A geometrical criterion for nonexistence of constant-sign solutions for some third-order two-point boundary value problems

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Abstract. We give a simple geometrical criterion for the nonexistence of constant-sign solutions for a certain type of third-order two-point boundary value problem in terms of the behavior of nonlinearity in the equation. We also provide examples to illustrate the applicability of our results.

Keywords: third-order two-point boundary value problems, nonexistence of solutions, comparison methods for the first zero functions.

1 Introduction

We study boundary value problem consisting of the nonlinear third-order differential equation

$$x''' = -p(t)f(x),\tag{1}$$

subject to the two-point boundary conditions

$$x(t_0) = x'(t_0) = 0 = x(b),\tag{2}$$

where $b > t_0$, $f : \mathbb{R} \to \mathbb{R}$ and $p : [t_0, +\infty) \to [0, +\infty)$ are continuous, $xf(x) > 0$ for $x \neq 0$, and for every $\varepsilon > 0$, there exists $\xi \in (t_0, t_0 + \varepsilon)$ such that $p(\xi) > 0$. Some additional assumptions on the function p will be introduced later.

It is the intent of this paper to provide a simple geometrical criterion for nonexistence of constant-sign solutions to boundary value problem (1), (2). By a constant-sign solution of (1), (2), we mean a solution $x(t)$ such that $x(t) \neq 0$ for $t \in (t_0, b)$.

Many papers deal with nonexistence results. We mention only few [1, 7, 8, 12] for the third-order problems and [5, 10, 11] for the second-order problems.

This investigation was motivated by the paper [2] in which the authors obtain sufficient conditions for the existence and nonexistence of positive solutions to the third-order

two-point boundary value problem. The nonexistence theorems are formulated there in terms of inequalities of the type we use. To prove their results, the authors use so-called Guo–Krasnosel'skii fixed point theorem [3,6], but in our proofs, we employ comparison methods for the first zero functions.

The key idea is the following. Let us consider solution $x(t, \gamma)$ of Eq. (1), which satisfies initial conditions

$$x(t_0) = x'(t_0) = 0, \qquad x''(t_0) = \gamma \neq 0. \tag{3}$$

Suppose that $x(t, \gamma)$ is unique with respect to initial data. Let us denote the first zero of $x(t, \gamma)$ (if it exists) by t_1 $(t_1 > t_0)$. The first zero t_1 is a function of γ.

If, for all $\gamma \neq 0$, $t_1(\gamma) \neq b$, then boundary value problem (1), (2) has no nontrivial constant-sign solutions (has only the trivial solution).

For some equations (for example, linear equations, Emden–Fowler-type equations [9]), it is possible to find analytic expressions for the first zero function, but mostly it is impossible. In these cases, we can employ comparison methods.

The idea is very simple. Comparing the right-hand side functions in equations, we can make a conclusion about the corresponding first zero functions. It is convenient to compare with linear equations because linear equations are well researched.

The rest of the paper is organized as follows. The next section contains some preliminary results. Section 3 contains nonexistence results as well as some examples.

2 Preliminary results

We start with an elementary observation.

Proposition 1. *If $x(t, \gamma)$ is a nontrivial solution of initial value problem* (1)*,* (3) *and $x(t_1, \gamma) = 0$ $(x(t, \gamma) \neq 0$ for $t \in (t_0, t_1))$, then $x'(t_1, \gamma) \neq 0$.*

Proof. In view of assumptions on functions f and p, we get

$$0 > \int_{t_0}^{t} xx''' \, ds = x(t, \gamma) \, x''(t, \gamma) - \int_{t_0}^{t} x'x'' \, ds$$

$$= x(t, \gamma) \, x''(t, \gamma) - \frac{1}{2}x'(t, \gamma)^2 + \frac{1}{2}x'(t_0, \gamma)^2.$$

If $x'(t_1, \gamma) = 0$, we have the contradiction $x'(t_0, \gamma)^2 < 0$. □

Since we want to compare nonlinear equation with linear one, let us consider auxiliary problem for linear equation

$$y''' = kp(t)y, \tag{4}$$

$$y(t_0) = 0, \qquad y(\tau_1) = y'(\tau_1) = 0, \tag{5}$$

where $k > 0$ and $\tau_1 > t_0$ is the first conjugate point of t_0 $(y(t) \neq 0$ for $t \in (t_0, \tau_1))$.

The concept of conjugate points for linear third-order equations was introduced by Hanan in his famous work [4]. To provide the existence of τ_1, we use the additional assumption [4, Thm. 5.7] on the function p

$$\liminf_{t \to \infty} \left(t^3 p(t) \right) > \frac{2}{3\sqrt{3}}.$$

Note that this condition holds throughout the paper.

The first conjugate point of t_0 is independent of initial data at the point t_0, but τ_1 depends on k or $\tau_1 = \tau_1(k)$. Moreover, τ_1 is a decreasing function of k.

Let us denote solution of auxiliary problem (4), (5) by $y(t)$.

The next proposition plays an important role in the proofs of our main results.

Proposition 2. *If $x(t, \gamma)$ is a solution of initial value problem* (1), (3) *and $y(t)$ is a solution of auxiliary problem* (4), (5), *then*

$$x(\tau_1, \gamma)\, y''(\tau_1) = \int_{t_0}^{\tau_1} p(t)y(t)\big[k\, x(t, \gamma) - f\big(x(t, \gamma)\big)\big]\, \mathrm{d}t. \tag{6}$$

Proof. Multiplying Eq. (1) by $y(t)$, Eq. (4) by $x(t, \gamma)$ and adding, we get

$$x'''(t, \gamma)\, y(t) + y'''(t)x(t, \gamma) = p(t)y(t)\big[kx(t, \gamma) - f\big(x(t, \gamma)\big)\big].$$

Then integrating from t_0 to τ_1, we obtain formula (6).   □

3   Main results

The next proposition is a straight consequence of formula (6).

Proposition 3. *If there exists a nontrivial solution $x(t, \gamma_0)$ of initial value problem* (1), (3) *such that $t_1(\gamma_0) = \tau_1(k)$, then there exists $x_0 \neq 0$ such that $kx_0 = f(x_0)$.*

Proof. Since $t_1(\gamma_0) = \tau_1(k)$, we have $x(\tau_1, \gamma_0) = 0$, and in view of formula (6), we get

$$\int_{t_0}^{\tau_1} p(t)y(t)\big[kx(t, \gamma_0) - f\big(x(t, \gamma_0)\big)\big]\, \mathrm{d}t = 0.$$

Since $p(t) \geqslant 0$, $y(t) \neq 0$ for $t \in (t_0, \tau_1)$, it follows that there exists $x_0 \neq 0$ such that $kx_0 - f(x_0) = 0$ (if not, then for all $x \neq 0$, $kx - f(x) \neq 0$, and we get the contradiction that the integral is not equal to zero).   □

By the law of contraposition, we get the next corollary.

Corollary 1. *If, for all $x \neq 0$, $kx \neq f(x)$, then, for all nontrivial solutions $x(t, \gamma)$ of initial value problem* (1), (3), *$t_1(\gamma) \neq \tau_1(k)$ ($t_1(\gamma) < \tau_1(k)$ or $t_1(\gamma) > \tau_1(k)$).*

We can formulate the last corollary in terms of solutions of boundary value problem (1), (2).

Theorem 1. *If, for all $x \neq 0$, $kx \neq f(x)$, then boundary value problem* (1), (2) *with* $b = \tau_1(k)$ *has no nontrivial constant-sign solutions.*

Our nonexistence Theorem 1 has a simple geometrical interpretation. Employing auxiliary problem (4), (5), we can find k for which $\tau_1(k) = b$, then we can construct the graph of the function $f(x)$ and the straight line kx, and if, for all $x \neq 0$, the straight line kx does not intersect the graph of $f(x)$, then boundary value problem (1), (2) has no nontrivial constant-sign solutions.

To illustrate the last theorem, let us consider the next example.

Example 1. Consider the problem

$$x''' = -25 \arctan x, \qquad x(0) = x'(0) = 0 = x(1) \tag{7}$$

and the auxiliary linear problem

$$y''' = ky, \qquad y(0) = 0 = y(1) = y'(1).$$

In view of $\tau_1(k) = 1$, we get $k \approx 75.8593$. Thus, constructing the graphs, we get that problem (7) has no nontrivial constant-sign solutions (see Fig. 1).

We can check this result constructing the first zero function for the problem by using program Mathematica 7.0 (see Fig. 2).
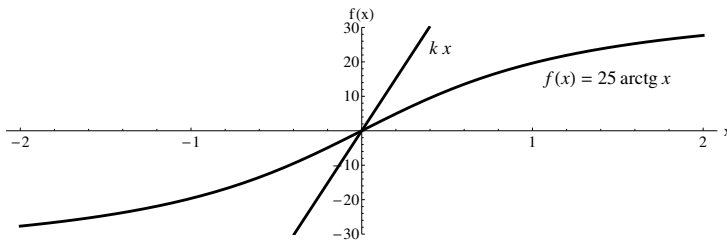

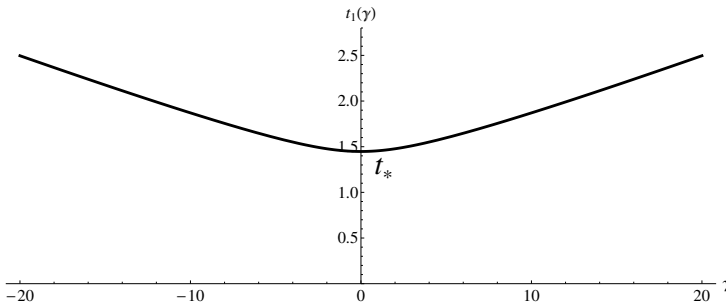**Figure 1.** The graphs of the functions $f(x)$ and kx.

Figure 2. The graph of the first zero function $t_1(\gamma)$.

As we can see, for every b from the interval $(0, t_*)$, the problem has no nontrivial constant-sign solutions.

In general, we can find intervals for b, where boundary value problem (1), (2) has no nontrivial constant-sign solutions using formula (6). Let us consider some auxiliary propositions.

Proposition 4. *If, for all x, $|f(x)| \geqslant k_2|x|$, then, for all nontrivial solutions $x(t, \gamma)$ of initial value problem (1), (3), $t_1(\gamma) \leqslant \tau_1(k_2)$.*

Proof. Let us suppose that there exists a nontrivial solution $x(t, \gamma_0)$ such that $t_1(\gamma_0) > \tau_1(k_2)$. Let $t_1(\gamma_0) = \tau_1(k_3)$, where $k_3 < k_2$ because $\tau_1(k)$ decreases in k. Then, by Proposition 3, there exists $x_0 \neq 0$ such that $k_3 x_0 = f(x_0)$. We get the contradiction since, for all x, $|f(x)| \geqslant k_2|x|$.   □

Proposition 5. *If, for all x, $|f(x)| \leqslant k_1|x|$, then, for all nontrivial solutions $x(t, \gamma)$ of initial value problem (1), (3), $t_1(\gamma) \geqslant \tau_1(k_1)$.*

Proof. The proof is analogous to that of Proposition 4.   □

Corollary 2. *If, for all x,*
$$k_2|x| \leqslant |f(x)| \leqslant k_1|x|,$$
then, for all nontrivial solutions $x(t, \gamma)$ of initial value problem (1), (3),

$$\tau_1(k_1) \leqslant t_1(\gamma) \leqslant \tau_1(k_2).$$

We can formulate the last corollary in terms of solutions of boundary value problem (1), (2).

Theorem 2. *If, for all x,*
$$k_2|x| \leqslant |f(x)| \leqslant k_1|x|,$$
then boundary value problem (1), (2) with $b < \tau_1(k_1)$ or $b > \tau_1(k_2)$ has no nontrivial constant-sign solutions.

Remark 1. Unfortunately, we cannot use the result on the boundedness of the first zero function (Corollary 2) to state the criterion for the existence of nontrivial constant-sign solutions because we cannot guarantee that the range of the first zero function is interval $[\tau_1(k_1), \tau_1(k_2)]$. As the examples show, very often the range of $t_1(\gamma)$ is only a subinterval of $[\tau_1(k_1), \tau_1(k_2)]$.

Example 2. Consider the problem

$$x''' = -x - \sin x,$$
$$x(0) = x'(0) = 0, \qquad x''(0) = \gamma.$$

As we can see, the graph of the function $f(x) = x + \sin x$ lies between two straight lines $2x$ and $0.78x$ (Fig. 3). By Corollary 2, we get that the first zero function is bounded by
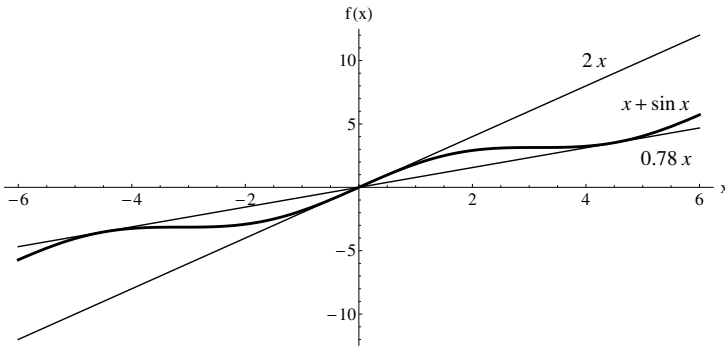
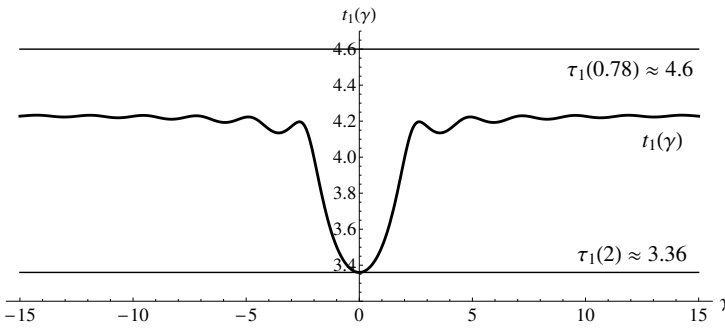**Figure 3.** The graph of the function $f(x) = x + \sin x$.

Figure 4. The first zero function.

two constants 4.6 and 3.36, where $\tau_1(0.78) \approx 4.6$ and $\tau_1(2) \approx 3.36$. Constructing the first zero function for the problem by using program Mathematica 7.0, we can make sure that the range of $t_1(\gamma)$ is only a subinterval of $[3.36, 4.6]$ (Fig. 4). So, if $b < \tau_1(2)$ or $b > \tau_1(0.78)$, then (by Theorem 2) boundary value problem

$$x''' = -x - \sin x,$$
$$x(0) = x'(0) = 0 = x(b)$$

has no nontrivial constant-sign solutions, but if, for example, $b = 3.5$, then the problem has exactly two nontrivial constant-sign solutions.

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