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SYNCHRONIZATION OF DELAY DYNAMICAL SYSTEMS

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INTRODUCTION

A wide class of dynamical systems can be described by the first-order delay differential equation (DDE) [1-4]:

$$dx/dt = N[x(t), x(t-\tau)]. \tag{1}$$

Here N is the nonlinear function and τ is the time delay.

In contrary to the ordinary differential equations (ODEs), where a certain but finite number of initial conditions $x_i(0)$ should be given, the evolution of the dynamical variable $x(t)$ in the DDEs is determined by the initial function $\varphi(t)$:

$$x(t) \equiv \varphi(t) \quad \text{for} \quad t \in [-\tau, 0],$$

that is the problem to be defined needs initial data over an interval of length τ. Since the continuous function $\varphi(t)$ contains an infinite number of data points, the systems given by Eq.(1) are referred to as the infinite-dimensional ones [1,4].

There are many practical examples of dynamical systems with delay in electronics, optics, laser physics, physiology, population biology, economics. Many of the references can be found in [5].

An interesting feature of the DDEs is that depending on the time delay τ they may exhibit periodic, chaotic and even very complex hyperchaotic behaviour characterized by multiple positive Lyapunov exponents [1,2].

In recent years synchronization of chaotic systems [6,7] has attracted much attention. This phenomenon is supposed to have intriguing applications in secure communications [8,9]. However, simple chaotic systems with only one positive Lyapunov exponent do not ensure a sufficient level of security. In this regard hyperchaotic systems are more advantageous ones [10-13] and the delay dynamical systems seem to be good candidates.

MACKEY-GLASS SYSTEM

In some cases Eq.(1) can be given in the following form:

$$dx/dt = -x(t) + N[x(t-\tau)]. \tag{2}$$

An example is the Mackey-Glass (MG) system [1-4,14]:

$$dx/dt = -x(t) + \frac{ax(t-\tau)}{1+x^c(t-\tau)} \tag{3}$$

introduced as a model of blood production. The common parameters are $a = 2$, $c = 10$, and τ varied [1,2,4]. A linear and a numerical analysis of Eq.(3) shows, that there is a stable fixed point attractor for $\tau<0.47$, a stable limit cycle for $0.47 < \tau < 1.33$, a period doubling sequence for $1.33 < \tau < 1.68$, and chaotic attractors for $\tau > 1.68$. Hyperchaotic behaviour characterized by more than two positive Lyapunov exponents is observed when τ approaches 4. A hyperchaotic solution of Eq.(3) is illustrated in Fig.1 and Fig.2.

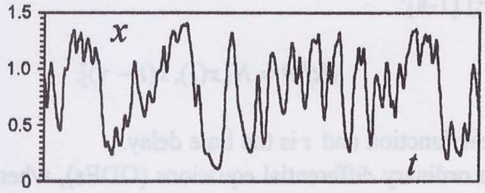

Fig. 1. Hyperchaotic time series $x(t)$ from Eq.(3). $a=2$, $c = 10$, $\tau = 6$.

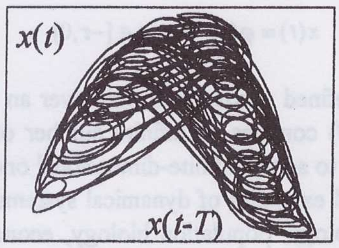

Fig. 2. Phase portrait $x(t)$ *versus* $x(t-T)$ from Eq.(3). $a=2$, $c = 10$, $\tau = 6$, $T = \tau$.

An electronic oscillator (Fig. 3) which simulates the MG model has been proposed [15], developed and investigated experimentally [16]. It contains a tunable delay line DEL, a nonlinear device ND and an RC filter. The circuit provides a convenient tool for modelling and studying chaotic phenomena in dynamical systems with time delays. The MG model along with its electronic analogue [16] have been intensively exploited as the test systems for various techniques developed to stabilize unstable fixed points [17-19], unstable periodic orbits [17,20], and to control chaotic states [21] in delay dynamical systems.

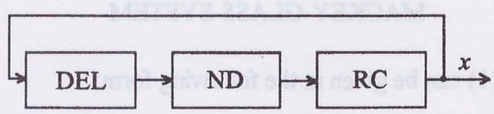

Fig. 3. Block diagram of an electronic analogue of the Mackey-Glass system.

SYNCHRONIZATION OF THE MACKEY-GLASS SYSTEM

One of the possible ways to synchronize two hyperchaotic oscillators is to apply a linear term in the form of the difference between of the state variable of the drive system and the state variable of the response one, $k(x_1 - x_2)$ [9,12,13]:

$$dx_2/dt = -x_2(t) + \frac{ax_2(t-\tau)}{1+x_2^c(t-\tau)} + k[x_1(t) - x_2(t)], \quad (4)$$

where $x_1(t)$ is the variable from Eq.(3) with a subscript "1" used to distinguish the drive system from the response system indicated with a subscript "2".

Actually, this method employs the idea known in the control theory as "closed-loop state feedback". It was used to control the unstable fixed points and unstable periodic orbits in the MG system [17-20], where the target fixed point x_0 or the target limit cycle $x_p(t)$ were taken instead of the variable $x_1(t)$.

The corresponding experimental setup to synchronize two chaotic oscillators is shown in Fig. 4 and the experimental results [22] are illustrated in Fig. 5.

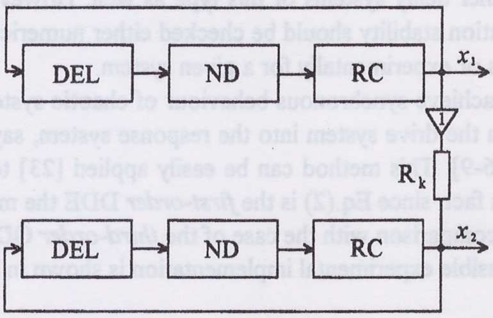

Fig. 4. Block diagram for synchronization of two active MG oscillators.

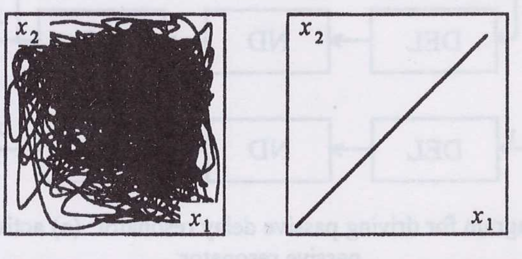

Fig. 5. Phase portraits of unsynchronized, $k = 0$ (left) and synchronized, $k > k_{cr}$ (right) MG systems, x_2 vs. x_1 from Eqs.(3,4) at $\tau = 6$ [22].

The stability properties of the synchronization and the critical value of the coupling coefficient k_{cr} can be obtained from the transversal Lyapunov exponents introduced in [7]. As it is seen from Fig. 6 the largest Lyapunov exponent turns to be negative at $k = k_{cr} = 0.65$, thus ensuring robust synchronization of the MG system for a given set of parameters.

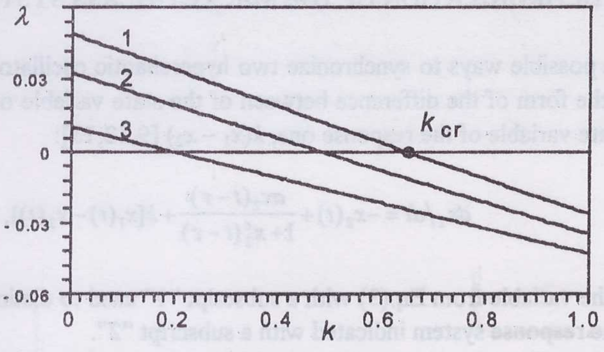

Fig. 6. Three largest transversal Lyapunov exponents $\lambda_{1,2,3}$ as the functions of the coupling coefficient k from Eqs.(3,4). $\tau = 6$.

SYNCHRONIZING OTHER DELAY DYNAMICAL SYSTEMS

The synchronization method considered in the previous section for the MG system can be applied to other delay systems of this type as well. However, in every specific case the synchronization stability should be checked either numerically by means of the Lyapunov exponents or experimentally for a given system.

Another way to achieve synchronous behaviour of chaotic systems is to plug some of the variables from the drive system into the response system, say into the *nonlinear terms* of the latter [6-9]. This method can be easily applied [23] to delay systems described by Eq.(2). In fact, since Eq.(2) is the *first-order* DDE the method appears to be extremely simple in comparison with the case of the *third-order* ODEs [8].

A diagram of possible experimental implementation is shown in Fig. 7.

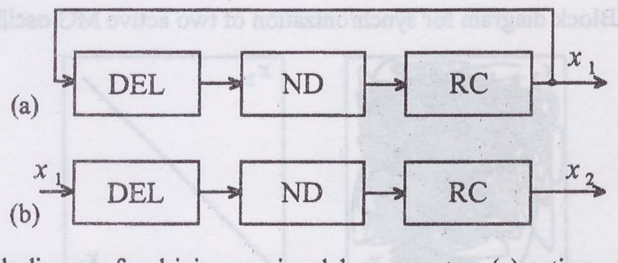

Fig. 7. Block diagram for driving passive delay resonator: (a) active oscillator, (b) passive resonator.

In contrast to the active oscillator (Fig. 7a) the circuit in Fig. 7b is a passive one similarly to the fourth-order nonlinear resonator considered in [24]. The corresponding differential equation (for $x_1 = 0$) is

$$dx_2/dt = -x_2(t).$$

It has a single stable stationary solution $x_2 = 0$.

Driving this circuit with the external signal $x_1(t)$ from the active oscillator yields an equation:

$$dx_2/dt = -x_2(t) + N[x_1(t-\tau)]. \tag{5}$$

Introducing the error variable $e = x_1 - x_2$ [8] a simple equation governing the error dynamics is obtained: $de/dt = -e$, which is globally asymptotically stable at the origin ($e = 0$). Consequently, synchronous response of the passive resonator is guaranteed ($x_2 \rightarrow x_1$).

This method of synchronization can be readily applied to secure communication. Suppose, that an information signal $i(t)$ is injected into the transmitter [25] and is masked with the chaotic signal $x_1(t)$ at the output $s(t) = i(t) + x_1(t)$ [8,25]. Then the $s(t)$ is transmitted into the channel. The receiver is easily synchronized to the transmitter no matter that the "intrinsic" signal $x_1(t)$ in the nonlinear terms is substituted with more complex signal $s(t)$ [25]:

$$dx_1/dt = -x_1(t) + N[s(t-\tau)], \tag{5a}$$

$$dx_2/dt = -x_2(t) + N[s(t-\tau)]. \tag{5b}$$

The synchronization error $e(t)$ vanishes also in this case since it is described by the same equation $de/dt = -e$. Once the receiver and the transmitter are synchronized ($x_2 = x_1$) extracting of the message is a straightforward procedure. The receiver output is $i^*(t) = s(t) - x_2(t) = i(t) + x_1(t) - x_2(t) = i(t)$, i.e. the information signal $i(t)$ is easily recovered.

A diagram of the experimental implementation of secure communication with the delay dynamical systems is presented in Fig. 8.

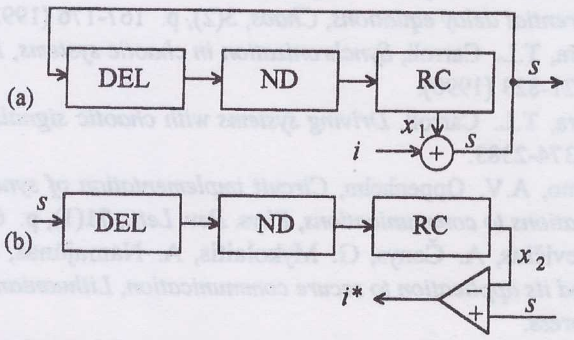

Fig. 8. Block diagram for secure communication: (a) transmitter, (b) receiver.

We note, however, that the level of $i(t)$ should be sufficiently lower than the level of $x_1(t)$. There are two reasons for such a requirement. The first one is that the $i(t)$ should not influence chaotic behaviour of the system too much. The second one is that the $i(t)$ itself should be efficiently masked with the $x_1(t)$ in the transmitted signal $s(t)$.

GENERAL CASE

In general case given by Eq.(1) more sophisticated synchronization methods should be used since simple replacing of $x_2(t)$ in the response system with $x_1(t)$ does not ensure robust synchronization. A simple way is just to add a linear control term $b[x_1(t) - x_2(t)]$:

$$dx_2/dt = N[x_1(t), x_1(t-\tau)] + b[x_1(t) - x_2(t)]. \qquad (6)$$

This method is similar in a sense to the double synchronization technique employed to synchronize third-order chaotic systems [9].

The error dynamics is governed by $de/dt = -be$. Consequently, synchronous response is achieved at any $b > 0$. In contrast to the common synchronization method *via* $k[x_1(t) - x_2(t)]$ described for the MG system there is no critical value for the coefficient b in Eq.(6). However, the rate of convergence does depend on the value of b.

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