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Dynamics for a stochastic delayed SIRS epidemic model*

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Abstract. In this paper, we consider a stochastic delayed SIRS epidemic model with seasonal variation. Firstly, we prove that the system is mathematically and biologically well-posed by showing the global existence, positivity and stochastically ultimate boundness of the solution. Secondly, some sufficient conditions on the permanence and extinction of the positive solutions with probability one are presented. Thirdly, we show that the solution of the system is asymptotical around of the disease-free periodic solution and the intensity of the oscillation depends of the intensity of the noise. Lastly, the existence of stochastic nontrivial periodic solution for the system is obtained.

Keywords: stochastic SIRS epidemic model, extinction and persistence, periodic solution, time delay.

1 Introduction

In the real world, epidemic spreads are always affected by demographic and environmental stochasticity. However, deterministic model cannot capture these fluctuating features. Hence, stochastic differential equation models, which can describe and predict the systems more accurately than that of their deterministic counterpart (see [6,7,10,19]), play an important role in many types of braches of applied sciences including epidemic dynamics. To make epidemic models even more realistic, combining the effects of disease latency or immunity, stochastic delay differential equations have attracted much attention in the last decade (see [1, 2, 18]). As is well known, due to the seasonal variation, individuals life cycle, hunting, food supplies, mating habits, harvesting and so on, the birth rate, the death rate of the population and other parameters will not remain unchanged, but exhibit

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a more or less periodicity. For humans, seasonal diseases are inherent in the organic growth of man from infancy to old age. Some examples of such disease are measles, cholera, rotaviruses, respiratory syncytial virus (RSV), etc. [8]. Therefore, some epidemic models were formulated as dynamical systems of nonautonomous differential equations (see [16, 20]).

Recently, many authors have studied epidemic models with the effect of seasonal variation and stochastically. Jin et al. [13] considered an SIRS epidemic model with general seasonal variation in the constant rate. Bai et al. considered a nonautonomous SIR model with periodic transmission rate and a constant removal rate in [3]. Xing et al. investigated the stochastic nonautonomous logistic system with time delays and obtained the persistence and extinction of system [17]. The asymptotic behavior of a nonautonomous predator–prey model with Hassell–Varley-type functional response and random perturbation was discussed by the Zhang et al. [21]. Cai et al. investigated the stochastic dynamics of a simple epidemic model incorporating the mean-reverting Ornstein–Uhlenbeck process analytically and numerically [5].

To the best of our knowledge, the existence of nontrivial positive periodic solution for stochastic nonautonomous epidemic models have been studied not by theoretical methods, but only by numerical methods [4, 9, 12]. Especially, few studies have discussed the dynamic behaviors of nonautonomous stochastic differential equation systems for epidemic models with time delays. In this paper, we will consider a stochastic SIRS epidemic model with the seasonal parameters and time delay, which takes the following form:

$$\begin{aligned} \mathrm{d}S(t) &= \bigl(\mu(t) - \mu(t)S(t) - \beta(t)S(t)I(t) + \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\bigr)\,\mathrm{d}t \\ &\quad - \sigma S(t)I(t)\,\mathrm{d}B(t), \\ \mathrm{d}I(t) &= \bigl(\beta(t)S(t)I(t) - \bigl(\mu(t)+\gamma(t)\bigr)I(t)\bigr)\,\mathrm{d}t + \sigma S(t)I(t)\,\mathrm{d}B(t), \\ \mathrm{d}R(t) &= \bigl(\gamma(t)I(t) - \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} - \mu(t)R(t)\bigr)\,\mathrm{d}t, \end{aligned} \tag{1}$$

where $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective and removed individuals at time t, respectively. The instantaneous birth rate $\mu(t)$ equals to the instantaneous death rate. For all the classes, it is assumed that the death rate $\mu(t)$ is the same because we assume that deaths associated with disease are small. $\beta(t)$ denotes the transmissions coefficient between compartment $S(t)$ and $I(t)$, and $\gamma(t)$ is the recovery rate of the infective individual. It is assumed that $\gamma(t) < \mu(t)$, for all $t \geqslant 0$, i.e., the per capita rate of recovery is smaller than the per capita rate of leaving the infective. $\mu(t), \beta(t), \gamma(t)$ are positive, nonconstant and continuous functions of period ω. The time delay $\tau > 0$ represents the duration of the immunity period, and the term $I(t-\tau)\mathrm{e}^{-\mu(t)\tau}$ reflects the fact that an individual has survived from natural death in a recovery pool before becoming susceptible again. Here, the perturbation to transmission coefficient $\beta(t)$ is introduced in the following form: $\beta(t) \to \beta(t) + \sigma\xi(t)$, and $\xi(t) = \mathrm{d}B(t)/\mathrm{d}t$, $B(t)$ is a standard Brownian motions defined on a complete probability space, σ is the intensity of the white noises. We also present some notations: if $f(t)$ is an integrable function on $[0, \infty)$, define

$$\langle f(t) \rangle_t = \frac{1}{t} \int_0^t f(s)\,\mathrm{d}s, \quad t > 0;$$

if $f(t)$ is a bounded function on $[0, \infty)$, define

$$f^\mu = \sup_{t \in [0,\infty)} f(t), \qquad f^l = \inf_{t \in [0,\infty)} f(t).$$

Throughout this paper, unless otherwise specified, let $(\Omega, \{F_t\}_{t \geqslant 0}, \mathbf{P})$ be a complete probability space with a filtration $\{F_t\}_{t \geqslant 0}$ satisfying the usual conditions. Let $X(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ and $\mathbb{R}^3_+ = \{X(t) \in \mathbb{R}^3\colon x_1(t) > 0,\ x_2(t) > 0,\ x_3(t) > 0,\ t \geqslant 0\}$ and $\mathbb{C} = \mathbb{C}([-\tau, 0], \mathbb{R}^3_+)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_+$ equipped by $\|\varphi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|$.

For biological reasons, we assume that the initial conditions of system (1) satisfy

$$\begin{aligned} &S(\theta) = \varphi_1(\theta), \qquad I(\theta) = \varphi_2(\theta), \qquad R(\theta) = \varphi_3(\theta), \\ &\varphi_i(\theta) > 0 \quad (i = 1, 2, 3),\ \theta \in [-\tau, 0], \\ &(\varphi_1, \varphi_2, \varphi_3) \in \mathbb{C} = \mathbb{C}\big([-\tau, 0], \mathbb{R}^3_+\big). \end{aligned} \tag{2}$$

The aim of this work is to investigate the dynamic behavior of the solutions of the disease-free state and the existence of positive periodic solutions of the proposed stochastic delayed epidemic model (1). For this purpose, we first prove that system (1) is mathematically and biologically well-posed by showing the global existence, positivity and stochastically ultimate boundness of the solution in Section 2. The disease extinction and persistence of disease are investigated in Section 3. The sufficient conditions for the dynamic behavior of the solutions of the disease-free state are obtained in Section 4. The existence of positive periodic solutions is proved by using the Khasmimskii's boundary periodic Markov processes in Section 5. Simulation and conclusion in Section 6 complete the paper.

2 Global positive solution

Model (1) describes the dynamics of a biological population. Hence, the population sizes should be nonnegative and bounded. For this reasons, we first establish the global existence, positivity, and boundedness of solutions.

Let $Z(t) = S(t) + I(t) + R(t)$. It is easy to see that $\mathrm{d}Z(t) = (\mu(t) - \mu(t)Z(t))\,\mathrm{d}t$. So,

$$Z(t) = \big(Z(0) - 1\big)\mathrm{e}^{-\int_0^t \mu(s)\,\mathrm{d}s} + 1.$$

Then one can obtain that

$$Z(t) \leqslant \begin{cases} 1, & Z(0) \leqslant 1, \\ Z(0), & Z(0) > 1 \end{cases}$$

for $t \in [0, T]$ a.s. We denote

$$\varGamma = \{(S, I, R) \in \mathbb{R}_+^3\colon\ S + I + R \leqslant 1\}. \tag{3}$$

It is clear that if $Z(0) \in \varGamma$, then $Z(t) \leqslant 1$, i.e., $S(t), I(t), R(t) \in (0, 1]$ for $t \in [0, T]$ a.s.

Theorem 1. *Assume that μ^u, γ^u, σ are positive real numbers. Let $X(0) = (S(0), I(0), R(0))^{\mathrm{T}} \in \varGamma$ be any initial condition. Then, there is a unique solution $X(t) = (S(t), I(t), R(t))$ of system* (1) *for $t \geqslant 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. It is clear that the coefficients of system (1) are locally Lipschitz continuous. It follows from [15] that for any initial value (2), system (1) admits a unique maximal local solution $(S(t), I(t), R(t))$ on $t \in [-\tau, \tau_e]$, where τ_e is the explosion time. To verify this solution is global, we only need to show that $\tau_e = \infty$ a.s. For this, we consider the following stopping time:

$$\tau^+ = \inf\{t \in [-\tau, \tau_e]\colon\ S(t) \leqslant 0 \text{ or } I(t) \leqslant 0 \text{ or } R(t) \leqslant 0\},$$

where we set $\inf \phi = \infty$ (ϕ denote the empty set). Clearly, $\tau^+ \leqslant \tau_e$, so if we prove that $\tau^+ = \infty$ a.s., then $\tau_e = \infty$, which means that $(S(t), I(t), R(t)) \in \mathbb{R}_+^3$ a.s. for all $t \geqslant 0$. Assume the $\tau^+ < \infty$, then there exists a $T > 0$ such that $\mathbf{P}(\tau^+ < T) > 0$.

Define the function $V_1 : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follows:

$$V_1\big(S(t), I(t)\big) = \ln S(t) + \ln I(t).$$

Applying the Itô formula, we obtain

$$\begin{aligned}
&\mathrm{d}V_1\big(S(t), I(t)\big) \\
&\quad = \left[\frac{1}{S(t)}\big(\mu(t) - \mu S(t) - \beta(t)S(t)I(t) + \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\big) + \frac{1}{2}\sigma^2 I^2(t)\right]\mathrm{d}t \\
&\qquad - \sigma I(t)\,\mathrm{d}B(t) + \left[\frac{1}{I(t)}\big(\beta(t)I(t)S(t) - (\gamma(t) + \mu(t))I(t)\big) - \frac{1}{2}\sigma^2 S^2(t)\right]\mathrm{d}t \\
&\qquad + \sigma S(t)\,\mathrm{d}B(t) \\
&\quad = \left[\frac{\mu(t)}{S(t)} - \mu(t) - \beta(t)I(t) + \frac{\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}}{S(t)} - \frac{1}{2}\sigma^2 I^2(t)\right]\mathrm{d}t - \sigma I(t)\,\mathrm{d}B(t) \\
&\qquad + \left[\beta(t)S(t) - \big(\gamma(t) + \mu(t)\big) - \frac{1}{2}\sigma^2 S^2(t)\right]\mathrm{d}t + \sigma S(t)\,\mathrm{d}B(t).
\end{aligned}$$

Then

$$\begin{aligned}
&\ln S(t) + \ln I(t) - \ln\varphi_1(0) - \ln\varphi_2(0) \\
&\quad = \int_0^t \left[\frac{\mu(s)}{S(s)} - \mu(s) - \beta(s)I(s) + \frac{\gamma(s)I(s-\tau)\mathrm{e}^{-\mu(t)\tau}}{S(s)}\right. \\
&\qquad\qquad \left. + \beta(s)S(s) - \big(\gamma(s) + \mu(s)\big) - \frac{1}{2}\sigma^2 S^2(s) - \frac{1}{2}\sigma^2 I^2(s)\right]\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \sigma\big(S(s)-I(s)\big)\,\mathrm{d}B(s)\\
&\geqslant \int_0^t \left[-2\mu^u-\beta^u I(s)-\gamma^u-\frac{1}{2}\big(\sigma^2+\sigma^2\big)\right]\mathrm{d}s+\int_0^t \sigma\big(S(s)-I(s)\big)\,\mathrm{d}B(s). \qquad (4)
\end{aligned}
$$

From the definition of τ^+ it follows that the solution of system (1) is positive on $[0,\tau^+)$ for almost all w in $\{\tau^+ < T\}$ and $S(\tau^+)I(\tau^+) = 0$. Therefore,

$$\lim_{t\to\tau^+}\big(\ln S(t)+\ln I(t)\big) = -\infty.$$

Letting $t \to \tau^+$ in (3), we get

$$
\begin{aligned}
-\infty &\geqslant -\left(2\mu^u+\gamma^u+\frac{1}{2}\big(\sigma^2+\sigma^2\big)\right)\tau^+\\
&\quad -\beta^u\int_0^{\tau^+} I(s)\,\mathrm{d}s+\int_0^{\tau^+}\sigma\big(S(s)-I(s)\big)\,\mathrm{d}B(s)\\
&> -\infty,
\end{aligned}
$$

which is a contradiction. Thus $\tau^+ = \tau_e = +\infty$ a.s. □

Corollary 1. *The set Γ is almost positive in invariant of system* (1). *That is, for $X(0) = (S(0), I(0), R(0)) \in \Gamma$ it holds that $\mathbf{P}[X(t) = (S(t), I(t), R(t)) \in \Gamma] = 1$ for all $t \geqslant 0$.*

Theorem 1 shows that the solutions of system (1) with positive initial value in Γ will remain positive in Γ. The properties of positivity and nonexplosion are essential for a population system. Once they have been established, we can discuss some other properties of the solutions of the system. From a biological point of view, due to the limitation of resources, the property of stochastically ultimate boundedness is more desirable than the nonexplosion property. In the following, we will show system (1) is stochastically ultimate bounded. Now, we give the following definition.

Definition 1. System (1) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0, 1)$, there exists a positive constant $H = H(\varepsilon)$ such that for any initial value $(S(0), I(0), R(0)) \in \Gamma$, the solution $X(t) = (S(t), I(t), R(t))$ of (1) has the property

$$\lim_{t\to\infty}\sup \mathbf{P}\big\{\big|X(t)\big| > H(\varepsilon)\big\} < \varepsilon.$$

In order to prove the stochastically ultimate boundness of system (1), we show that system (1) is asymptotically bounded firstly.

Theorem 2. *If $\theta \in [1,\infty)$, and there exists a positive constant H_0, which is independent of the initial value $X(0) = (S(0), I(0), R(0)) \in \Gamma$, the solution $X(t) = (S(t), I(t), R(t))$ of system* (1) *has the following property*:

$$\lim_{t\to\infty}\sup \mathbf{E}\big[\big|X(t)\big|^\theta\big] \leqslant H_0.$$

Proof. From Theorem 1 the solutions will remain in Γ for all $t \geqslant 0$ with probability 1 a.s. Let us choose $V_2(t) = S^\theta(t) + I^\theta(t) + R^\theta(t)$ with $\theta \geqslant 1$, $k_1 > 0$ such that $k_1 - \mu^l\theta < 0$ sufficiently small.

Applying the Itô's formula, we get

$$\begin{aligned}
&\mathrm{d}\big(\mathrm{e}^{k_1 t} V_2\big(X(t)\big)\big)\\
&\quad= \big[k_1 \mathrm{e}^{k_1 t}\big(S^\theta + I^\theta + R^\theta\big)\big]\,\mathrm{d}t\\
&\qquad+ \mathrm{e}^{k_1 t}\Big[\theta S^{\theta-1}\big(\mu(t) - \mu(t)S(t) - \beta(t)S(t)I(t) + \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\big)\\
&\qquad+ \theta I^{\theta-1}\big(\beta(t)S(t)I(t) - (\mu(t)+\gamma(t))I(t)\big)\\
&\qquad+ \theta R^{\theta-1}\big(\gamma(t)I(t) - \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} - \mu(t)R(t)\big)\\
&\qquad+ \frac{1}{2}\sigma^2\theta(\theta-1)S^\theta(t)I^2(t) + \frac{1}{2}\sigma^2\theta(\theta-1)S^2(t)I^\theta(t)\Big]\,\mathrm{d}t\\
&\qquad- \mathrm{e}^{k_1 t}\sigma\theta S^\theta(t)I(t)\,\mathrm{d}B(t) + \mathrm{e}^{k_1 t}\sigma\theta I^\theta(t)S(t)\,\mathrm{d}B(t)\\
&\quad\leqslant \Big[k_1\mathrm{e}^{k_1 t}S^\theta(t) + k_1\mathrm{e}^{k_1 t}I^\theta(t) + k_1\mathrm{e}^{k_1 t}R^\theta(t) + \mathrm{e}^{k_1 t}\theta S^{\theta-1}(t)\mu^u - \mathrm{e}^{k_1 t}\mu^l\theta S^\theta(t)\\
&\qquad+ \mathrm{e}^{k_1 t}\theta\gamma^u\mathrm{e}^{-\mu^l\tau} + \mathrm{e}^{k_1 t}\theta I^{\theta-1}(t)\beta^u - \mathrm{e}^{k_1 t}\theta(\mu^l+\gamma^l)I^\theta(t) + \mathrm{e}^{k_1 t}\gamma^u\theta R^{\theta-1}(t)\\
&\qquad- \mathrm{e}^{k_1 t}\theta\mu^l R^\theta(t) + \mathrm{e}^{k_1 t}\frac{1}{2}\sigma^2\theta(\theta-1) + \mathrm{e}^{k_1 t}\frac{1}{2}\sigma^2\theta(\theta-1)\Big]\,\mathrm{d}t\\
&\qquad+ \mathrm{e}^{k_1 t}\sigma\theta\big(I^\theta(t)S(t) - S^\theta(t)I(t)\big)\,\mathrm{d}B(t)\\
&\quad= \mathrm{e}^{k_1 t}\big[S^\theta(t)\big(k_1 - \mu^l\theta\big) + I^\theta(t)\big(k_1 - \theta\big(\mu^l+\gamma^l\big)\big) + R^\theta\big(k_1 - \mu^l\theta\big)\\
&\qquad+ \theta\mu^u S^{\theta-1}(t) + \theta\beta^u I^{\theta-1}(t) + \theta\gamma^u R^{\theta-1}(t) + \theta\gamma^u\mathrm{e}^{-\mu^l\tau} + \sigma^2\theta(\theta-1)\big]\,\mathrm{d}t\\
&\qquad+ \mathrm{e}^{k_1 t}\sigma\theta\big(I^\theta(t)S(t) - I(t)S^\theta(t)\big)\,\mathrm{d}B(t)\\
&\quad\leqslant \mathrm{e}^{k_1 t}\big[\theta\mu^u + \theta\beta^u + \theta\gamma^u + \theta\gamma^u\mathrm{e}^{-\mu^l\tau} + \sigma^2\theta(\theta-1)\big]\,\mathrm{d}t\\
&\qquad+ \mathrm{e}^{k_1 t}\sigma\theta\big(I^\theta(t)S(t) - I(t)S^\theta(t)\big)\,\mathrm{d}B(t)\\
&\quad\triangleq \mathrm{e}^{k_1 t}H_1\,\mathrm{d}t + \mathrm{e}^{k_1 t}\sigma\theta\big(I(t)^\theta S(t) - I(t)S^\theta(t)\big)\,\mathrm{d}B(t),
\end{aligned}\tag{5}$$

where $H_1 = \theta\mu^u + \theta\beta^u + \theta\gamma^u + \theta\gamma^u\mathrm{e}^{-\mu^l\tau} + \sigma^2\theta(\theta-1)$. Let $k_0 > 0$ be sufficiently large such that for every component of $X(0)$ stays in the interval $[1/k_0, 1]$. For each integer$k \geqslant k_0$, define the stoping time by

$$\tau_k = \inf\left\{t \geqslant 0 \;\middle|\; S(t) \leqslant \frac{1}{k} \text{ or } I(t) \leqslant \frac{1}{k}, \text{ or } R(t) \leqslant \frac{1}{k}\right\}.$$

Clearly, $\tau_k \to \infty$ a.s. as $k \to \infty$. It follows from (5) that

$$\mathbf{E}\big[\mathrm{e}^{k_1 t\wedge\tau_k}V_2\big(X(t\wedge\tau_k)\big)\big] \leqslant V_2\big(X(0)\big) + H_1\mathbf{E}\int_0^{t\wedge\tau_k}\mathrm{e}^{k_1 s}\,\mathrm{d}s.$$

Letting $k \to \infty$, we have

$$e^{k_1 t}\mathbf{E}\left[V_2\left(X(t)\right)\right] \leqslant V_2\left(X(0)\right) + H_1\left(e^{k_1 t} - 1\right),$$

therefore,

$$\mathbf{E}\left[V_2\left(X(t)\right)\right] < e^{-k_1 t}V_2\left(X(0)\right) + H_1.$$

Since

$$\left|X(t)\right|^2 \leqslant 3\max\left\{S^2(t), I^2(t), R^2(t)\right\},$$

then

$$\left|X(t)\right|^\theta \leqslant 3^{\theta/2}\max\left\{S^\theta(t), I^\theta(t), R^\theta(t)\right\} \leqslant 3^{\theta/2}V_2\left(X(t)\right).$$

Thus,

$$\mathbf{E}\left[\left|X(t)\right|^\theta\right] \leqslant 3^{\theta/2}\left(e^{-k_1 t}V_2\left(X(0) + H_1\right)\right).$$

Therefore, we can obtain

$$\lim_{t\to\infty}\sup \mathbf{E}\left[\left|X(t)\right|^\theta\right] \leqslant 3^{\theta/2}H_1 \triangleq H_0. \qquad \square$$

Theorem 3. *Under the assumption of Theorem* 2*, system* (1) *is stochastically ultimately bounded.*

Proof. By Theorem 2, for $\theta = 3/2$, there is a $H_2 > 0$ such that

$$\lim_{t\to\infty}\sup \mathbf{E}\left[\left|X(t)\right|^{3/2}\right] \leqslant H_2.$$

Let ϵ be given, we can choose $H(\epsilon) = (H_2/\epsilon)^{2/3}$. Therefore, by the application of the Chebyshev's inequality, one gets that

$$\mathbf{P}\left\{\left|X(t)\right| > H\right\} \leqslant H(\epsilon)^{-3/2}\mathbf{E}\left[\left|X(t)\right|^{3/2}\right] < \epsilon.$$

This completes the proof of the theorem. □

3 Extinction and persistence of the disease

In this section, we shall investigate the persistence and extinction of system (1) and obtain a threshold, which determines whether the disease dies out or persists. Now, we present two auxiliary lemmas, which will be used in the proof of the main results of this section.

Lemma 1. (*See* [11].) *Let* $f \in \mathbb{C}[[0,\infty)\times\Omega,(0,\infty)]$. *If there exist positive constants* λ_0, λ *such that*

$$\ln f(t) \geqslant \lambda t - \lambda_0 \int_0^t f(s)\,\mathrm{d}s + F(t) \quad \textit{a.s.}$$

for all $t \geqslant 0$, *where* $F \in \mathbb{C}[[0,\infty)\times\Omega,\mathbb{R}]$ *and* $\lim_{t\to\infty} F(t)/t = 0$ *a.s., then*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \geqslant \frac{\lambda}{\lambda_0} \quad a.s.$$

Lemma 2. (*See* [11].) *Let* $f \in \mathbb{C}[[0,\infty)\times\Omega,(0,\infty)]$. *If there exist positive constants* λ_0, λ *such that*

$$\ln f(t) \leqslant \lambda t - \lambda_0 \int_0^t f(s)\,\mathrm{d}s + F(t) \quad \textit{a.s.}$$

for all $t \geqslant 0$, *where* $F \in \mathbb{C}[[0,\infty)\times\Omega,\mathbb{R}]$ *and* $\lim_{t\to\infty} F(t)/t = 0$ *a.s., then*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \leqslant \frac{\lambda}{\lambda_0} \quad \textit{a.s.}$$

Theorem 4. *Assume* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ *when* $\langle R_0\rangle_\omega - \sigma^2/2 > 0$ *and* $\int_0^\omega (\mu(t) - \sigma^2/2)\,\mathrm{d}t > 0$, *the disease* $I(t)$ *will persist in the sense that*

$$\frac{\langle R_0\rangle_\omega - \frac{1}{2}\sigma^2}{v^u\gamma^u + \beta^u} \leqslant \lim_{t\to\infty}\inf\bigl\langle I(t)\bigr\rangle_t \leqslant \lim_{t\to\infty}\sup\bigl\langle I(t)\bigr\rangle_t \leqslant \frac{\langle R_0\rangle_\omega}{\beta^l},$$

where $R_0(t) = \mu(t)v(t) - (\mu(t) + \gamma(t))$, *and* $v(t)$ *is the unique positive* ω*-periodic solution of the equation*

$$v'(t) = \mu(t)v(t) - \beta(t),$$

where

$$v(t) = \frac{\int_t^{t+\omega} \exp\{\int_s^t \mu(\tau)\,\mathrm{d}\tau\}\beta(s)\,\mathrm{d}s}{1 - \exp\{-\int_0^\omega \mu(\tau)\,\mathrm{d}\tau\}}, \quad t \geqslant 0.$$

Proof. By using similar arguments in Zhao and Jing [22], we can get that if $\int_0^\omega (\mu(t) - \sigma^2/2)\,\mathrm{d}t > 0$, then

$$\frac{1}{t}\int_0^t \sigma S(s)\,\mathrm{d}B(s) = 0 \quad \text{a.s.} \tag{6}$$

and

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0 \quad \text{a.s.} \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0 \quad \text{a.s.} \quad \lim_{t\to\infty}\frac{R(t)}{t} = 0 \quad \text{a.s.} \tag{7}$$

By applying Itô formula, we have

$$\begin{aligned}
&\mathrm{d}\big(v(t)\big(S(t)+I(t)\big)\big)\\
&\quad= v(t)d\big(S(t)+I(t)\big)+v'(t)\big(S(t)+I(t)\big)\,\mathrm{d}t\\
&\quad= \big[v(t)\big(\mu(t)-\mu(t)S(t)+\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}-\big(\mu(t)+\gamma(t)\big)I(t)\\
&\qquad+\big(\mu(t)v(t)-\beta(t)\big)\big(S(t)+I(t)\big)\big]\,\mathrm{d}t\\
&\quad= \big[\mu(t)v(t)-\beta(t)S(t)-\big(v(t)\gamma(t)+\beta(t)\big)I(t)+v(t)\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\big]\,\mathrm{d}t.
\end{aligned}$$

This, together with (7), implies

$$\begin{aligned}
0&=\lim_{t\to\infty}\frac{v(t)(S(t)+I(t))}{t}\\
&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\mu(s)v(s)\,\mathrm{d}s-\frac{1}{t}\int_0^t\beta(s)S(s)\,\mathrm{d}s-\lim_{t\to\infty}\frac{1}{t}\int_0^t\big(v(s)\gamma(s)+\beta(s)\big)I(s)\,\mathrm{d}s\\
&\quad+\lim_{t\to\infty}\frac{1}{t}\int_0^t v(s)\gamma(s)I(s-\tau)\mathrm{e}^{-\mu(s)\tau}\,\mathrm{d}s\\
&=\langle\mu v\rangle_\omega-\lim_{t\to\infty}\langle\beta S\rangle_t-\lim_{t\to\infty}\big\langle(v\gamma+\beta)I\big\rangle_t+\lim_{t\to\infty}\big\langle v\gamma I(t-\tau)\mathrm{e}^{-\mu\tau}\big\rangle_t.
\end{aligned}\tag{8}$$

Also applying Itô formula, we have

$$\begin{aligned}
&\mathrm{d}\big(v(t)\big(S(t)+I(t)+R(t)\big)\big)\\
&\quad= v(t)\mathrm{d}\big(S(t)+I(t)+R(t)\big)+v'(t)\big(S(t)+I(t)+R(t)\big)\,\mathrm{d}t\\
&\quad= v(t)\big(\mu(t)-\mu(t)S(t)-\mu(t)I(t)-\mu(t)R(t)\big)\\
&\qquad+\big(\mu(t)v(t)-\beta(t)\big)\big(S(t)+I(t)+R(t)\big)\,\mathrm{d}t\\
&\quad= \big[v(t)\mu(t)-\beta(t)S(t)-\beta(t)I(t)-\beta(t)R(t)\big]\,\mathrm{d}t.
\end{aligned}$$

Then

$$\begin{aligned}
0&=\lim_{t\to\infty}\frac{v(t)(S(t)+I(t)+R(t))}{t}\\
&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\mu(s)v(s)\,\mathrm{d}s-\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(s)S(s)\,\mathrm{d}s-\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(s)I(s)\,\mathrm{d}s\\
&\quad-\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(s)R(s)\,\mathrm{d}s\\
&=\langle\mu v\rangle_\omega-\lim_{t\to\infty}\langle\beta S\rangle_t-\lim_{t\to\infty}\langle\beta I\rangle_t-\lim_{t\to\infty}\langle\beta R\rangle_t.
\end{aligned}$$

By Itô formula, we have

$$\mathrm{d}\ln I(t)=\beta(t)S(t)-\big(\mu(t)+\gamma(t)\big)-\frac{1}{2}\sigma^2S^2(t)+\sigma S(t)\,\mathrm{d}B(t).$$

Thus,

$$\ln I(t) = \ln I(0) + \int_0^t \beta(s)S(s)\,\mathrm{d}s - \int_0^t \big(\mu(s) + \gamma(s)\big)\,\mathrm{d}s - \frac{1}{2}\sigma^2 S^2(s)\,\mathrm{d}s + \int_0^t \sigma S(s)\,\mathrm{d}B(s).$$

On the one hand,

$$\ln I(t) = \langle R_0\rangle_\omega t - \int_0^t \big(v(s)\gamma(s) + \beta(s)\big)I(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2 S^2(s)\,\mathrm{d}s + \int_0^t v(s)\gamma(s)I(s-\tau)\mathrm{e}^{-\mu(s)\tau}\,\mathrm{d}s + \varphi_1(t), \tag{9}$$

where

$$\varphi_1(t) = \ln I(0) + \int_0^t \beta(s)S(s)\,\mathrm{d}s - \int_0^t \big(\mu(s) + \gamma(s)\big)\,\mathrm{d}s + \int_0^t \big(v(s)\gamma(s) + \beta(s)\big)I(s)\,\mathrm{d}s - \int_0^t v(s)\gamma(s)I(s-\tau)\mathrm{e}^{-\mu(s)\tau}\,\mathrm{d}s - \langle R_0\rangle_\omega t.$$

From (6) and (8) we have that

$$\lim_{t\to\infty} \frac{\varphi_1(t)}{t} = 0 \quad \text{a.s.}$$

In view of Eq. (9), we have

$$\ln I(t) \geqslant \langle R_0\rangle_\omega t - \big(v^u\gamma^u + \beta^u\big)\int_0^t I(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2\,\mathrm{d}s + \varphi_1(t).$$

From Lemma 1 we obtain

$$\lim_{t\to\infty}\inf \frac{1}{t}\int_0^t I(s)\,\mathrm{d}s \geqslant \frac{\langle R_0\rangle_\omega - \frac{1}{2}\sigma^2}{v^u\gamma^u + \beta^u}. \tag{10}$$

On the other hand,

$$\ln I(t) = \langle R_0\rangle_\omega t - \int_0^t \beta(s)I(s)\,\mathrm{d}s - \int_0^t \beta(s)R(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2 S^2(s)\,\mathrm{d}s + \varphi_2(t), \tag{11}$$

where

$$\varphi_2(t) = \ln I(0) + \int_0^t \beta(s)S(s)\,\mathrm{d}s + \int_0^t \beta(s)I(s)\,\mathrm{d}s + \int_0^t \beta(s)R(s)\,\mathrm{d}s - \int_0^t \big(\mu(s)+\gamma(s)\big)\,\mathrm{d}s + \int_0^t \sigma S(s)\,\mathrm{d}B(s) - \langle R_0\rangle_\omega t.$$

In view of Eq. (11), we have

$$\ln I(t) \leqslant \langle R_0\rangle_\omega t - \int_0^t \beta(s)I(s)\,\mathrm{d}s + \varphi_2(t) \leqslant \langle R_0\rangle_\omega t - \beta^l \int_0^t I(s)\,\mathrm{d}s + \varphi_2(t).$$

Equation (8) implies that $\lim_{t\to\infty} \varphi_2(t)/t = 0$ a.s. From Lemma 2 we have

$$\lim_{t\to\infty} \sup \frac{1}{t}\int_0^t I(s)\,\mathrm{d}s \leqslant \frac{\langle R_0\rangle_\omega}{\beta^l}. \tag{12}$$

From (10) and (12) we complete the proof of the theorem. □

Theorem 5. *Assume* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. *The disease will become extinct exponentially almost surely when* $\langle R_0\rangle_\omega < 0$ *and*

$$\int_0^\omega \left(\mu(t) - \frac{\sigma^2}{2}\right)\mathrm{d}t > 0.$$

Proof. It follows from Eq. (11) that

$$\ln I(t) \leqslant \langle R_0\rangle_\omega t + \varphi_2(t) \quad \text{a.s.}$$

Noting $\lim_{t\to\infty}\varphi_2(t)/t = 0$ a.s., we get

$$\lim_{t\to\infty} \sup \frac{\ln I(t)}{t} \leqslant \langle R_0\rangle_\omega \quad \text{a.s.},$$

which completes the proof. □

4 The dynamic behaviors of the solution in the disease-free state

In this section, we analyze the dynamic behavior of the solutions of the disease-free state. The following theorem shows that the solution of system (1) is asymptotical around of the bounding periodic solution $(1, 0, 0)$ if $\beta^u - \gamma^l < 0$. Furthermore, the intensity of the oscillations depends of σ.

Theorem 6. *If* $\beta^u - \gamma^l < 0$, *then system* (1) *is globally asymptotically stable, i.e., for any initial value* $X_0 \in \Gamma$, *the solution* $X(t)$ *will tend to the disease-free periodic solution* $(1, 0, 0)$ *asymptotically with probability* 1.

Proof. From the proof of Theorem 4 we obtain

$$\ln I(t) = \ln I(0) + \int_0^t \beta(s)S(s)\,\mathrm{d}s - \int_0^t \bigl(\mu(s) + \gamma(s)\bigr)\,\mathrm{d}s - \frac{1}{2}\sigma^2 S^2(s)\,\mathrm{d}s + \int_0^t \sigma S(s)\,\mathrm{d}B(s).$$

From (4) we have

$$\ln I(t) < \ln I(0) + \int_0^t \bigl(\beta^u - \mu^l - \gamma^l\bigr)\,\mathrm{d}s + \frac{1}{t}\int_0^t \sigma S(s)\,\mathrm{d}B(s).$$

Simplifying and dividing the above inequality by $t > 0$, it follows that

$$\frac{1}{t}\ln I(t) < \frac{1}{t}\ln I(0) + \frac{1}{t}\int_0^t \bigl(\beta^u - \mu^l - \gamma^l\bigr)\,\mathrm{d}s + \frac{1}{t}\int_0^t \sigma S(s)\,\mathrm{d}B(s).$$

Using the fact that $\lim_{t\to\infty}\int_0^t \sigma S(s)\,\mathrm{d}B(s)/t = 0$ a.s., we get

$$\lim_{t\to\infty}\frac{1}{t}\ln I(t) < \beta^u - \mu^l - \gamma^l < 0.$$

Thus,

$$I(t) \leqslant \mathrm{e}^{(\beta^u - \mu^l - \gamma^l)t} \tag{13}$$

for all $t > 0$. Since $I(t) > 0$ for all $t > 0$ a.s., we get

$$\lim_{t\to\infty} I(t) = 0 \quad \text{a.s.}$$

Now, from the third equation of system (1) it follows that

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} \leqslant \gamma^u I(t) - \mu^l R(t).$$

Thus,

$$R(t) \leqslant \mathrm{e}^{-\mu^l t}\gamma^u \int_0^t I(s)\mathrm{e}^{\mu^l s}\,\mathrm{d}s + R(0)\mathrm{e}^{-\mu^l t} \quad \text{a.s.},$$

and from (13) we have that

$$\begin{aligned} R(t) &\leqslant \mathrm{e}^{-\mu^l t}\gamma^u \int_0^t \mathrm{e}^{(\beta^u - \mu^l - \gamma^l)s}\mathrm{e}^{\mu^l s}\,\mathrm{d}s + R(0)\mathrm{e}^{-\mu^l t} \\ &= \mathrm{e}^{-\mu^l t}\gamma^u \int_0^t \mathrm{e}^{(\beta^u - \gamma^l)s}\,\mathrm{d}s + R(0)\mathrm{e}^{-\mu^l t}. \end{aligned}$$

Therefore, it follows that

$$\lim_{t\to\infty} R(t) = 0 \quad \text{a.s.}$$

Finally, we show that $\lim_{t\to\infty} S(t) = 1$. From the proof of Theorem 1 we know that

$$\lim_{t\to\infty} Z(t) \leqslant \lim_{t\to\infty} \bigl(Z(0)\mathrm{e}^{-\mu^l t} + 1\bigr) = 1.$$

Since $Z(t) = S(t) + I(t) + R(t)$, it is easy to know that

$$\lim_{t\to\infty} S(t) = 1.$$

Then we complete the proof. □

Next, we show the solution of (1) is oscillatory around of the disease-free periodic solution $(1, 0, 0)$, and the intensity of this oscillation depends on the parameter σ (which is the intensity of the noise).

Theorem 7. *Under the hypothesis of Theorem* 6*, and if $X_0 \in \Gamma$, then the solution $X(t)$ of system* (1) *has the property*

$$\lim_{t\to\infty} \sup \frac{1}{t}\mathbf{E} \int_0^t \bigl[\bigl(S(s) - 1\bigr)^2 + I^2(s) + R^2(s)\bigr]\,\mathrm{d}s \leqslant \frac{2\sigma^2 + 2\gamma^u \mathrm{e}^{-\mu^l \tau}}{k}.$$

Proof. Let $P(t) = S(t) - 1$, then system (1) can be rewritten as

$$\begin{aligned}
\mathrm{d}P(t) &= \bigl(-\mu(t)P(t) - \beta(t)I(t) - \beta(t)I(t)P(t) + \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\bigr)\,\mathrm{d}t \\
&\quad - \sigma\bigl(P(t) + 1\bigr)I(t)\,\mathrm{d}B(t), \\
\mathrm{d}I(t) &= \beta(t)P(t)I(t) - \bigl(\mu(t) + \gamma(t) - \beta(t)\bigr)I(t) + \sigma I(t)\bigl(P(t) + 1\bigr)\,\mathrm{d}B(t), \\
\mathrm{d}R(t) &= \gamma(t)I(t) - \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} - \mu R(t).
\end{aligned}$$

Define $V_3 = (P(t) + I(t))^2 + c_1 I(t) + c_2 R(t)$, where $P(t) \in \mathbb{R}$, $I(t), R(t) \in \Gamma$, and c_1, c_2 are positive constants to be determined later. Then

$$\begin{aligned}
\mathrm{d}V_3(t) &= 2\bigl(P(t) + I(t)\bigr)\,\mathrm{d}P + 2\bigl(P(t) + I(t)\bigr)\,\mathrm{d}I + c_1\,\mathrm{d}I + c_2\,\mathrm{d}R \\
&\quad + 2\sigma^2\bigl(P(t) + 1\bigr)^2 I^2\,\mathrm{d}t \\
&= \bigl[2\bigl(P(t) + I(t)\bigr)\bigl(-\mu(t)P(t) - \beta(t)I(t) - \beta(t)I(t)P(t) \\
&\quad + \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau}\bigr) + 2\bigl(P(t) + I(t)\bigr)\bigl(\beta(t)P(t)I(t) \\
&\quad - \bigl(\mu(t) + \gamma(t) - \beta(t)\bigr)I(t)\bigr) + c_1\bigl(\beta(t)P(t)I(t) \\
&\quad - \bigl(\mu(t) + \gamma(t) - \beta(t)\bigr)I(t)\bigr) + c_2\bigl(\gamma(t)I(t) - \gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} \\
&\quad - \mu(t)R(t)\bigr)\bigr]\,\mathrm{d}t + c_1\sigma I(t)\bigl(P(t) + 1\bigr)\,\mathrm{d}B(t) \\
&\quad + 2\sigma^2\bigl(P(t) + 1\bigr)^2 I^2(t)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \big[-2\mu(t)P^2(t) - 2\mu(t)P(t)I(t) - 2\beta(t)P(t)I(t) - 2\beta(t)I^2(t) \\
&\quad - 2\beta(t)I(t)P^2(t) - 2\beta(t)I^2(t)P(t) + 2P(t)\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} \\
&\quad + 2I(t)\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} + 2\beta(t)P^2(t)I(t) + 2\beta(t)I^2(t)P(t) \\
&\quad - 2\big(\mu(t)+\gamma(t)-\beta(t)\big)P(t)I(t) - 2\big(\mu(t)+\gamma(t)-\beta(t)\big)I^2(t) \\
&\quad + c_1\beta(t)P(t)I(t) - c_1\big(\mu(t)+\gamma(t)-\beta(t)\big)I(t) + c_2\gamma(t)I(t) \\
&\quad - c_2\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} - c_2\mu(t)R(t)\big]\,\mathrm{d}t \\
&\quad + c_1\sigma I(t)(P(t)+1)\,\mathrm{d}B(t) + 2\sigma^2\big(P(t)+1\big)^2 I^2(t)\,\mathrm{d}t \\
&= \big[-2\mu(t)P^2(t) - \big(2\beta(t) + 2\big(\mu(t)+\gamma(t)-\beta(t)\big)\big)I^2(t) - c_2\mu(t)R(t) \\
&\quad - \big(2\mu(t) + 2\beta(t) + 2\big(\mu(t)+\gamma(t)-\beta(t)\big) - c_1\beta(t)\big)P(t)I(t) \\
&\quad + \big(c_2\gamma(t) - c_1\big(\mu(t)+\gamma(t)-\beta(t)\big)\big)I(t) \\
&\quad + 2\gamma(t)\big(P(t)+I(t)\big)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} - c_2\gamma(t)I(t-\tau)\mathrm{e}^{-\mu(t)\tau} \\
&\quad + 2\sigma^2\big(P(t)+1\big)^2 I^2(t)\big]\,\mathrm{d}t + c_1\sigma I(t)\big(P(t)+1\big)\,\mathrm{d}B(t) \\
&\leqslant \big[-2\mu^l P^2(t) - 2\big(\mu^l+\gamma^l\big)I^2(t) - c_2\mu^l R^2(t) \\
&\quad - \big(4\mu^l + 2\gamma^l - c_1\beta^u\big)P(t)I(t) + \big(c_2\gamma^u - c_1\big(\mu^l+\gamma^l-\beta^u\big)\big)I(t) \\
&\quad + 2\sigma^2 + 2\gamma^u\mathrm{e}^{-\mu^l\tau}\big]\,\mathrm{d}t + c_1\sigma I(t)\big(P(t)+1\big)\,\mathrm{d}B(t).
\end{aligned}
$$

From the proof of Theorem 1 we know that $0 < S(t), I(t), R(t) \leqslant 1$ a.s., then $P(t) \leqslant 0$ a.s. And from the hypothesis we can choose $c_1 > 0$ and $c_2 > 0$ such that $4\mu^l + 2\gamma^l - c_1\beta^u = 0$, and $c_2\gamma^u - c_1(\mu^l + \gamma^l - \beta^u) = 0$, it follows that

$$
\begin{aligned}
\mathrm{d}V_3(t) &\leqslant \big[-2\mu^l P^2(t) - 2\big(\mu^l+\gamma^l\big)I^2(t) - c_2\mu^l R^2(t) + 2\sigma^2 + 2\gamma^u\mathrm{e}^{-\mu^l\tau}\big]\,\mathrm{d}t \\
&\quad + c_1\sigma I(t)\big(P(t)+1\big)\,\mathrm{d}B(t) \quad \text{a.s.}
\end{aligned} \tag{14}
$$

Integrating both sides of (14) from 0 to t and taking expectation, we obtain that

$$
\begin{aligned}
&\mathbf{E}\int_0^t \big[2\mu^l P^2(s) + 2\big(\mu^l+\gamma^l\big)I^2(s) + c_2\mu^l R^2(s)\big]\,\mathrm{d}s \\
&\quad \leqslant \mathbf{E}\big[V_3\big(P(0), I(0), R(0)\big)\big] + \big(2\sigma^2 + 2\gamma^u\mathrm{e}^{-\mu^l\tau}\big)t \quad \text{a.s.}
\end{aligned}
$$

Let $K = \min(2\mu^l, c_2\mu^l)$, we get that

$$
\limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\int_0^t \big[\big(S(s)-1\big)^2 + I^2(s) + R^2(s)\big]\,\mathrm{d}s \leqslant \frac{2\sigma^2 + 2\gamma^u\mathrm{e}^{-\mu^l\tau}}{K} \quad \text{a.s.}
$$

We complete the proof. □

5 Existence of ω-periodic solution

In this section, we shall investigate the existence of nontrivial positive periodic solution of system (1). First, we introduce some results concerning the periodic Markov process.

Definition 2. (See [14].) A Markov process $x(t)$ is ω-periodic if and only if its transition probability function is ω-periodic and the function $P_0(t, A) = \mathbf{P}\{X(t) \in A\}$ satisfies the equation $P_0(s, A) = \int_{\mathbb{R}^n} P_0(s, \mathrm{d}x)\mathbf{P}(s, x, s + \omega, A) \equiv P_0(s + \omega, A)$.

Consider the following integral equation:

$$X(t) = X(t_0) + \int_{t_0}^{t} b\big(s, X(s)\big) + \int_{t_0}^{t} \sigma_r\big(s, X(s)\big)\,\mathrm{d}B_r(s), \quad X \in \mathbb{R}^n. \tag{15}$$

Lemma 3. (*See* [22].) *Suppose that the coefficient of* (15) *is ω-periodic in t and satisfies the conditions*

$$\big|b(s, x) - b(s, y)\big| + \sum_{r=1}^{k} \big|\sigma_r(s, x) - \sigma_r(s, y)\big| \leqslant B|x - y|,$$
$$b(s, x) + \sum_{r=1}^{k} \big|\sigma_r(s, x)\big| \leqslant B\big(1 + |x|\big)$$

in every cylinder $I \times U$, where B is a constant. And suppose further that there exists a function $V(t, x) \in \mathbb{C}^2$ in $\mathbb{R}^n$, which is ω-periodic in t and satisfies the following conditions:

$$\inf_{|x|>R} V(t, x) \to \infty \quad \textit{as } R \to \infty \tag{16}$$

and

$$LV \leqslant -1 \quad \textit{out side some compact set}, \tag{17}$$

where the operator L is given by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i(t, x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$a_{ij} = \sum_{r=1}^{k} {\sigma_r}^i(t, x){\sigma_r}^j(t, x).$$

Then there exists a solution of (15)*, which is an ω-periodic Markov process.*

Remark 1. In view of the proof of Theorem 1, we can see that the local Lipschitz condition and linear growth condition are only used to guarantee the existence and uniqueness of the solution of system (1).

Now, we prove the existence of nontrivial positive periodic solution to stochastic system (1).

Theorem 8. *If* $\langle R_0\rangle_\omega - \sigma^2/2 > 0$*, then there exists an* ω*-periodic solution of system* (1).

Proof. For any $(S(0), I(0), R(0)) \in \Gamma$, system (1) has a unique global positive solution. Since the first two equations do not depend on the third equation of system (1), we equivalently consider the positive periodic solution of the first two equations. From (4) we denote $\Gamma_1 = \{(S, I)\colon\ S > 0,\ I > 0,\ 0 < S + I \leqslant 1\}$. Thus, we only need to present the proof for Theorem 8 in region Γ_1. According to Lemma 3, in order to prove Theorem 8, it suffices to find a $\mathbb{C}^2$-function $V(t, x)$ and a closed set $U \subset \mathbb{R}^2_+$ such that (16) and (17) hold. Take $0 < \alpha < \min\{\mu/\sigma^2, 1\}$ and choose constant $r_1 > 0$ such that

$$\begin{aligned} &\mu^l - \alpha\sigma^2 > 0, \qquad \mu^l + \gamma^l - \alpha\sigma^2 > 0,\\ &\max_{S\in\Gamma} f(S) - r_1\biggl(\langle R_0\rangle_\omega - \frac{1}{2}\sigma^2\biggr) \leqslant -2, \end{aligned} \tag{18}$$

where the function $f(x)$ is given in (19). The auxiliary function $V(t, x)$ is given as follows:

$$V(t, S, I) = \frac{1}{\alpha+1}(S + I)^{\alpha+1} - r_1\bigl(\ln I + v(t)(S + I) + \omega(t)\bigr) - \ln S,$$

where $v(t)$ is the unique positive ω-periodic solution of the equation $v'(t) = \mu v(t) - \beta(t)$. $\omega(t)$ is the function defined on $[0, +\infty)$ satisfying $\omega'(t) = \langle R_0\rangle_w - R_0(t)$ and $\omega(0) = 0$, where $R_0(t)$ is defined as Theorem 4. Obviously, $\omega(t)$ is an ω-periodic function on $[0, +\infty)$. Hence, $V(t, S, I)$ is ω-periodic in t and satisfies (16).

Next, we will find a closed set $U \subset \mathbb{R}^2_+$ such that $LV(t, S, I) \leqslant -1$, $(S, I) \in \Gamma_1 - U$. Denote $V_3 = ((S+I)^{\alpha+1}/(\alpha+1)$, $V_4 = -\ln I - v(t)(S+I) - \omega(t)$, $V_5 = -\ln S$. Then $LV = LV_3 + r_1LV_4 + LV_5$. In the rest of the proof, for simplicity, we always use μ to denote the function $\mu(t)$, the other parameter functions are the same. Direct calculation implies that

$$\begin{aligned} LV_3 &= (S + I)^\alpha\bigl(\mu - \mu S + \gamma I(t - \tau)\mathrm{e}^{-\mu\tau} - (\mu + \gamma)I\bigr)\\ &\quad + \sigma^2S^2I^2\alpha(S + I)^{\alpha-1}\\ &\leqslant 2^\alpha\mu\bigl(S^\alpha + I^\alpha\bigr) - \mu S^{1+\alpha} - (\mu + \gamma)I^{1+\alpha}\\ &\quad + 2^\alpha\bigl(S^\alpha + I^\alpha\bigr)\gamma I(t - \tau)\mathrm{e}^{-\mu\tau} + \alpha\sigma^2S^{1+\alpha} + \alpha\sigma^2I^{1+\alpha}\\ &\leqslant 2^\alpha\mu^u\bigl(S^\alpha + I^\alpha\bigr) - \mu S^{1+\alpha} - \bigl(\mu^l + \gamma^l\bigr)I^{1+\alpha}\\ &\quad + 2^\alpha\bigl(S^\alpha + I^\alpha\bigr)\gamma^u\mathrm{e}^{-\mu^l\tau} + \alpha\sigma^2S^{1+\alpha} + \alpha\sigma^2I^{1+\alpha}\\ &= \bigl(S^\alpha + I^\alpha\bigr)\bigl(2^\alpha\mu^u + 2^\alpha\gamma^u\mathrm{e}^{-\mu^l\tau}\bigr) - \bigl(\mu^l - \alpha\sigma^2\bigr)S^{1+\alpha}\\ &\quad - \bigl(\mu^l + \gamma^l - \alpha\sigma^2\bigr)I^{1+\alpha}, \end{aligned}$$

$$\begin{aligned}
LV_4 &= -\frac{1}{I}\big(\beta SI - (\mu+\gamma)I\big) - v'(S+I) \\
&\quad - v\big(\mu - \mu S + \gamma I(t-\tau)\mathrm{e}^{-\mu\tau} - (\mu+\gamma)I\big) + \frac{1}{2}\sigma^2 S^2 - \omega'(t) \\
&= -(\beta - \mu v + v')S - \big(v' - v(\mu+\gamma)\big)I + (\mu+\gamma) - \mu v \\
&\quad - v\gamma I(t-\tau)\mathrm{e}^{-\mu\tau} + \frac{1}{2}\sigma^2 S^2 - \omega'(t) \\
&= -\mu v + (\mu+\gamma) + (\beta + v\gamma)I - v\gamma I(t-\tau)\mathrm{e}^{-\mu\tau} + \frac{1}{2}\sigma^2 S^2 - \omega'(t) \\
&\leqslant -v\mu + (\mu+\gamma) + (\beta + v\gamma)I + \frac{1}{2}\sigma^2 S^2 - \langle R_0\rangle_\omega + R_0(t) \\
&\leqslant -\langle R_0\rangle_\omega + \big(\beta^u + v^u\gamma^u\big)I + \frac{1}{2}\sigma^2, \\
LV_5 &= -\frac{1}{S}\big(\mu - \mu S - \beta SI + \gamma I(t-\tau)\mathrm{e}^{-\mu\tau}\big) + \frac{1}{2}\sigma^2 I^2 \\
&= -\frac{1}{S}\mu + \mu + \beta I - \frac{\gamma I(t-\tau)\mathrm{e}^{-\mu\tau}}{S} + \frac{1}{2}\sigma^2 I^2 \\
&\leqslant -\frac{\mu^l}{S} + \mu^u + \beta^u I + \frac{1}{2}\sigma^2.
\end{aligned}$$

Hence, $LV \leqslant f(S) + g(I)$, where

$$\begin{aligned}
f(S) &= \big(2^\alpha\mu^u + 2^\alpha\gamma^u\mathrm{e}^{-\mu^l\tau}\big)S^\alpha - \big(\mu^l - \alpha\sigma^2\big)S^{1+\alpha} - \frac{\mu^l}{S} + \mu^u + \frac{1}{2}\sigma^2, \qquad (19)\\
g(I) &= \big(2^\alpha\mu^u + 2^\alpha\gamma^u\mathrm{e}^{-\mu^l\tau}\big)I^\alpha - \big(\mu^l + \gamma^l - \alpha\sigma^2\big)I^{1+\alpha} + \beta^u I \\
&\quad + r_1\left[\frac{1}{2}\sigma^2 - \langle R_0\rangle_\omega + \big(\beta^u + v^u\gamma^u\big)I\right].
\end{aligned}$$

In view of (18), we can obtain

$$f(S) + \max_{I\in\Gamma_1} g(I) \to -\infty \quad \text{as } S \to 0^+,$$

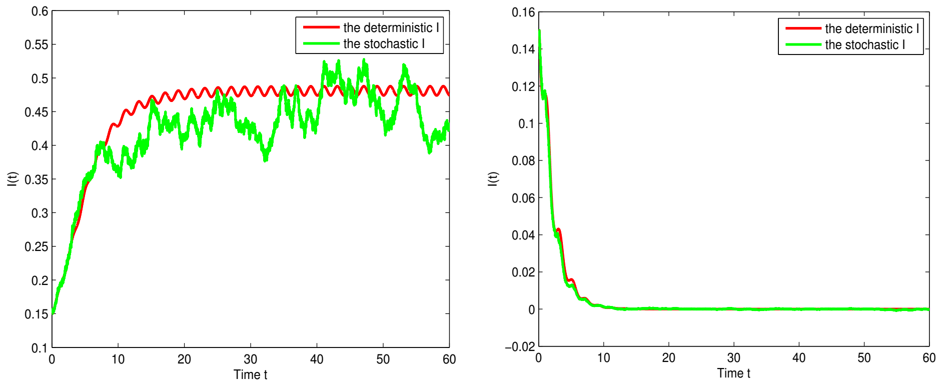
and

$$\max_{S\in\Gamma} f(S) + g(I) \to \max_{S\in\Gamma_1} f(S) - r_1\left(\langle R_0\rangle_\omega - \frac{1}{2}\sigma^2\right) \leqslant -2 \quad \text{as } I \to 0^+.$$

Take $k > 0$ small enough, and let $U := [k, 1/k] \times [k, 1/k]$. It follows that $LV < -1$, $(S, I) \in \Gamma_1 - U$. The proof is completed. □

6 Simulation and conclusion

In this section, we will illustrate our analytical results by some examples with the help of numerical simulations firstly.



(a) (b)

Figure 1. The path of $I(t)$ for the stochastic model (1) and its corresponding deterministic model with parameters in Example 1 (a) and Example 2 (b).

Example 1. For numerical simulations of the stochastic model (1), we choose $\mu(t) = 0.06 + 0.012\cos\pi t$, $\gamma(t) = 0.05 + 0.14\sin\pi t$, $\beta(t) = 0.15 + 0.05\sin\pi t$, $\tau = 0.1$, and the initial values are taken as $S(0) = 0.65$, $I(0) = 0.15$, $R(0) = 0.05$. We carry out the numerical simulation with noise intensities $\sigma = 0.1$ for the stochastic differential equation model (SDE), compare to its deterministic model with $\sigma = 0$ (i.e., the SDE model (1) without noise leads to ordinary differential equation (ODE) model). By computing, $\langle R_0\rangle_\omega - \sigma^2/2 = 0.044 > 0$, and $\int_0^\omega(\mu(t) - \sigma^2/2)\,\mathrm{d}t = 0.39 > 0$. Thus, all conditions in Theorem 4 hold, and the disease of SDE model (1) is persistent in the mean; see Fig. 1(a). The green curve is the path of disease $I(t)$ of SDE model, and the red curve is the path of disease $I(t)$ of the corresponding deterministic model.

Example 2. For numerical simulations of the stochastic model (1), we choose $\mu(t) = 0.4 + 0.55\cos t$, $\gamma(t) = 0.15 + 0.05\sin t$, $\beta(t) = 0.05 + 0.4\sin t$, $\tau = 0.12$, and the initial values are taken as $S(0) = 0.55$, $I(0) = 0.15$, $R(0) = 0.15$. We start our numerical simulation with noise intensities $\sigma = 0$ and $\sigma = 0.076$, respectively. By computing, we obtain $\langle R_0\rangle_\omega = -0.954 < 0$ and $\int_0^\omega(\mu(t) - \sigma^2/2)\,\mathrm{d}t = 0.79 > 0$, then the conditions in Theorem 5 hold, and the disease of SDE model (1) will become extinct exponentially almost surely; see Fig. 1(b). The green curve is the path of disease $I(t)$ of SDE model, and the red curve is the path of disease $I(t)$ of the corresponding deterministic model.

Example 3. For numerical simulations of the stochastic model (1), we choose $\mu(t) = 0.24 + 0.6\cos\pi t$, $\gamma(t) = 0.88 + 0.04\sin\pi t$, $\beta(t) = 2.6 + 1.2\sin\pi t$, $\tau = 1$, and the initial values are taken as $S(0) = 0.4$, $I(0) = 0.4$, $R(0) = 0.3$. We carry out our numerical simulation with noise intensities $\sigma = 0$ and $\sigma = 0.068$, respectively. By computing, we obtain $\langle R_0\rangle_\omega - \sigma^2/2 = 2.1514 > 0$. The corresponding numerical simulations are given in Fig. 2. The right ones show that the model (1) has a stochastic positive 2-periodic solution, and the left ones are the numerical simulations on the deterministic counterpart of system (1).

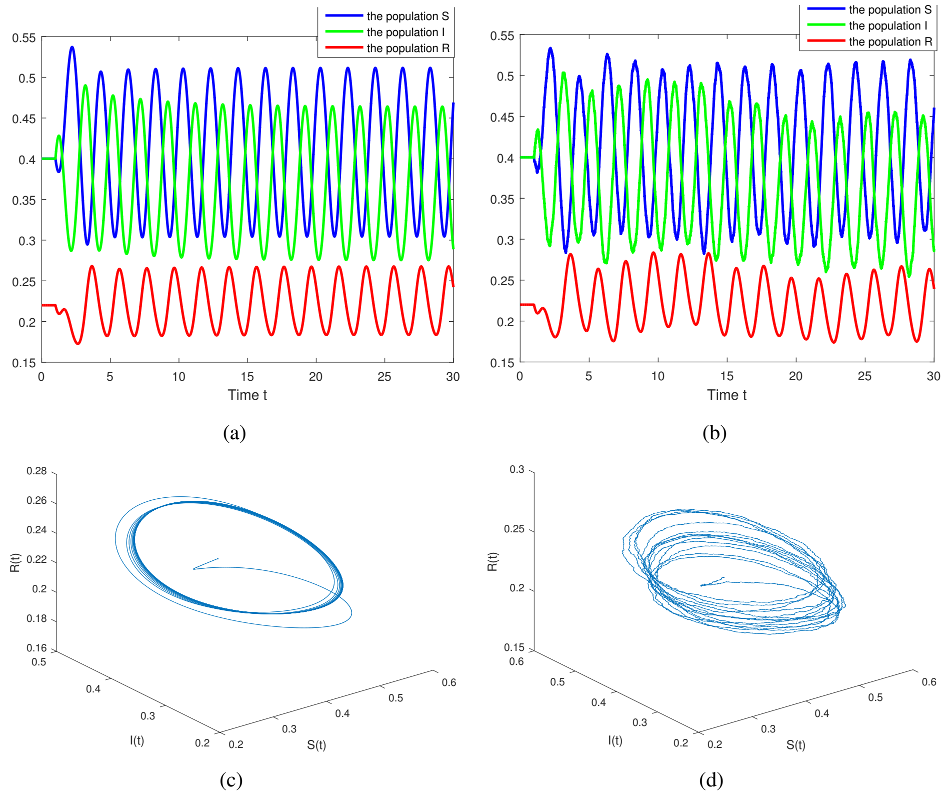


Figure 2. The paths and the trajectories of periodic solution $(S(t), I(t), R(t))$ for the stochastic model (1) with parameters in Example 3 compared to the corresponding deterministic model: (a), (c) are for deterministic model, and (b), (d) are for stochastic model (1).

In this paper, we consider a stochastic seasonal epidemic model with time delay in a population, where the stochasticity is introduced on the baseline transmission rate. We investigate the existence and uniqueness of the solution of the stochastic model and prove positivity and boundedness. In addition, we show the system is stochastically ultimately bounded. We obtain the threshold of stochastic system, which determines whether the disease occurs or not, i.e., when $\langle R_0\rangle_\omega - \sigma^2/2 > 0$, the disease will persist, and the disease will become extinct exponentially almost surely when $\langle R_0\rangle_\omega < 0$. Especially, the dynamic behavior of the solution in the disease-free state is discussed, we prove that the solution of system (1) is oscillatory around of the disease-free state $(1, 0, 0)$, and the intensity of this oscillation depends on the intensity of the noise. Furthermore, by using the Khasmimskii's boundary periodic Markov processes, the existence of stochastic nontrivial periodic solutions for the model is also obtained.

During the process of proving the persistence and extinction and the existence of positive periodic solutions of the stochastic system, we extend the method of proving nonautonomous stochastic systems to nonautonomous stochastic differential systems with time delays. It has been illustrated that this method is suitable for models with a stationnary population. We believe that this approach can also be applied to some nonautonomous and deleyed models in different areas.

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