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Controllability of conformable differential systems*

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Abstract. This paper deals with complete controllability of systems governed by linear and semilinear conformable differential equations. By establishing conformable Gram criterion and rank criterion, we give sufficient and necessary conditions to examine that a linear conformable system is null completely controllable. Further, we apply Krasnoselskii's fixed point theorem to derive a completely controllability result for a semilinear conformable system. Finally, three numerical examples are given to illustrate our theoretical results.

Keywords: complete controllability, conformable fractional differential systems, Gram and rank criterion.

1 Introduction

Conformable calculus and equations has a rapid development in basic theory and application in many fields. For example, Khan and Khan [9] concerned the open problem in Abdeljawad [1] and introduced the generalized conformable operators, which are the generalizations of Katugampola, Riemann–Liouville, and Hadamard fractional operators. Bendouma and Hammoudi [3] established the conformable dynamic equations on time scales with nonlinear functional boundary value conditions and obtained the existence

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of solutions. Bohner and Hatipoğlu [4] used conformable derivatives to establish thenew dynamic cobweb models and obtained the general solutions and stability criteria. Abdeljawad et al. [2] proposed conformable quadratic and cubic logistic models and obtained existence theorems and stability of solutions. Jaiswal and Bahuguna [7] proposed conformable abstract Cauchy problems via semigroup theory, introduced the concept of mild and strong solution, and obtained existence and uniqueness theorem. Bouaouid et al. [5] investigated nonlocal problems for second-order evolution differential equation in the frame of sequential conformable derivatives and presented Duhamel's formula and existence, stability, and regularity of mild solutions. Martínez et al. [11] applied this new conformable derivative to analyze RC, LC, and RLC electric circuits described by linear differential equations with noninteger power variable coefficients derivative. However, there are quite a few papers on controllability of systems governed by conformable differential equations.

In this paper, we study controllability of linear and semilinear conformable control systems governed by

$$\mathfrak{D}_\alpha^0 x(t) = Mx(t) + Qu_1(t), \quad t \in J := [0, t_1],\ t_1 > 0, \qquad x(0) = x_0, \tag{1}$$

$$\mathfrak{D}_\alpha^0 x(t) = Mx(t) + f\bigl(t, x(t)\bigr) + Qu(t), \quad t \in J, \qquad x(0) = x_0, \tag{2}$$

where $\mathfrak{D}_\alpha^0$ $(0 < \alpha < 1)$ denotes the conformable derivative with lower index zero (see Definition 1), $M \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times r}$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$. The state $x(\cdot)$ take values from $\mathbb{R}^n$, the control functions $u_1(\cdot)$ and $u(\cdot)$ belong to $L^2(J, \mathbb{R}^r)$.

The main contributions are stated as follows: (i) We establish conformable Gram criterion and rank criterion to give the necessary and sufficient conditions to guarantee (1) is null completely controllable. The corresponding control function is also presented. (ii) We construct a suitable control function and apply Krasnoselskii's fixed point to derive complete controllability of (2).

2 Preliminaries and notation

Let $\mathbb{R}^n$ be the n-dimensional Euclid space with the vector norm $\|\cdot\|$ and $\mathbb{R}^{n\times n}$ be the $n\times n$ matrix space with real value elements. Denote by $\mathcal{C}(J, \mathbb{R}^n)$ the Banach space of vector-value continuous functions from $J \to \mathbb{R}^n$ endowed with the norm $\|x\|_{\mathcal{C}} = \sup_{t\in J} \|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. Let X, Y be two Banach spaces, $L_b(X, Y)$ denotes the space of all bounded linear operators from X to Y, and $L^p(J, Y)$ denotes the Banach space of all the Bochner-integrable functions endowed with $\|\cdot\|_{L^p(J,Y)}$ for some $1 < p < \infty$. For $M : \mathbb{R}^n \to \mathbb{R}^n$, we consider its matrix norm $\|M\| = \sup_{\|x\|=1} \|Mx\|$ generated by $\|\cdot\|$. $\mathbf{0}$ denotes the n-dimensional zero vector.

Definition 1. (See [8, Def. 2.1].) The conformable derivative with lower index a of a function $x : [a, \infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_\beta^a x(t) = \lim_{\varepsilon\to 0} \frac{x(t + \varepsilon(t-a)^{1-\beta}) - x(t)}{\varepsilon}, \quad t > a,\ 0 < \beta \leqslant 1,$$

$$\mathfrak{D}_\beta^a x(a) = \lim_{t\to a^+} \mathfrak{D}_\beta^a x(t).$$

Remark 1. If $\mathfrak{D}_\beta^a x(t_0)$ exists and is finite, we consider that x is β-differentiable at t_0. If $x : [a, \infty) \to \mathbb{R}$ is a once continuous differential function, then $\mathfrak{D}_\beta^a x(t) = (t-a)^{1-\beta} \times x'(t)$.

Definition 2. (See [10, Thm. 3.3].) A solution $x \in \mathcal{C}(J, \mathbb{R}^n)$ of system (1) has the following form:

$$x(t) = \mathrm{e}^{Mt^\alpha/\alpha} x_0 + \int_0^t \mathrm{e}^{M(t^\alpha - \tau^\alpha)/\alpha} Q u_1(\tau)\, \mathrm{d}\frac{\tau^\alpha}{\alpha}. \tag{3}$$

Obviously, a solution $x \in \mathcal{C}(J, \mathbb{R}^n)$ of system (2) has the following form:

$$x(t) = \mathrm{e}^{Mt^\alpha/\alpha} x_0 + \int_0^t \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} \bigl(f\bigl(\tau, x(\tau)\bigr) + Qu(\tau)\bigr)\, \mathrm{d}\frac{\tau^\alpha}{\alpha}. \tag{4}$$

Definition 3. System (1) is called null completely controllable on J if for an arbitrary initial vector function x_0, the terminal state vector $\mathbf{0} \in \mathbb{R}^n$, and terminal time t_1, there exists a control $u_1 \in L^2(J, \mathbb{R}^r)$ such that the state $x \in \mathcal{C}(J, \mathbb{R}^n)$ of system (1) satisfies $x(t_1) = \mathbf{0}$.

Definition 4. System (2) is called completely controllable if for an arbitrary initial vector function x_0, for the terminal state of vector $x_1 \in \mathbb{R}^n$ and time t_1, there exists a control $u \in L^2(J, \mathbb{R}^r)$ such that the state $x \in \mathcal{C}(J, \mathbb{R}^n)$ of system satisfies $x(t_1) = x_1$.

Lemma 1 [Krasnoselskii's fixed point theorem]. *Let $\mathcal{B}$ be a bounded closed and convex subset of Banach space X, and let F_1, F_2 be maps of $\mathcal{B}$ into X such that $F_1 x + F_2 y \in \mathcal{B}$ for every pair $x, y \in \mathcal{B}$. If F_1 is a contraction and F_2 is compact and continuous, then the equation $F_1 x + F_2 x = x$ has a solution on $\mathcal{B}$.*

3 Controllability results

3.1 Linear systems

In this section, we are going to investigate the null completely controllable of system (1).

We introduce a notation of a conformable Gram matrix as follows:

$$W_c[0, t_1] := \int_0^{t_1} \mathrm{e}^{-M\tau^\alpha/\alpha} Q Q^\top \mathrm{e}^{M(t_1^\alpha - \tau^\alpha)/\alpha}\, \mathrm{d}\frac{\tau^\alpha}{\alpha}, \tag{5}$$

where the $\top$ denotes the transpose of the matrix. Then we will give the first controllability result.

Theorem 1. *System* (1) *is null completely controllable if and only if $W_c[0, t_1]$ defined in* (5) *is nonsingular.*

Proof. Sufficiency. Owing to $W_c[0, t_1]$ is nonsingular, its inverse $W_c^{-1}[0, t_1]$ is well defined. For any nonzero state x in the state space, the corresponding control input $u_1(\tau)$ can be constructed as:

$$u_1(\tau) = -Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} W_c^{-1}[0, t_1] x_0,\ \tau \in [0, t_1]. \tag{6}$$

According to (3), for all $x_0 \in \mathbb{R}^n$, one can get

$$\begin{aligned}
x(t_1) &= \mathrm{e}^{M t_1^\alpha/\alpha} x_0 + \int_0^{t_1} \mathrm{e}^{M(t_1^\alpha - \tau^\alpha)/\alpha} Q u_1(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&= \mathrm{e}^{M t_1^\alpha/\alpha} x_0 + \int_0^{t_1} \mathrm{e}^{M(t_1^\alpha - \tau^\alpha)/\alpha} Q\bigl(-Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} W_c^{-1}[0, t_1] x_0\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&= \mathrm{e}^{M t_1^\alpha/\alpha} x_0 - \mathrm{e}^{M t_1^\alpha/\alpha} \int_0^{t_1} \mathrm{e}^{-M \tau^\alpha/\alpha} Q Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} W_c^{-1}[0, t_1] x_0\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&= \mathrm{e}^{M t_1^\alpha/\alpha} x_0 - \mathrm{e}^{M t_1^\alpha/\alpha} W_c[0, t_1] W_c^{-1}[0, t_1] x_0 \\
&= \mathrm{e}^{M t_1^\alpha/\alpha} x_0 - \mathrm{e}^{M t_1^\alpha/\alpha} x_0 = \mathbf{0}.
\end{aligned}$$

Necessity. Assume $W_c[0, t_1]$ is singular. There exists at least one nonzero state $\bar{x}_0 \in \mathbb{R}^n$ such that $\bar{x}_0^\top W_c[0, t_1] \bar{x}_0 = 0$.

Consequently, we can get

$$\begin{aligned}
0 = \bar{x}_0^\top W_c[0, t_1] \bar{x}_0 &= \int_0^{t_1} \bar{x}_0^\top \mathrm{e}^{-M \tau^\alpha/\alpha} Q Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \bar{x}_0\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&= \int_0^{t_1} \bigl[Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \bar{x}_0\bigr]^\top \bigl[Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \bar{x}_0\bigr]\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&= \int_0^{t_1} \bigl\|Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \bar{x}_0\bigr\|^2\,\mathrm{d}\frac{\tau^\alpha}{\alpha},
\end{aligned}$$

which implies that

$$Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \bar{x}_0 = \mathbf{0} \quad \forall \tau \in [0, t_1]. \tag{7}$$

Owing to system (1) is relatively controllable, according to Definition 3, there exists a control $u(\tau)$ that drives the initial state to zero at t_1, that is,

$$x(t_1) = \mathrm{e}^{M t_1^\alpha/\alpha} \bar{x}_0 + \int_0^{t_1} \mathrm{e}^{M t_1^\alpha/\alpha} \mathrm{e}^{-M \tau^\alpha/\alpha} Q u_1(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} = \mathbf{0}. \tag{8}$$

According to (8), we can have

$$\bar{x}_0 = -\int_0^{t_1} \mathrm{e}^{-M\tau^\alpha/\alpha} Q u_1(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}.$$

Then we get

$$\begin{aligned}\|\bar{x}_0\|^2 = \bar{x}_0^\top \bar{x}_0 &= \left[-\int_0^{t_1} \mathrm{e}^{-M\tau^\alpha/\alpha} Q u_1(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\right]^\top \bar{x}_0 \\ &= -\int_0^{t_1} u_1^\top(\tau)\big[Q^\top \mathrm{e}^{-M^\top\tau^\alpha/\alpha}\bar{x}_0\big]\,\mathrm{d}\frac{\tau^\alpha}{\alpha}. \end{aligned} \tag{9}$$

By (7) and (9), one can get $\|\bar{x}_0\|^2 = 0$, that is

$$\bar{x}_0 = \mathbf{0}, \tag{10}$$

which contradicts the conditions of the $\bar{x}_0 \neq \mathbf{0}$. Thus, $W_c[0, t_1]$ is nonsingular. The proof is complete. □

Now, we introduce a notation of a rank criterion as follows:

$$\Gamma_c = \big[Q \;\; MQ \;\; \ldots \;\; M^{n-1}Q\big]. \tag{11}$$

Then we are ready to give the second controllability result.

Theorem 2. *The necessary and sufficient condition for null complete controllability of* (1) *is* $\operatorname{rank}\Gamma_c = n$.

Proof. Sufficiency. Assuming that system (1) is not controllable. By Theorem 1, $W_c[0, t_1]$ is nonsingular. Namely, there exists at least one nonzero state vector β such that

$$\begin{aligned}0 = \beta^\top W_c[0, t_1]\beta &= \int_0^{t_1} \beta^\top \mathrm{e}^{-M\tau^\alpha/\alpha} Q Q^\top \mathrm{e}^{-M\tau^\alpha/\alpha}\beta\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &= \int_0^{t_1} \big[\beta^\top \mathrm{e}^{-M\tau^\alpha/\alpha} Q\big]\big[\beta^\top \mathrm{e}^{-M\tau^\alpha/\alpha}\big]^\top \mathrm{d}\frac{\tau^\alpha}{\alpha},\end{aligned}$$

which implies that

$$\beta^\top \mathrm{e}^{-M\tau^\alpha/\alpha} Q = \mathbf{0} \quad \forall \tau \in [0, t_1]. \tag{12}$$

For (12), find the derivative of $z = \tau^\alpha/\alpha$ to $n-1$ times and then take $\tau = 0$. We have

$$\beta^\top Q = \mathbf{0}, \quad \beta^\top M Q = \mathbf{0}, \quad \ldots, \quad \beta^\top M^{n-1} Q = \mathbf{0},$$

that is,

$$\beta^\top \begin{bmatrix} Q & MQ & \dots & M^{n-1}Q \end{bmatrix} = \mathbf{0}. \tag{13}$$

According to $\beta \neq 0$, we can know $\mathit{\Gamma}_c < n$, which contradicts to our assumption. So system (1) is controllable.

Necessity. Assume $\operatorname{rank} \mathit{\Gamma}_c < n$. Namely, there exists at least one nonzero state β in $\mathbb{R}^n$ such that

$$\beta^\top \mathit{\Gamma}_c = \beta^\top \begin{bmatrix} Q & MQ & \dots & M^{n-1}Q \end{bmatrix} = \mathbf{0}. \tag{14}$$

From (14) we obtain

$$\beta^\top M^i Q = \mathbf{0}, \quad i = 0, 1, \dots, n-1. \tag{15}$$

According to Cayley–Hmilton theorem, M^n, M^{n+1} can be expressed as a linear combination of I, M, ..., M^{n-1}. Then the upper form can be expanded to

$$\beta^\top M^i Q = \mathbf{0}, \quad i = 0, 1, \dots .$$

For all $t > 0$, one can obtain

$$\pm\beta^\top \frac{M^i(\frac{\tau^\alpha}{\alpha})^i}{i!} Q = \mathbf{0} \quad \forall \tau \in [0, t_1],\ i = 0, 1, \dots,$$

or

$$\begin{aligned} \mathbf{0} &= \beta^\top \left[I - M\left(\frac{\tau^\alpha}{\alpha}\right) + \frac{1}{2!}M^2\left(\frac{\tau^\alpha}{\alpha}\right)^2 - \frac{1}{3!}M^3\left(\frac{\tau^\alpha}{\alpha}\right)^3 + \cdots \right] Q \\ &= \beta^\top \mathrm{e}^{-M\tau^\alpha/\alpha} Q \quad \forall \tau \in [0, t_1]. \end{aligned}$$

That is,

$$0 = \beta^\top \int_0^{t_1} \mathrm{e}^{-M\tau^\alpha/\alpha} Q Q^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \,\mathrm{d}\frac{\tau^\alpha}{\alpha} \beta = \beta^\top W_c[0, t_1]\beta. \tag{16}$$

From (16) we can know $W_c[0, t_1]$ is singular, namely, the system is not controllable, which is contradictory to what is known. Thus, $\operatorname{rank} \mathit{\Gamma}_c = n$. The proof is complete. □

3.2 Semilinear systems

We introduce the following assumptions:

(A1) The operator $W : L^2(J, \mathbb{R}^r) \longrightarrow \mathbb{R}^n$ defined by

$$Wu = \int_0^{t_1} \mathrm{e}^{M(t_1^\alpha/\alpha - \tau^\alpha/\alpha)} Q u(\tau) \,\mathrm{d}\frac{\tau^\alpha}{\alpha}$$

has an inverse operator W^{-1}, which takes values in $L^2(J, \mathbb{R}^r) \setminus \ker W$.

Then we set

$$H = \left\| W^{-1} \right\|_{L_b(\mathbb{R}^n, L^2(J, \mathbb{R}^r) \setminus \ker W)}.$$

Remark 2. Obviously, W must be surjective to satisfy (A1). We recommend the reader to see the demonstrated examples in [6, 12]. On the other side, if W is surjective, then we can define an inverse $W^{-1}:\mathbb{R}^n\to L^2(J,\mathbb{R}^r)\backslash\ker W$. We present its natural construction as follows. Let $(\cdot,\cdot)$ denote the Euclid scalar product in $\mathbb{R}^n$. Since $L^2(J,\mathbb{R}^r)$ is a Hilbert space, we can use $\ker W=\operatorname{im} W^{*\perp}$. We need to find W^*. Let $\tilde{W}(\tau)=\mathrm{e}^{M(t_1^\alpha-\tau^\alpha)/\alpha}Q$, and for any $w\in\mathbb{R}^n$ and $u\in L^2(J,\mathbb{R}^r)$, we derive

$$(Wu,w)=\left(\int_0^{t_1}\tilde{W}(\tau)u(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha},\,w\right)=\int_0^{t_1}\bigl(u(\tau),\tilde{W}(\tau)^\top w\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha},$$

which gives $W^*w=\tilde{W}(\tau)^\top w$. Thus $\ker W^*=\{0\}$ if and only if

$$\int_0^{t_1}\bigl\|\tilde{W}(\tau)^\top w\bigr\|^2\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\neq 0$$

for any $0\neq w\in\mathbb{R}^n$. But

$$\begin{aligned}\int_0^{t_1}\bigl\|\tilde{W}(\tau)^\top w\bigr\|^2\,\mathrm{d}\frac{\tau^\alpha}{\alpha}&=\int_0^{t_1}\bigl(\tilde{W}(\tau)^\top w,\tilde{W}(\tau)^\top w\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\\&=\int_0^{t_1}\bigl(\tilde{W}(\tau)\tilde{W}(\tau)^\top w,w\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}=\bigl(W_c[0,t_1]w,w\bigr).\end{aligned}\tag{17}$$

So the surjectivity of W is equivalent to the regularity of $W_c[0,t_1]$, and we assume this. To solve $Wu=v$, $u\in\ker W^\perp=\operatorname{im} W^*$, we take $u(t)=W(t)^\top w$ and then solve

$$v=W\bigl(\tilde{W}(\cdot)^\top w\bigr)=\int_0^{t_1}\tilde{W}(\tau)\tilde{W}(\tau)^\top w\,\mathrm{d}\frac{\tau^\alpha}{\alpha}=W_c[0,t_1]w,$$

which gives $w=W_c[0,t_1]^{-1}v$, and this implies

$$u(t)=W^{-1}w=\tilde{W}(t)^\top W_c[0,t_1]^{-1}v.$$

In addition, by (17), we derive

$$\begin{aligned}\int_0^{t_1}\bigl\|u(\tau)\bigr\|^2\,\mathrm{d}\frac{\tau^\alpha}{\alpha}&=\int_0^{t_1}\bigl\|\tilde{W}(\tau)^\top W_c[0,t_1]^{-1}v\bigr\|^2\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\\&=\int_0^{t_1}\bigl(\tilde{W}(\tau)^\top W_c[0,t_1]^{-1}v,\tilde{W}(\tau)^\top W_c[0,t_1]^{-1}v\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\end{aligned}$$

$$= \int_0^{t_1} \left(\left(W_c[0,t_1]^{-1} \right)^\top \tilde{W}(\tau) \tilde{W}(\tau)^\top W_c[0,t_1]^{-1} v,\, v \right) \mathrm{d}\frac{\tau^\alpha}{\alpha}$$

$$= \left(\left(W_c[0,t_1]^{-1} \right)^\top \int_0^{t_1} \tilde{W}(\tau) \tilde{W}(\tau)^\top \mathrm{d}\frac{\tau^\alpha}{\alpha} W_c[0,t_1]^{-1} v,\, v \right)$$

$$= \left(\left(W_c[0,t_1]^{-1} \right)^\top v,\, v \right) = \left(w,\, W_c[0,t_1]^{-1} v \right),$$

which gives

$$H = \left\| W_c[0,t_1]^{-1} \right\|^{1/2}. \tag{18}$$

We note that (17) also implies

$$\|W\| = \|W^*\| = \left\| W_c[0,t_1] \right\|^{1/2}.$$

(A2) The function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists $L_f(\cdot) \in L^q_\alpha(J, \mathbb{R}^+)$, $q > 1$, i.e., $\int_0^t L_f^q(\tau)\, \mathrm{d}(\tau^\alpha/\alpha) < \infty$, such that

$$\left\| f(t,x_1) - f(t,x_2) \right\| \leqslant L_f(t) \|x_1 - x_2\|, \quad x_i \in \mathbb{R}^n,\ t \in J,\ i = 1,2.$$

In viewing of (A1), for arbitrary $x(\cdot) \in \mathcal{C}(J, \mathbb{R}^n)$, consider a control function $u_x(t)$ given by

$$u_x(t) = W^{-1} \left[x_1 - \mathrm{e}^{M t_1^\alpha/\alpha} x_0 - \int_0^{t_1} \mathrm{e}^{M(t_1^\alpha/\alpha - \tau^\alpha/\alpha)} f\big(\tau, x(\tau)\big)\, \mathrm{d}\frac{\tau^\alpha}{\alpha} \right](t), \quad t \in J. \tag{19}$$

Next, we state our main idea to prove our main result via fixed point method. We firstly show that, using control (19), the operator $\mathcal{P} : \mathcal{C}(J, \mathbb{R}^n) \to \mathcal{C}(J, \mathbb{R}^n)$ defined by

$$(\mathcal{P}x)(t) = \mathrm{e}^{M t^\alpha/\alpha} x_0 + \int_0^{t} \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} f\big(\tau, x(\tau)\big)\, \mathrm{d}\frac{\tau^\alpha}{\alpha}$$
$$+ \int_0^{t} \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} Q u_x(\tau)\, \mathrm{d}\frac{\tau^\alpha}{\alpha}, \quad t \in J,$$

has a fixed point x, which is just a solution of system (2). Then we check $(\mathcal{P}x)(t_1) = x_1$ and $(\mathcal{P}x)(0) = x_0$, which means that u_x steers system (2) from x_0 to x_1 in finite time t_1. This implies system (2) is relatively controllable on J.

For each positive number r, we define $\mathcal{B}_r = \{x \in \mathcal{C}(J, \mathbb{R}^n) \colon \|x\|_{\mathcal{C}} \leqslant r\}$, which is obviously a bound, closed and convex set of $\mathcal{C}(J, \mathbb{R}^n)$. For the sake of brevity, we set $R_f = \sup_{t \in J} \|f(t,0)\|$ and $N = \|M\|$.

In the following, we apply Krasnoselskii's fixed point theorem to derive the relative controllability result for system (2).

Theorem 3. *Assumptions* (A1) *and* (A2) *are satisfied. Then system* (2) *is completely controllable provided that*

$$H_2\left[1+\frac{H}{(2N)^{1/2}}\left(\mathrm{e}^{2Nt_1^\alpha/\alpha}-1\right)^{1/2}\|Q\|\right]<1, \tag{20}$$

where $H_2 = [(\mathrm{e}^{Npt_1^\alpha/\alpha}-1)/(Np)]^{1/p}\|L_f\|_{L^q_\alpha(J,\mathbb{R}^+)}$, $1/p+1/q=1$, $p,q>1$.

Proof. To examine the conditions for Lemma 1, we divide our proof into several steps.

Step 1. We attest that there exists a positive number r such that $\mathcal{P}(\mathcal{B}_r)\subseteq\mathcal{B}_r$. Note

$$\begin{aligned}
&\int_0^t \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}L_f(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\\
&\quad\leqslant\left(\int_0^t \mathrm{e}^{Np(t^\alpha/\alpha-\tau^\alpha/\alpha)}\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\right)^{1/p}\left(\int_0^t L_f^q(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\right)^{1/q}\\
&\quad\leqslant\left[\frac{1}{Np}\left(\mathrm{e}^{Npt^\alpha/\alpha}-1\right)\right]^{1/p}\|L_f\|_{L^q_\alpha(J,\mathbb{R}^+)},\quad t\in J,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\big\|f(\tau,0)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\\
&\quad\leqslant R_f\int_0^t \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\leqslant\frac{R_f}{N}\left(\mathrm{e}^{Nt^\alpha/\alpha}-1\right),\quad t\in J.
\end{aligned}$$

In consideration of (19), using (A1), (A2), and $\|\mathrm{e}^{At}\|\leqslant\mathrm{e}^{\|A\|t}$, $t\in\mathbb{R}$, we have

$$\begin{aligned}
&\|u_x\|_{L^2(J,\mathbb{R}^r)\backslash\ker W}\\
&\quad=\inf_{u_x\in[u_x]}\|u_x\|_{L^2(J,\mathbb{R}^r)}\\
&\quad\leqslant\big\|W^{-1}\big\|_{L_b(\mathbb{R}^n,L^2(J,\mathbb{R}^r)\backslash\ker W)}\\
&\qquad\times\left\|x_1-\mathrm{e}^{Mt_1^\alpha/\alpha}x_0-\int_0^{t_1}\mathrm{e}^{M(t_1^\alpha/\alpha-\tau^\alpha/\alpha)}f\big(\tau,x(\tau)\big)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\right\|\\
&\quad\leqslant H\|x_1\|+H\mathrm{e}^{Nt_1^\alpha/\alpha}\|x_0\|\\
&\qquad+H\int_0^{t_1}\mathrm{e}^{N(t_1^\alpha/\alpha-\tau^\alpha/\alpha)}\big\|f\big(\tau,x(\tau)\big)-f(\tau,0)+f(\tau,0)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\\
&\quad\leqslant H\|x_1\|+H\mathrm{e}^{Nt_1^\alpha/\alpha}\|x_0\|+H\int_0^{t_1}\mathrm{e}^{N(t_1^\alpha/\alpha-\tau^\alpha/\alpha)}L_f(\tau)\big\|x(\tau)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+ H\int\limits_0^{t_1} \mathrm{e}^{N(t_1^\alpha/\alpha-\tau^\alpha/\alpha)}\big\|f(\tau,0)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant H\|x_1\| + H\mathrm{e}^{Nt_1^\alpha/\alpha}\|x_0\| + H\left[\frac{1}{Np}\big(\mathrm{e}^{Npt_1^\alpha/\alpha}-1\big)\right]^{1/p}\|L_f\|_{L_\alpha^q}\|x\|_{\mathcal{C}} \\
&+ \frac{HR_f}{N}\big(\mathrm{e}^{Nt_1^\alpha/\alpha}-1\big) \\
&\leqslant H\|x_1\| + Ha + HH_2r,
\end{aligned}
$$

where $a = \mathrm{e}^{Nt_1^\alpha/\alpha}\|x_0\| + (R_f/N)(\mathrm{e}^{Nt_1^\alpha/\alpha}-1)$, and H_2 is defined in the above.

According to (A1) and (A2), we have

$$
\begin{aligned}
\big\|P(x)(t)\big\| &\leqslant \mathrm{e}^{Nt^\alpha/\alpha}\|x_0\| + \int\limits_0^{t} \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\big\|f\big(\tau,x(\tau)\big)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\quad + \int\limits_0^{t} \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\|Q\|\big\|u_x(\tau)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \mathrm{e}^{Nt^\alpha/\alpha}\|x_0\| + \int\limits_0^{t} \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}L_f(\tau)\big\|x(\tau)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\quad + \int\limits_0^{t} \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\big\|f(\tau,0)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\quad + \left[\int\limits_0^{t} \mathrm{e}^{2N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\right]^{1/2}\|u_x\|_{L^2(J,\mathbb{R}^r)}\|Q\| \\
&\leqslant \mathrm{e}^{Nt^\alpha/\alpha}\|x_0\| + \left[\frac{1}{Np}\big(\mathrm{e}^{Npt^\alpha/\alpha}-1\big)\right]^{1/p}\|L_f\|_{L_\alpha^q}\|x\|_{\mathcal{C}} + \frac{R_f}{N}\big(\mathrm{e}^{Nt^\alpha/\alpha}-1\big) \\
&\quad + \left[\frac{1}{2N}\big(\mathrm{e}^{2Nt^\alpha/\alpha}-1\big)\right]^{1/2}\|Q\|\big[H\|x_1\| + Ha + HH_2r\big] \\
&\leqslant a\left[1 + \left[\frac{H}{(2N)^{1/2}}\big(\mathrm{e}^{2Nt_1^\alpha/\alpha}-1\big)\right]^{1/2}\|Q\|\right] \\
&\quad + \frac{H}{(2N)^{1/2}}\big(\mathrm{e}^{2Nt_1^\alpha/\alpha}-1\big)^{1/2}\|Q\|\|x_1\| \\
&\quad + H_2\left[1 + \frac{H}{(2N)^{1/2}}\big(\mathrm{e}^{2Nt_1^\alpha/\alpha}-1\big)^{1/2}\|Q\|\right]r \\
&= r
\end{aligned}
$$

for

$$r = \frac{a[1 + \frac{H}{(2N)^{1/2}}(\mathrm{e}^{2Nt_1^\alpha/\alpha} - 1)]^{1/2}\|Q\| + \frac{H}{(2N)^{1/2}}(\mathrm{e}^{2Nt_1^\alpha/\alpha} - 1)^{1/2}\|Q\|\|x_1\|}{1 - H_2[1 + \frac{H}{(2N)^{1/2}}(\mathrm{e}^{2Nt_1^\alpha/\alpha} - 1)^{1/2}\|Q\|]}.$$

Therefore, we obtain $\mathcal{P}(\mathcal{B}_r) \subseteq \mathcal{B}_r$ for such an r.

Next, we split $\mathcal{P}$ into two operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\mathcal{B}_r$ as, respectively,

$$(\mathcal{P}_1 x)(t) = \mathrm{e}^{Mt^\alpha/\alpha}x_0 + \int_0^t \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} Q u_x(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}, \quad t \in J,$$

$$(\mathcal{P}_2 x)(t) = \int_0^t \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}, \quad t \in J.$$

Step 2. We prove that $\mathcal{P}_1$ is a contraction mapping.

Let $x, y \in \mathcal{B}_r$. In light of (A1) and (A2), for each $t \in J$, we obtain

$$\begin{aligned}
&\|u_x - u_y\|_{L^2(J,\mathbb{R}^r)\setminus\ker W} \\
&\quad = \left\| W^{-1}\left[\int_0^{t_1} \mathrm{e}^{M(t_1^\alpha/\alpha - \tau^\alpha/\alpha)}\big(f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big)\big)\right]\right\| \\
&\quad \leqslant \big\|W^{-1}\big\|_{L(\mathbb{R}^n, L^2(J,\mathbb{R}^r)\setminus\ker W)} \\
&\qquad \times \left\|\int_0^{t_1} \mathrm{e}^{M(t_1^\alpha/\alpha - \tau^\alpha/\alpha)}\big(f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big)\big)\right\|_{\mathbb{R}^n} \\
&\quad \leqslant H\int_0^{t_1} \mathrm{e}^{N(t_1^\alpha/\alpha - \tau^\alpha/\alpha)} L_f(\tau)\big\|x(\tau) - y(\tau)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\quad \leqslant H\left[\frac{1}{Np}\big(\mathrm{e}^{Npt_1^\alpha/\alpha} - 1\big)\right]^{1/p}\|L_f\|_{L_\alpha^q}\|x - y\|_{\mathcal{C}} \leqslant HH_2\|x - y\|_{\mathcal{C}}.
\end{aligned}$$

From the above fact we get

$$\begin{aligned}
&\big\|(\mathcal{P}_1 x)(t) - (\mathcal{P}_1 y)(t)\big\| \\
&\quad \leqslant \int_0^t \mathrm{e}^{N(t^\alpha/\alpha - \tau^\alpha/\alpha)}\|Q\|\big\|u_x(\tau) - u_y(\tau)\big\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\quad \leqslant \|Q\|\left[\int_0^t \mathrm{e}^{2N(t^\alpha/\alpha - \tau^\alpha/\alpha)}\,\mathrm{d}\frac{\tau^\alpha}{\alpha}\right]^{1/2}\|u_x - u_y\|_{L^2} \\
&\quad \leqslant \|Q\|\left[\frac{1}{2N}\big(\mathrm{e}^{2Nt_1^\alpha/\alpha} - 1\big)\right]^{1/2} HH_2\|x - y\|_{\mathcal{C}},
\end{aligned}$$

which gives that

$$\|\mathcal{P}_1 x - \mathcal{P}_1 y\|_{\mathcal{C}} \leqslant L\|x-y\|_{\mathcal{C}}, \qquad L := \|Q\| \frac{HH_2}{(2N)^{1/2}} \bigl(\mathrm{e}^{2Nt_1^\alpha/\alpha} - 1\bigr)^{1/2}.$$

According to (20), we conclude that $L<1$, which implies $\mathcal{P}_1$ is a contraction.

Step 3. We show that $\mathcal{P}_2$ is a compact and continuous operator.

Let $x_n \in \mathcal{B}_r$ with $x_n \to x$ in $\mathcal{B}_r$. Denote $F_n(\cdot) = f(\cdot, x_n(\cdot))$ and $F(\cdot) = f(\cdot, x(\cdot))$. Using (A2), we have $F_n \to F$ in $\mathcal{C}(J, \mathbb{R}^n)$ and thus

$$\begin{aligned}\bigl\|(\mathcal{P}_2 x_n)t - (\mathcal{P}_2 x)(t)\bigr\| &\leqslant \int_0^t \mathrm{e}^{N(t^\alpha/\alpha - \tau^\alpha/\alpha)} \bigl\|F_n(\tau) - F(\tau)\bigr\| \,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &\leqslant \frac{1}{N}\bigl(\mathrm{e}^{Nt_1^\alpha/\alpha} - 1\bigr)\|F_n - F\|_{\mathcal{C}} \to 0 \quad \text{as } n\to\infty\end{aligned}$$

uniformly for $t \in J$, which implies that $\mathcal{P}_2$ is continuous on $\mathcal{B}_r$.

In order to check the compactness of $\mathcal{P}_2$, we prove that $\mathcal{P}_2(\mathcal{B}_r) \subset \mathcal{C}(J, \mathbb{R}^n)$ is equicontinuous and bounded.

In fact, for any $x \in \mathcal{B}_r$, $t_1 \geqslant t+h \geqslant t > 0$, it holds

$$\begin{aligned}&(\mathcal{P}_2 x)(t+h) - (\mathcal{P}_2 x)(t) \\ &\quad= \int_0^{t+h} \mathrm{e}^{M(t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} F(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} - \int_0^t \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)} F(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &\quad= I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_t^{t+h} \mathrm{e}^{M((t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} F(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}, \\ I_2 &= \int_0^t \bigl[\mathrm{e}^{M((t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} - \mathrm{e}^{M(t^\alpha/\alpha - \tau^\alpha/\alpha)}\bigr] F(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha}.\end{aligned}$$

Combining the previous derivations, we have

$$\bigl\|(\mathcal{P}_2 x)(t+h) - (\mathcal{P}_2 x)(t)\bigr\| \leqslant \|I_1\| + \|I_2\|.$$

Then we check $\|I_i\| \to 0$ as $h \to 0$, $i = 1, 2$, uniformly for t.

For I_1, using (A2),

$$\begin{aligned}\|I_1\| &\leqslant \int_t^{t+h} \mathrm{e}^{N((t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} \bigl\|F(\tau)\bigr\| \,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &\leqslant \int_t^{t+h} \mathrm{e}^{N((t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} L_f(\tau)\bigl\|x(\tau)\bigr\| \,\mathrm{d}\frac{\tau^\alpha}{\alpha} + \int_t^{t+h} \mathrm{e}^{N((t+h)^\alpha/\alpha - \tau^\alpha/\alpha)} \bigl\|f(\tau, 0)\bigr\| \,\mathrm{d}\frac{\tau^\alpha}{\alpha}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|x\|_{\mathcal{C}} \int\limits_t^{t+h} \mathrm{e}^{N((t+h)^\alpha/\alpha-\tau^\alpha/\alpha)} L_f(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} + R_f \int\limits_t^{t+h} \mathrm{e}^{N((t+h)^\alpha/\alpha-\tau^\alpha/\alpha)}\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \left[\frac{1}{Np}\bigl(\mathrm{e}^{pN((t+h)^\alpha/\alpha-t^\alpha/\alpha)}-1\bigr)\right]^{1/p} \|L_f\|_{L^q_\alpha} r \\
&\quad + \frac{R_f}{N}\bigl(\mathrm{e}^{N((t+h)^\alpha/\alpha-t^\alpha/\alpha)}-1\bigr) \to 0 \quad \text{as } h \to 0.
\end{aligned}
$$

For I_2, it is easy to get that

$$
\begin{aligned}
\|I_2\| &\leqslant \int\limits_0^t \bigl\|\mathrm{e}^{M((t+h)^\alpha/\alpha-\tau^\alpha/\alpha)} - \mathrm{e}^{M(t^\alpha/\alpha-\tau^\alpha/\alpha)}\bigr\|\bigl\|F(\tau)\bigr\|\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \int\limits_0^t \mathrm{e}^{N\tau^\alpha/\alpha}\bigl\|\mathrm{e}^{M(t+h)^\alpha/\alpha} - \mathrm{e}^{Mt^\alpha/\alpha}\bigr\|\bigl(L_f(\tau)\|x\|_{\mathcal{C}} + \bigl\|f(\tau,0)\bigr\|\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \int\limits_0^t \mathrm{e}^{N\tau^\alpha/\alpha}\bigl\|\mathrm{e}^{M(t+h)^\alpha/\alpha} - \mathrm{e}^{Mt^\alpha/\alpha}\bigr\|\bigl(L_f(\tau)r + R_f\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \bigl\|\mathrm{e}^{M(t+h)^\alpha/\alpha} - \mathrm{e}^{Mt^\alpha/\alpha}\bigr\|\left[r\int\limits_0^t \mathrm{e}^{N\tau^\alpha/\alpha} L_f(\tau)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} + R_f\int\limits_0^t \mathrm{e}^{N\tau^\alpha/\alpha}\mathrm{e}\frac{\tau^\alpha}{\alpha}\right] \\
&\leqslant \bigl\|\mathrm{e}^{M(t+h)^\alpha/\alpha} - \mathrm{e}^{Mt^\alpha/\alpha}\bigr\| \\
&\quad \times \left[\frac{1}{(Np)^{1/p}}\bigl(\mathrm{e}^{Npt_1^\alpha/\alpha}-1\bigr)^{1/p}\|L_f\|_{L^q_\alpha} r + \frac{R_f}{N}\bigl(\mathrm{e}^{Nt_1^\alpha/\alpha}-1\bigr)\right] \to 0 \quad \text{as } h \to 0.
\end{aligned}
$$

From above we obtain

$$
\bigl\|(\mathcal{P}_2x)(t+h) - (\mathcal{P}_2x)(t)\bigr\| \to 0 \quad \text{as } h \to 0,
$$

uniformly for all t and $x \in \mathcal{B}_r$. Thus, $\mathcal{P}_2(\mathcal{B}_r) \subset \mathcal{C}(J,\mathbb{R}^n)$ is equicontinuous.

According to the above computations, one can get

$$
\begin{aligned}
\bigl\|(\mathcal{P}_2x)(t)\bigr\| &\leqslant \int\limits_0^t \mathrm{e}^{N(t^\alpha/\alpha-\tau^\alpha/\alpha)}\bigl(L_f(\tau)r + R_f\bigr)\,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\
&\leqslant \frac{1}{(Np)^{1/p}}\bigl(\mathrm{e}^{Npt_1^\alpha/\alpha}-1\bigr)^{1/p}\|L_f\|_{L^q_\alpha} r + \frac{R_f}{N}\bigl(\mathrm{e}^{Nt_1^\alpha/\alpha}-1\bigr).
\end{aligned}
$$

Thus $\mathcal{P}_2(\mathcal{B}_r)$ is bounded. By Arzela–Ascoli theorem, $\mathcal{P}_2(\mathcal{B}_r) \subset \mathcal{C}(J,\mathbb{R}^n)$ is relatively compact.

Hence, $\mathcal{P}_2$ is a compact and operator. Then Krasnoselskii's fixed point theorem gives that $\mathcal{P}$ has a fixed point x on $\mathcal{B}_r$. Apparently, x is a solution of system (2) satisfying $x(t_1) = x_1$. The proof is completed. □

4 Numerical examples

Example 1. Let $n = r = 2$. Consider the following linear control system:

$$\mathfrak{D}_{\alpha}^{0}x(t) = Mx(t) + Qu_1(t), \quad t \in J_1 := [0,1], \qquad x(0) = x_0, \tag{21}$$

where

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \tag{22}$$

and $\alpha = 0.9$.

By elementary calculation, the conformable Gram matrix of system (21) with (22) via (5) can be written into:

$$\begin{aligned} W_c[0,1] &= \int_0^1 \mathrm{e}^{-M\tau^{\alpha}/\alpha} QQ^{\top} \mathrm{e}^{-M^{\top}\tau^{\alpha}/\alpha} \,\mathrm{d}\frac{\tau^{\alpha}}{\alpha} \\ &= \int_0^1 \begin{pmatrix} \mathrm{e}^{-2\tau^{\alpha}/\alpha} & 1 \\ 1 & \mathrm{e}^{-3\tau^{\alpha}/\alpha} \end{pmatrix} \begin{pmatrix} 0.36 & 0 \\ 0 & 0.36 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2\tau^{\alpha}/\alpha} & 1 \\ 1 & \mathrm{e}^{-3\tau^{\alpha}/\alpha} \end{pmatrix} \mathrm{d}\frac{\tau^{\alpha}}{\alpha} \\ &= \begin{pmatrix} 0.489 & 0.276 \\ 0.276 & 0.460 \end{pmatrix}. \end{aligned}$$

Then we can get

$$W_c^{-1}[0,1] = \begin{pmatrix} 3.095 & -1.859 \\ -1.859 & 3.291 \end{pmatrix}.$$

Obviously, $W_c[0,1]$ is nonsingular.

Therefore, according to (6), we obtain

$$\begin{aligned} u_1(\cdot) &= -Q^{\top} \mathrm{e}^{-M^{\top}(\cdot^{\alpha}/\alpha)} W_c^{-1}[0,1] x_0 \\ &= \begin{pmatrix} -0.6 & 0 \\ 0 & -0.6 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2(\cdot^{\alpha}/\alpha)} & 1 \\ 1 & \mathrm{e}^{-3(\cdot^{\alpha}/\alpha)} \end{pmatrix} \begin{pmatrix} 3.095 & -1.859 \\ -1.859 & 3.291 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\ &= \begin{pmatrix} -0.371\mathrm{e}^{-2(\cdot^{\alpha}/\alpha)} - 0.430 \\ -0.430\mathrm{e}^{-3(\cdot^{\alpha}/\alpha)} - 0.380 \end{pmatrix}. \end{aligned}$$

Finally, by Theorem 1, system (21) with (22) is null completely controllable on $[0,1]$.

Example 2. Let $n = 3$ and $r = 2$. Consider the following linear control system:

$$\mathfrak{D}_{\alpha}^{0}x(t) = Mx(t) + Qu_1(t), \quad t \in J_1 := [0,1], \tag{23}$$

where

$$M = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 6 & 7 \\ 2 & 4 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix}, \tag{24}$$

and $\alpha = 0.9$.

By elementary calculation, the rank criterion of system (23) with (24) can be written into

$$\Gamma_c = \begin{pmatrix} 0 & 1 & 7 & * & * & * \\ 2 & 0 & 19 & * & * & * \\ 1 & 2 & 9 & * & * & * \end{pmatrix}.$$

From the first three columns of the above matrix it can be determined

$$\det \begin{pmatrix} 0 & 1 & 7 \\ 2 & 0 & 19 \\ 1 & 2 & 9 \end{pmatrix} \neq 0,$$

that is, $\operatorname{rank} \Gamma_c = 3 = n$. From Theorem 2, system (23) with (24) is null completely controllable on $[0, 1]$.

Example 3. Let $n = r = 2$. Consider the following semilinear controlled system:

$$\mathfrak{D}_\alpha^0 x(t) = Mx(t) + f\bigl(t, x(t)\bigr) + Qu(t), \quad x(t) \in \mathbb{R}^2,\ t \in [0, 1] := J_1, \tag{25}$$

$u \in L^2(J_1, \mathbb{R}^2)$, where

$$M = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad M^\top = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \tag{26}$$

and

$$f\bigl(t, x(t)\bigr) = \begin{pmatrix} \frac{1}{5}(t + 0.1)x_1(t) \\ \frac{1}{5}(t + 0.1)x_2(t) \end{pmatrix}, \qquad \alpha = 0.5. \tag{27}$$

By elementary calculation, we have $N = \|M\| = 0.2$. Now we use (18) to estimate H. For this purpose, we need to obtain $W_c[0, 1]$ and then derive $W_c[0, 1]^{-1}$. By computation, the Gram matrix (5) can be written into

$$\begin{aligned} W_c[0, 1] &= \int_0^1 \mathrm{e}^{-M\tau^\alpha/\alpha} QQ^\top \mathrm{e}^{-M^\top \tau^\alpha/\alpha} \,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &= \int_0^1 \begin{pmatrix} \mathrm{e}^{-0.2\tau^\alpha/\alpha} & 1 \\ 1 & \mathrm{e}^{-0.2\tau^\alpha/\alpha} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-0.2\tau^\alpha/\alpha} & 1 \\ 1 & \mathrm{e}^{-0.2\tau^\alpha/\alpha} \end{pmatrix} \,\mathrm{d}\frac{\tau^\alpha}{\alpha} \\ &= \begin{pmatrix} 13.507 & 13.187 \\ 13.187 & 13.507 \end{pmatrix}. \end{aligned}$$

Therefore, we derive

$$W_c[0, 1]^{-1} = \begin{pmatrix} 1.584 & -1.546 \\ -1.546 & 1.584 \end{pmatrix}. \tag{28}$$

Consequently, we get

$$H = \left\| W_c[0,1]^{-1} \right\|^{1/2} = 3.130^{1/2} = 1.769.$$

Hence, W satisfies assumption (A1).

Then it is easy to see that for any $x(t), y(t) \in \mathbb{R}^2$ and $t \in J_1$,

$$\begin{aligned} \left\| f(t,x) - f(t,y) \right\| &= \frac{1}{5}(t+0.1)\left(\left(x_1(t) - y_1(t)\right)^2 + \left(x_2(t) - y_2(t)\right)^2 \right)^{1/2} \\ &\leqslant \frac{1}{5}(t+0.1)\|x-y\|. \end{aligned}$$

Therefore, f satisfies assumption (A2), where we set

$$L_f(\cdot) = \frac{\cdot + 0.1}{5} \in L^q_\alpha\left(J_1, \mathbb{R}^+\right).$$

Obviously, $(1/5)\|L_f\|_{L^q_\alpha(J_1,\mathbb{R}^+)} = ((1.1^{q+1} - 0.1^{q+1})/(q+1))^{1/q}$ and $R_f = \sup_{t\in J_1} \|f(t,0)\| = 0$. Next, $\|Q\| = 2$, $\|L_f\|_{L^q_\alpha(J_1,\mathbb{R}^+)} = 0.133$, and

$$H_2 = \left[\frac{1}{2N}\left(\mathrm{e}^{2Nt^\alpha/\alpha} - 1\right) \right]^{1/2} \|L_f\|_{L^q_\alpha(J_1,\mathbb{R}^+)} = 0.233$$

when we choose $p = q = 2$. Therefore,

$$\begin{aligned} \gamma &= H_2\left[1 + \frac{H}{(2N)^{1/2}}\left(\mathrm{e}^{2Nt^\alpha/\alpha} - 1\right)^{1/2}\|Q\| \right] \\ &= 0.233\left[1 + 0.5\frac{1.769}{0.4^{1/2}}\left(\mathrm{e}^{0.8} - 1\right)^{1/2} \right] \\ &= 0.864 < 1, \end{aligned}$$

which guarantees that condition (20) holds.

Thus all the conditions of Theorem 3 are satisfied. Hence, system (25) with (26) and (27) is completely controllable on $[0,1]$.

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