}t, \quad r \geqslant 0. \end{aligned} \tag{10}$$

We want to generate two positive increasing sequences $\{y_m\}_{m\geqslant 1}$ and $\{z_m\}_{m\geqslant 0}$, which are bounded above on $[0, L]$ for fixed $L > 0$. Let $\{y_m\}_{m\geqslant 1}$ and $\{z_m\}_{m\geqslant 0}$ be as defined in Theorem 1 because of the monotonic increasing property of ψ and φ, we already know that the sequences $\{y_m\}$ and $\{z_m\}$ are increasing. Next, what we need to prove is that the sequences $\{z_m(r)\}_{m\geqslant 0}$ and $\{y_m(r)\}_{m\geqslant 1}$ are bounded on $[0, L]$ for fixed $L > 0$.

We can see that $y_m'(t) \geqslant 0$ and $z_m'(t) \geqslant 0$ from Theorem 1. By (A6) and Lemma 1 we fix $L > 0$ and have

$$\begin{aligned} z_m(L) &= \gamma + \int_0^L \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \varphi\big(y_m(s)\big)\,\mathrm{d}s\right) \mathrm{d}t \\ &\leqslant \gamma + \int_0^L \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \big(\lambda_2 y_m^\beta(s) + \mu_2\big)\,\mathrm{d}s\right) \mathrm{d}t \\ &\leqslant \gamma + \int_0^L \mathcal{R}^{-1}\left(\big(\lambda_2 y_m^\beta(t) + \mu_2 + 1\big) \frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s)\,\mathrm{d}s\right) \mathrm{d}t \end{aligned}$$

$$\leqslant \gamma + \left(\lambda_2 y_m^\beta(L) + \mu_2 + 1\right)^{1/(p-1)} H(L)$$
$$\leqslant \gamma + \left(\lambda_1 z_m^\alpha(L) + \lambda_2 y_m^\beta(L) + \mu_1 + \mu_2 + 1\right)^{1/(p-1)} H(L).$$

Similarly,

$$y_m(L) \leqslant \gamma + \left(\lambda_1 z_m^\alpha(L) + \lambda_2 y_m^\beta(L) + \mu_1 + \mu_2 + 1\right)^{1/(p-1)} B(L).$$

Thus,

$$z_m(L) + y_m(L) \leqslant 2\gamma + \left(\lambda_1 z_m^\alpha(L) + \lambda_2 y_m^\beta(L) + \mu_1 + \mu_2 + 1\right)^{1/(p-1)} \times \left\{B(L) + H(L)\right\}. \tag{11}$$

Denote

$$M(L) := \lim_{m\to\infty} \left(z_m(L)\right) \quad \text{and} \quad N(L) := \lim_{m\to\infty} \left(y_m(L)\right).$$

It is sure that $M(L)$ and $N(L)$ are finite. Otherwise, by (11) one has

$$1 \leqslant \frac{2\gamma}{z_m(L) + y_m(L)} + \frac{(\lambda_1 z_m^\alpha(L) + \lambda_2 y_m^\beta(L) + \mu_1 + \mu_2 + 1)^{1/(p-1)}}{z_m(L) + y_m(L)} \times \left\{H(L) + B(L)\right\} \to 0$$

as $m \to \infty$, which is obviously a mistake. Hence $M(L)$ and $N(L)$ are finite. It follows from the fact, $z_m(r)$ and $y_m(r)$ are two increasing functions, that M and N are also increasing mappings on $(0, \infty)$. Thus, for all $r \in [0, L]$ and $m \geqslant 1$, one can know that

$$\gamma \leqslant z_m(r) \leqslant z_m(L) \leqslant M(L) \quad \text{and} \quad \gamma \leqslant y_m(r) \leqslant y_m(L) \leqslant N(L),$$

which imply that the sequences $\{z_m(r)\}_{m\geqslant 0}$ and $\{y_m(r)\}_{m\geqslant 1}$ are bounded on $[0, L]$. Let

$$z(r) := \lim_{m\to\infty} z_m(r) \geqslant 0 \quad \text{and} \quad y(r) := \lim_{m\to\infty} y_m(r) \quad \forall r \geqslant 0.$$

Now, we take the limit in (6) and conclude that (y, z) is a positive solution of (10). Next, what we need to prove is that $y(r)$ and $z(r)$ are blow-up. In fact, it follows from (10) that

$$z(r) = z(0) + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \varphi\left(y(s)\right) \mathrm{d}s\right) \mathrm{d}t$$
$$\geqslant \gamma + \int_0^r \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \varphi(\gamma) \, \mathrm{d}s\right) \mathrm{d}t$$
$$\geqslant \gamma + \left(\min\left\{\frac{1}{2}, \varphi(\gamma)\right\}\right)^{1/(p-1)} H(r).$$

Similarly,

$$y(r) \geqslant \gamma + \left(\min\left\{\frac{1}{2}, \psi(\gamma)\right\}\right)^{1/(p-1)} B(r).$$

Since $B(\infty) = \infty$ and $H(\infty) = \infty$, the right-hand side of the above two inequalities tends to $+\infty$ as $r \to +\infty$, which suggests that (y, z) is a blow-up solution of system (1). According to the arbitrariness of the initial value $\gamma \in (0, \infty)$, we can know that the Schrödinger system (1) has infinitely many positive entire blow-up radial solutions.

The proof is completed. □

Theorem 5. *Assume that* (A4) *and* (A6) *hold. Then the Schrödinger system* (1) *has no positive entire blow-up radial solution.*

Proof. We assume that system (1) has a positive entire blow-up radial solution (y, z). Since $B(\infty) < \infty$, $H(\infty) < \infty$, we see that there exists a large enough $r_0 > 0$ such that

$$\int_{r_0}^{\infty} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s)\,\mathrm{d}s\right) \mathrm{d}t < \frac{1}{2},$$

$$\int_{r_0}^{\infty} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s)\,\mathrm{d}s\right) \mathrm{d}t < \frac{1}{2}.$$

Thus, by Lemma 1 and (A6), for $r > r_0 > 0$, one has

$$\begin{aligned}
y(r) &= y(r_0) + \int_{r_0}^{r} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s) \psi\bigl(z(s)\bigr)\,\mathrm{d}s\right) \mathrm{d}t \\
&\leqslant y(r_0) + \int_{r_0}^{r} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s)\bigl(\lambda_1 z^{\alpha}(s) + \mu_1\bigr)\,\mathrm{d}s\right) \mathrm{d}t \\
&\leqslant y(r_0) + \int_{r_0}^{r} \bigl(\lambda_1 z^{\alpha}(t) + \mu_1 + 1\bigr)^{1/(p-1)} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s)\,\mathrm{d}s\right) \mathrm{d}t \\
&\leqslant y(r_0) + \bigl(\lambda_1 z^{\alpha}(r) + \lambda_2 y^{\beta}(r) + \mu_1 + \mu_2 + 1\bigr)^{1/(p-1)} \\
&\quad \times \int_{r_0}^{r} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s)\,\mathrm{d}s\right) \mathrm{d}t.
\end{aligned}$$

Similarly,

$$\begin{aligned}
z(r) &\leqslant z(r_0) + \bigl(\lambda_1 z^{\alpha}(r) + \lambda_2 y^{\beta}(r) + \mu_1 + \mu_2 + 1\bigr)^{1/(p-1)} \\
&\quad \times \int_{r_0}^{r} \mathcal{R}^{-1}\left(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s)\,\mathrm{d}s\right) \mathrm{d}t.
\end{aligned}$$

Thus,

$$
\begin{aligned}
y(r)+z(r) &\leqslant y(r_0)+z(r_0)+\left(\lambda_1 z^\alpha(r)+\lambda_2 y^\beta(r)+\mu_1+\mu_2+1\right)^{1/(p-1)} \\
&\quad\times\left\{\int_{r_0}^{r}\mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\,\mathrm{d}s\right)\mathrm{d}t\right. \\
&\quad\left.+\int_{r_0}^{r}\mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\,\mathrm{d}s\right)\mathrm{d}t\right\} \\
&\leqslant y(r_0)+z(r_0)+\left(\lambda_1 z^\alpha(r)+\lambda_2 y^\beta(r)+\mu_1+\mu_2+1\right)^{1/(p-1)}.
\end{aligned}
$$

Since $0<\alpha/(p-1)<1$ and $0<\beta/(p-1)<1$, we can see that $y(r)$ and $z(r)$ are bounded, which is obviously contradictory to the assumption. Hence problem (1) has no positive entire blow-up radial solution.

The proof is completed. □

4 Examples

Example 1. Consider the following nonlinear Schrödinger elliptic system:

$$
\begin{aligned}
&\operatorname{div}\left(\mathcal{G}\left(|\nabla y|^3\right)\nabla y\right)=b\left(|x|\right)\psi(z), \quad x\in\mathbb{R}^4,\\
&\operatorname{div}\left(\mathcal{G}\left(|\nabla z|^3\right)\nabla z\right)=h\left(|x|\right)\varphi(y), \quad x\in\mathbb{R}^4,
\end{aligned}
\tag{12}
$$

where $b(s)=s^3/5$, $h(s)=s^2+\mathrm{e}^s$, $\psi(s)=s^{17}/(2s^4+1)^4$ and $\varphi(s)=\arctan s$. $\psi(s)$ and $\varphi(s)$ are increasing on $[0,\infty)$, which satisfies (A5). Note $\mathcal{G}(s)=s^3$, $p=5$, then $\mathcal{G}\in\Theta$. After a simple calculation, one has

$$
\begin{aligned}
B(\infty)&=\int_0^\infty\mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\,\mathrm{d}s\right)\mathrm{d}t=\int_0^\infty\left(\frac{1}{t^3}\int_0^t s^3\frac{s^3}{5}\,\mathrm{d}s\right)^{1/10}\mathrm{d}t\\
&=\int_0^\infty\left(\frac{1}{t^3}\int_0^t\frac{s^6}{5}\,\mathrm{d}s\right)^{1/10}\mathrm{d}t=\frac{1}{\sqrt[10]{35}}\int_0^\infty t^{2/5}\,\mathrm{d}t=\infty
\end{aligned}
$$

and

$$
\begin{aligned}
H(\infty)&=\int_0^\infty\mathcal{R}^{-1}\left(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\,\mathrm{d}s\right)\mathrm{d}t=\int_0^\infty\left(\frac{1}{t^3}\int_0^t s^3(s^2+\mathrm{e}^s)ds\right)^{1/10}\mathrm{d}t\\
&\geqslant\int_0^\infty\left(\frac{1}{t^3}\int_0^t s^3(s^2+s^2)ds\right)^{1/10}\mathrm{d}t=2^{1/10}\int_0^\infty\left(\frac{1}{t^3}\int_0^t s^5\,\mathrm{d}s\right)^{1/10}\mathrm{d}t\\
&=\frac{1}{\sqrt[10]{3}}\int_0^\infty t^{3/10}\,\mathrm{d}t=\infty,
\end{aligned}
$$

which means that (A3) is satisfied.

$$A(\infty) = \int\limits_a^\infty \frac{\mathrm{d}s}{\psi(s)+\varphi(s)+1} = \int\limits_a^\infty \frac{\mathrm{d}s}{\frac{s^{17}}{(2s^4+1)^4} + \arctan s + 1} \geqslant \int\limits_a^\infty \frac{\mathrm{d}s}{2s+1} = \infty,$$

which means that (A1) is satisfied. Thus, by Theorem 3 one can see that the Schrödinger system (12) has infinitely many positive blow-up radial solutions.

Example 2. Consider the following nonlinear Schrödinger elliptic system:

$$\begin{aligned} \operatorname{div}\bigl(\mathcal{G}\bigl(|\nabla y|^5\bigr)\nabla y\bigr) &= \frac{|x|}{6}\ln(z+1), \quad x \in \mathbb{R}^5, \\ \operatorname{div}\bigl(\mathcal{G}\bigl(|\nabla z|^5\bigr)\nabla z\bigr) &= \bigl(|x|^5+1\bigr)\arctan y, \quad x \in \mathbb{R}^5. \end{aligned} \tag{13}$$

Note $\mathcal{G}(s) = s^5$, $p = 7$, $b(s) = s/6$, $h(s) = s^5+1$, $\psi(s) = \ln(s+1)$ and $\varphi(s) = \arctan s$, then $\mathcal{G} \in \Theta$. We see that $\psi(s)$ and $\varphi(s)$ are increasing on $[0,\infty)$, which satisfies (A5). After a simple calculation, one has

$$\begin{aligned} B(\infty) &= \int\limits_0^\infty \mathcal{R}^{-1}\Biggl(\frac{1}{t^{n-1}}\int\limits_0^t s^{n-1}b(s)\,\mathrm{d}s\Biggr)\mathrm{d}t = \int\limits_0^\infty \Biggl(\frac{1}{t^4}\int\limits_0^t s^4\frac{s}{6}\,\mathrm{d}s\Biggr)^{1/26}\mathrm{d}t \\ &= \int\limits_0^\infty \Biggl(\frac{1}{t^4}\int\limits_0^t \frac{s^5}{6}\,\mathrm{d}s\Biggr)^{1/26}\mathrm{d}t = \int\limits_0^\infty t^{1/13}\,\mathrm{d}t = \infty \end{aligned}$$

and

$$\begin{aligned} H(\infty) &= \int\limits_0^\infty \mathcal{R}^{-1}\Biggl(\frac{1}{t^{n-1}}\int\limits_0^t s^{n-1}h(s)\,\mathrm{d}s\Biggr)\mathrm{d}t = \int\limits_0^\infty \Biggl(\frac{1}{t^4}\int\limits_0^t s^4(s^5+1)\,\mathrm{d}s\Biggr)^{1/26}\mathrm{d}t \\ &= \int\limits_0^\infty \biggl(\frac{t^6}{10}+\frac{t}{5}\biggr)^{1/26}\mathrm{d}t \geqslant \int\limits_0^\infty \biggl(\frac{t^6}{10}\biggr)^{1/26}\mathrm{d}t = \infty, \end{aligned}$$

which means that (A3) is satisfied. Obviously, $0 \leqslant \psi(x), \varphi((x)) \leqslant x$ for all $x \in [0,\infty)$, where $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 0$, $\alpha = \beta = 1$. So, (A6) is satisfied. Thus, by Theorem 4 one can see that the Schrödinger system (13) has infinitely many positive entire blow-up radial solutions.

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