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SOME PROPERTIES OF THE p-BRANE MODEL WITH THE QUADRATIC LAGRANGIAN

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Abstract

Some general properties of the relativistic p-dimensional surface imbedded into D-dimensional space-time and its reduction to the simplest case of the quadratic Lagrangian are considered. The solutions of the equations of motion of such model for the p-brane with arbitrary topology and massless eigenstates, as well as with critical dimension after quantization are presented. Some generalizations for the supermembrane are discussed.

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INTRODUCTION

Nowadays not only one-dimensional relativistic objects – strings, but also the objects of higher dimension – p-dimensional (super)p-branes are suggested as substantial physical and mathematical objects. As for their properties, much less about those of p-branes is known this far [1–7].

The necessity to consider multidimensional objects with more than one space dimensions arises in various parts of the field theory. In particular, we may try to consider the (super)p-brane theory as fundamental, like the (super)string theory ($p = 1$) [8], as well as an effective model of supergravity, as shown in [9]. A possible correlation between ordinary and rigid (super)p-branes and, in particular, the correlation between the rigid string and the ordinary membrane at p=2 has been considered in [10, 11]. The calculation of the static potential for the p-brane compactified on the space-times of the various forms has been considered in [12, 13].

For the supermembrane ($p=2$), action is a direct multidimensional generalization of the string action [8]:

$$S=-\frac{T}{2}\int d^3\xi[\sqrt{h}h^{ij}\Pi_i^a\Pi_j^b\eta_{ab}-\sqrt{h}+2\varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA}], \tag{1}$$

where T is the parameter of tension with the dimension $[M]^{(p+1)}$ or $[L]^{-(p+1)}$, ξ^i $(i=0,1,...,p)$ are the world-volume coordinates, h_{ij} is the metric of the world-volume, $h=-\det(h_{ij})$, η_{ab} is the Minkowski space-time metric, and $\Pi_i^A=\partial_i Z^M E_M^A$, $A=a,\alpha$; $M=\mu,\dot{\alpha}$. Here, Z^M are the coordinates of the D-dimensional curved superspace, and E_M^A is the supervielbein. The 3-form $B=\frac{1}{6}E^A E^B E^C B_{CBA}$, $E^A=dZ^M E_M^A$ is the potential for the closed 4-form $H=dB$. (See Appendix for details of the conventions).

The action (1) is invariant respecting the global D-dimensional Poincaré transformations, as well as it is invariant respecting local parametrizations of the world-volume with the parameters $\eta^i(\xi)$:

$$\delta Z^M=\eta^i(\xi)\partial_i Z^M, \quad \delta h_{ij}=\eta^k\partial_k h_{ij}+2\partial_{(i}\eta^k h_{j)k}\ . \tag{2}$$

It is also invariant under local fermionic "k-transformations":

$$\delta Z^M E_M^a=0, \tag{3}$$

$$\delta Z^M E_M^\alpha=(1+\Gamma)^\alpha_\beta k^\beta, \tag{4}$$

$$\begin{aligned}\delta(\sqrt{h}h^{ij}) &= -2i(1+\Gamma)^\alpha_\beta k^\beta(\Gamma_{ab})_{\alpha\gamma}\Pi_n^\gamma h^{n(i}\varepsilon^{j)kl}\Pi_k^a\Pi_l^b- \\ &\quad -\frac{2i}{3\sqrt{h}}k^\alpha(\Gamma_c)_{\alpha\beta}\Pi_k^\beta\Pi_l^c h^{kl}\varepsilon^{mn(i}\varepsilon^{j)pq}\times \\ &\quad \times(\Pi_m^a\Pi_{pa}\Pi_n^b\Pi_{qb}+\Pi_m^a\Pi_{pa}h_{nq}+h_{mp}h_{nq}),\end{aligned} \tag{5}$$

with an anticommuting space-time spinor $k^\alpha(\xi)$, and the matrix Γ defined by

$$\Gamma=\frac{1}{6\sqrt{h}}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\Gamma_{abc}\ . \tag{6}$$

Unlike the two-dimensional string action, the action (1) at $p\neq 1$ is not invariant respecting local conformal transformations with the parameter $\Lambda(\xi)$:

$$\delta Z^M=0; \tag{7}$$

$$\delta h^{\alpha\beta}=\Lambda(\xi)h^{\alpha\beta}\ . \tag{8}$$

Varying the initial action leads to essentially non-linear field equations

$$\begin{aligned}\partial_i(\sqrt{h}h^{ij}\Pi_j^a)+\sqrt{h}h^{ij}\Pi_j^b\Pi_i^C\Omega_{Cb}^a+i\varepsilon^{ijk}\Pi_{ib}(\Pi_j^\alpha\Gamma^{ab}_{\alpha\beta}\Pi_k^\beta)+ \\ +\varepsilon^{ijk}\Pi_i^b\Pi_j^c\Pi_k^d H^a_{bcd}=0,\end{aligned} \tag{9}$$

$$[(1-\Gamma)h^{ij}\Pi_i^\mu\Gamma_\mu]^\alpha_\beta\Pi_j^\beta=0, \tag{10}$$

where Ω_B^A is the 1-form connection in the D-dimensional curved superspace, and to the "embedding" equation

$$h_{ij} = \Pi_i^a \Pi_j^b \eta_{ab} \ , \tag{11}$$

which remains non-linear at any gauge. Their solution is known for certain simplest cases [1].

For open membranes, or for the existing open dimensions, at $\sigma_i = \sigma_i^a, \sigma_i = \sigma_i^b$ the border condition is observed on the coordinates $Z^M(\xi)$:

$$\int d^3\xi \partial_i (\delta Z^a \sqrt{h} h^{ij} \Pi_{ja} + 3\varepsilon^{ijk} \delta Z^A \Pi_j^B \Pi_k^C B_{CBA}) = 0, \tag{12}$$

where h_{ij} is given by equation (11).

Any new solution of the equations of motion (9) and (10) describing the motion of a multidimensional relativistic object, on one hand, is of interest in itself, and on the other hand, it serves as a starting point for semiclassical quantization, when the minor variations respecting the known classical solution are investigated.

We have considered a mathematically simpler case at p=2. M.Duff in [3] presents a p-dimensional generalization of the supermembrane action, which has similar properties.

In the general case motion of the (super)p-brane is complicate. There are no gauge condition when equation of motion become linear. This is in a contrary to (super)-string model when conformal or orthogonal gauge conditions turn equations of motion to linear ones. In the general case, when non-linear dynamic system is too complicated, it seems reasonable to start from the some simpifier model. This work aims to investigate a special type of action corresponding to the quadratic Lagrangian model of the relativistic (super)p-brane. Such approach is possible in all cases when the (super)p-brane model appears.

THE MODEL OF THE BOSONIC p-BRANE NEAR STATIONARY POINT OF ACTION

Let us consider as a less complicated the case of the bosonic relativistic p-brane. This means that we are considering the action

$$S = -T \int d^{p+1}\xi |det(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})|^{\frac{1}{2}}, \tag{13}$$

where $\xi = (\tau, \sigma_1, \ldots, \sigma_p)$, $\xi_\alpha \in [\xi_\alpha^a, \xi_\alpha^b]$, $\xi_\alpha^a, \xi_\alpha^b$ are the initial (a) and final (b) meanings of the parameter ξ_α, $X^\mu = X^\mu(\tau, \sigma_1, \ldots, \sigma_p)$, $\mu = 0, \ldots, D-1$, where

D is the dimension of the Minkowski space-time with the metric $g_{\mu\nu}$; $\alpha = 0, \dots, p$, where p is the space dimension of p-brane.

The equation of motion

$$\partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta X^\mu) = 0, \tag{14}$$

resulting from (13), in the case the border conditions are taken into account, may be obtained from the classically equivalent action

$$S = -\frac{T}{2} \int d^{p+1}\xi \sqrt{h} [h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} - (p-1)] \ , \tag{15}$$

where an auxiliary metric $h_{\alpha\beta}$ on the world-volume of the membrane is introduced. The actions (13) and (15) to be equivalent, the metric $h_{\alpha\beta}$ must obeys the imbedding condition:

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \ , \tag{16}$$

like the embedding condition (11) in the supersymmetric case.

Besides, we must check if the constraint conditions $p+1$ are observed:

$$P^\mu_\tau X_{\mu;i} = 0, \qquad P^2 + T^2 det h_{ij} = 0, \tag{17}$$

where $P^\mu_\tau = \delta\mathcal{L}/\delta\dot{\mathrm{X}}^\mu, \quad 1 \le i, j \le p$.

There are at least two ways: to investigate small variations respecting the classical solutions and to introduce the quadratic action as a new independent action of the linearized version of the p-brane. Let us consider both of these possibilities.

We cannot quantize action (15) at $p > 1$, but we can introduce a certain simplification. Let Y^μ be a variation respecting the classical solution X^μ_0:

$$X^\mu = X^\mu_0 + \varepsilon Y^\mu \ . \tag{18}$$

Then the equation of motion (14) turns into

$$\partial_\alpha P^{\mu\alpha} = \partial_\alpha P^{\mu\alpha}_0 + \varepsilon \partial_\alpha C^{\mu\alpha} + o(\varepsilon) = 0 \ . \tag{19}$$

The requirement of the X^μ-solution of the equation of motion being the first order in ε leads to the equation $\partial_\alpha C^{\mu\alpha} = 0$:

$$\partial_\alpha \frac{\partial A}{\partial X_{\mu;\alpha}} + \frac{3}{2h^0} \sum_{\alpha=0}^{p} \partial_\alpha (A \frac{\partial h^0}{\partial \dot{X}_{0\mu}}) = 0, \tag{20}$$

where $A = \sum_{i,j=0}^{p} \partial_i X_0^\mu \partial_j Y_\mu \bar{h}^0_{ij}, \quad h^0_{ij} = \partial_i X_0^\mu \partial_j X_{0\mu}, \quad h^0 = deth^0_{ij}$.

The exact expression for the equation of motion (20) depends on the solution $X_0^\mu(\xi)$. For instance we may consider special type of the solution with one or few compactified dimensions. The solution for the toroidal membrane on the space-time with the topology $R^{D-2} \times S^1 \times S^1$ is

$$X^1 = l_1 R_1 \sigma, \quad X^2 = l_2 R_2 \rho, \quad X^I = 0, \quad I = 3, ..., D, \tag{21}$$

where $0 \leq \sigma \leq 2\pi,\ 0 \leq \rho \leq 2\pi,\ R_1$ and R_2 are the radii of the two circles, and l_1 and l_2 are the integers characterizing the winding numbers of the membrane around the two circles.

In the light cone gauge, $X^+ = p^+ \tau$. The world-volume metric on this background is flat,

$$g_{ij} = diag(-(l_1 l_2 R_1 R_2)^2, (l_1 R_1)^2, (l_2 R_2)^2), \tag{22}$$

and X^- is

$$X^- = \frac{1}{2p^+}(l_1 l_2 R_1 R_2)^2 \tau\ . \tag{23}$$

If we consider the fluctuations Z^μ of the transverse coordinate around this classical solution

$$X^1 = \sigma + Z^1, \quad X^2 = \rho + Z^2, \quad X^I = Z^I,\ I = 3, ..., D, \tag{24}$$

then, keeping only the terms of the linear order in Z, we find

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\sigma \partial_\rho Z^2, \ddot{Z}^2 = \partial_\rho \partial_\rho Z^2 + \partial_\sigma \partial_\rho Z^1, \ddot{Z}^I = \partial_\sigma \partial_\sigma Z^I + \partial_\rho \partial_\rho Z^I, \tag{25}$$

We may fix the remaining gauge invariance. The gauge choice $g_{0\alpha} = 0$ can be solved for $\partial_a X^-$. Upon linearization on our background, this constraint gives

$$\partial_\rho \dot{Z}^1 = \partial_\sigma \dot{Z}^2, \tag{26}$$

from which follows the possibility

$$\partial_\rho Z^1 = \partial_\sigma Z^2. \tag{27}$$

This allows us to rewrite (24) in the form of the standard wave equations:

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\rho \partial_\rho Z^1, \quad \ddot{Z}^2 = \partial_\sigma \partial_\sigma Z^2 + \partial_\rho \partial_\rho Z^2, \tag{28}$$

$$\ddot{Z}^I = \partial_\sigma \partial_\sigma Z^I + \partial_\rho \partial_\rho Z^I. \tag{29}$$

Equations of motion (28) and (29) are a special case of the equations (20). But here it should be noted that, as follows from (22) and (26), there is a special gauge condition, in which the general equation (20) turns into the ordinary wave equation.

The way described above is the investigation of small variations considering the classical solution. We may as well try to investigate the original action (13). Let us introduce new variables $\bar{X}^{\mu}$:

$$\partial^{\alpha}\bar{X}^{\mu} = \sqrt{|h|}h^{\alpha\beta}\partial_{\beta}X^{\mu} \ . \tag{30}$$

This means that

$$\bar{h} = det(\partial_{\alpha}\bar{X}^{\mu}\partial_{\beta}\bar{X}_{\mu}) = sign(h)|h|^{(p+1)^2+1}. \tag{31}$$

With these variables, the equation of motion (14) turns into the wave equation

$$\partial_{\alpha}\partial^{\alpha}\bar{X}^{\mu} = 0 \ , \tag{32}$$

and the conditions of the constrains (17) turn into

$$\bar{P}^2 + T^2|\bar{h}|^{-\frac{p^2}{(p+1)^2+1}}det(\partial_i\bar{X}^{\mu}\partial_j\bar{X}_{\mu}) = 0, \tag{33}$$

where $\bar{P}^{\mu} \equiv \dot{\bar{X}}^{\mu}$ and $i, j = 1, \ldots, p$ are space indexes of the membrane.

For a p-brane homeomorphous to the direct product $D^{p_0} \times T^{p_1} \times S^{p_2}(p = p_0 + p_1 + p_2)$, the coordinates σ_i parametrizing the disk D^{p_0} are $\sigma_i \in [0;\pi]$, for which

$$X^{\mu}(\tau, \ldots, \sigma_i = 0, \ldots) \neq X^{\mu}(\tau, \ldots, \sigma_i = \pi, \ldots) \, , \tag{34}$$

unlike for the coordinates σ_i parametrizing the torus T^{p_1} or sphere S^{p_2}, where the condition of the periodicity is observed:

$$X^{\mu}(\tau, \ldots, \sigma_i, \ldots) = X^{\mu}(\tau, \ldots, \sigma_i + \pi, \ldots) \tag{35}$$

or

$$X^{\mu}(\tau, \ldots, \sigma_i, \ldots) = X^{\mu}(\tau, \ldots, \sigma_i + 2\pi, \ldots) \, . \tag{36}$$

The values of parameters σ_i for the sake of convenience belong to areas $[0;\pi]$ or $[0;2\pi]$ to get direct correspondence to the string $(p = 1)$, where we have $\sigma \in [0;\pi]$ for open string and $\sigma \in [0;2\pi]$ for closed one.

The border conditions for the p-brane in bar variables $\bar{X}^{\mu}$ are the same like ordinary variables X^{μ}. At the same time, equations of motion in the X^{μ} variables is much more complicated then in $\bar{X}^\mu$ variables. If we can express the motion of the p-brane in $\bar{X}^\mu$ variables with the equation of motion (32) or in the case when X^μ variables obeys the same equation of motion, then the solution of this equation may be written.

Thus, for the p-brane homeomorphous $D^{p_0} \times T^{p_1} \times S^{p_2}$ ($p = p_0 + p_1 + p_2$) the solution of the equation of motion may be as follows:

$$
\begin{aligned}
X^\mu(\xi) = X^\mu + \frac{1}{\pi^p T} p^\mu \tau + \\
+ i\sqrt{\frac{2^{p_0-1}}{\pi^{p_0} T}} \sum_{\mathbf{n}} n^{-1} (\alpha^\mu_{\mathbf{n}} e^{-in\tau} - \alpha^{*\mu}_{\mathbf{n}} e^{-in\tau} \prod_{i=1}^{p_0} \cos n_i \sigma_i + \\
+ i\sqrt{\frac{2^{p_1-1}}{\pi^{p_1} T}} \sum_{\mathbf{m}} m^{-1} \left[(\alpha^\mu_{\mathbf{m}} e^{-2im\tau} - \alpha^{*\mu}_{\mathbf{m}} e^{2im\tau}) e^{-2i\bar{m}\bar{\sigma}} + \right. \\
\left. + (\beta^\mu_{\mathbf{m}} e^{-2im\tau} - \beta^{*\mu}_{\mathbf{m}} e^{2im\tau}) e^{2i\bar{m}\bar{\sigma}} \right] + \\
+ i\sqrt{\frac{2^{p_2-1}}{\pi^{p_2} T}} \sum_{\mathbf{k}} k^{-1} \left[(\alpha^\mu_{\mathbf{k}} e^{-ik\tau} - \alpha^{*\mu}_{\mathbf{k}} e^{ik\tau} e^{-i\bar{k}\bar{\sigma}} + \right. \\
\left. + (\beta^\mu_{\mathbf{k}} e^{-ik\tau} - \beta^{*\mu}_{\mathbf{k}} e^{ik\tau}) e^{i\bar{k}\bar{\sigma}} \right] ,
\end{aligned}
\tag{37}
$$

where X^μ are the initial coordinates of the mass centrum and p^μ is the impulse of the mass centrum of the membrane at

$$
\begin{aligned}
\mathbf{n} \in \bar{\mathbf{N}}^{p_0} \backslash 0, \quad n = \sqrt{n_1^2 + ... + n_{p_0}^2} \ ; \\
\mathbf{m} \in \bar{\mathbf{N}}^{p_1} \backslash 0, \quad m = \sqrt{m_1^2 + ... + m_{p_1}^2} \ ; \\
\mathbf{k} \in \bar{\mathbf{N}}^{p_1} \backslash 0, \quad k = \sqrt{k_1^2 + ... + k_{p_2}^2} \ ; \\
\bar{m}\bar{\sigma} \equiv m_{p_0+1}\sigma_{p_0+1} + ... + m_{p_0+p_1}\sigma_{p_0+p_1} \ , \\
\bar{k}\bar{\sigma} \equiv k_{p_0+p_1+1}\sigma_{p_0+p_1+1} + ... + k_p\sigma_p \ .
\end{aligned}
\tag{38}
$$

where $\bar{\mathbf{N}}^m \backslash 0$ means m-dimensional grade of natural numbers $\bar{\mathbf{N}} = 0, 1, \ldots$, without zero.

QUANTIZATION OF THE MODEL

To investigate the quantum properties of the p-brane we would like to have at our disposal the appropriate classical properties of the original p-brane. The motion of the p-brane in the $\bar{X}^\mu$ variables is the same as described by the original action (13), where all difficulties are hidden in the constraint conditions (33). Finding the solution of the wave equation obeying these constraint conditions is an intricate task in itself, and its solution is yet unknown. As a first step, let us consider the quadratic action under X^μ variables, which may be interpreted as an action in the original variables X^μ:

$$S = -\frac{T}{2}\int d^{p+1}\xi h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \tag{39}$$

where $h^{\alpha\beta} = \eta^{\alpha\beta},\ \alpha, \beta = 0, ..., p;\ g_{\mu\nu} = \eta_{\mu\nu},\ \mu, \nu = 0, ..., D-1.$

The action (39) is invariant respecting the global D-dimensional Poincaré transformations, but not invariant under local conformal and reparametrization transformations.

The absence of reparametrizations means the absence of the constraints. This allows an easy quantization of the quadratic action.

Consider $X^\mu(\xi)$ for D^p p-brane. Then the solution of the equation of motion (32) is like that of (37), and the density of the energy-momentum tensor

$$P^\mu_\tau = -\frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} =$$

$$= -i\sqrt{\frac{2^{p-1}T}{\pi^p}}\sum_{\mathbf{n}}(\alpha^\mu_{\mathbf{n}} e^{-in\tau} - \alpha^{*\mu}_{\mathbf{n}} e^{in\tau})\prod_{i=1}^{p} cos n_i \sigma_i, \tag{40}$$

$$\alpha^\mu_0 = \frac{1}{\sqrt{2^{p+1}\pi^p T}} p^\mu, \qquad \mathbf{n} \in \bar{\mathbf{N}}^p.$$

In the light-cone coordinates with assumption that tangent components $\alpha^\pm_{\mathbf{n}}$ are physical meaningless, like in the string case, we have from the commutation relations

$$[X^\mu(\tau,\sigma), P^\nu_\tau(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{41}$$

on the quantum level

$$[\alpha^i_{\mathbf{m}}, \alpha^{+j}_{\mathbf{n}}] = n\eta^{ij}\delta_{\mathbf{m},\mathbf{n}} \quad , \tag{42}$$

where $\alpha^{*\nu}_{\mathbf{n}} \to \alpha^{+\nu}_{\mathbf{n}}$.

The quantum Hamiltonian $H = \int_0^\pi d^p\sigma(P^\mu_\tau \dot{X}_\mu - \mathcal{L})$ is

$$H = \frac{T}{2}\int_0^\pi (\dot{X}^2 + X_1^2 + ... + X_p^2) d^p\sigma =$$

$$= \alpha_0^2 + \sum_{\mathbf{n}} \alpha^+_{\mathbf{n}}\alpha_{\mathbf{n}} + \frac{D-p-1}{2}\sum_{\mathbf{n}} n, \quad \mathbf{n} \in \bar{\mathbf{N}}^p \backslash 0. \tag{43}$$

As could be expected, the excitations of the model with the quadratic Lagrangian are an ordinary sum of the infinite number of harmonic oscillations described by creating and annihilating operators.

The zero-point energy of the infinite number oscillators (the Casimir energy) diverges, and for correct definition it must be regularized by using at least one of the existing methods. In string theory, various methods of regularization, including Riemann zeta-function regularization, lead to the same physical results. In advance it is not obvious that results of application of these methods to the p-brane are equivalent, but for quantization we must use at least one of them.

Let consider the regularization by the contracted Riemann zeta-function:

$$\zeta_p^{'}(s) = \sum_{\mathbf{n}} (n_1^2 + n_2^2 + ... + n_p^2)^{-s}, \qquad \mathbf{n}_i \in \bar{\mathbf{N}}^p \backslash 0, \tag{44}$$

for which the following properties are known :

$$\zeta_p^{'}(s) = \frac{\pi^p}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \sum_{\mathbf{n}} exp[-\pi(n_1^2 + n_2^2 + ... + n_p^2)t] \tag{45}$$

and

$$\zeta_p^{'}(s) = \pi^{2s-p/2} \frac{\Gamma(-s+p/2)}{\Gamma(s)} \zeta_p^{'}(-s+p/2) \quad . \tag{46}$$

In our case $s = -\frac{1}{2}$. According to the definition and the above-mentioned properties, we can find the first meanings of the $\zeta_p^{'}(-\frac{1}{2})$:

p	2	3	4	5	6	7	8
$\zeta_p^{'}(-\frac{1}{2})$	0.026	0.053	0.048	0.036	0.025	0.017	0.011
$-D_{cr}$	73.923	33.736	36.667	49.556	73.000	109.647	172.818

Table 1. The contracted Riemann zeta-function $\zeta_p^{'}(-\frac{1}{2})$ and the values of the critical dimensions D_{cr} for p-brane homeomorphous to D^p for p from 2 to 8.

Then, substituting the quantities $\zeta_p^{'}(-\frac{1}{2})$ in (43), we obtain the undiverging meanings of the Casimir energy and, correspondingly, good properties of the Hamiltonian H.

We remember that in the quantum case we have no constraints for this model. But we may impose "by hand" an additional condition $H|\phi\rangle = 0$. In this case, we obtain that for the existence of a massless vector, the coefficients at the second term in (43) must equal to minus one. This condition gives $D = D_{cr} = 1 + p - 2\zeta_p^{'}(-\frac{1}{2})$. Hence, the ground state of this model is a tachyon.

Now, let us consider $X^{\mu}(\xi)$ coordinates of T^p-type p-brane, for which space-like parameters $\sigma_i \in [0; 2\pi]$. Then, the solution of the equations of motion is like that of (37), and the density of the energy-momentum tensor

$$P^{\mu}_{\tau} = -\frac{\partial \mathcal{L}}{\partial \dot{X}_{\mu}} =$$

$$= -i\sqrt{\frac{2^{p-1}T}{\pi^p}} \sum_{\mathbf{k}} [(\alpha^{\mu}_{\mathbf{k}} e^{-ik\tau} - \alpha^{*\mu}_{\mathbf{k}} e^{ik\tau}) e^{-i\bar{k}\bar{\sigma}} + \tag{47}$$

$$+(\beta^{\mu}_{\mathbf{k}} e^{-ik\tau} - \beta^{*\mu}_{\mathbf{k}} e^{ik\tau}) e^{i\bar{k}\bar{\sigma}}],$$

$$\alpha^{\mu}_0 = \beta^{\mu}_0 \frac{1}{2\sqrt{2^{p+1}\pi^p T}} p^{\mu}, \qquad \mathbf{k} \in \bar{\mathbf{N}}^p.$$

The left-right symmetry condition gives us the correlation between the coefficients $\alpha^{\mu}_{\mathbf{k}}$ and $\beta^{\nu}_{\mathbf{k}}$

$$\beta^{\mu}_{\mathbf{k}} = \alpha^{\mu}_{-\mathbf{k}} \ . \tag{48}$$

In this case, from the commutation relations (41) it follows that

$$[\alpha^{\mu}_{\mathbf{m}}, \alpha^{+\nu}_{\mathbf{n}}] = n\eta^{\mu\nu}\delta_{\mathbf{m},\mathbf{n}} \quad , \quad [\beta^{\mu}_{\mathbf{m}}, \beta^{+\nu}_{\mathbf{n}}] = n\eta^{\mu\nu}\delta_{\mathbf{m},\mathbf{n}} \quad , \tag{49}$$

$$[\alpha^{\mu}_{\mathbf{m}}, \beta^{+\nu}_{\mathbf{n}}] = [\alpha^{\mu}_{\mathbf{m}}, \beta^{+\nu}_{\mathbf{n}}] = 0 \ . \tag{50}$$

The quantum Hamiltonian

$$H = \frac{T}{2}\int_0^{2\pi} (\dot{X}^2 + X_1^2 + ... + X_p^2) d^p\sigma =$$

$$\alpha_0^2 + \beta_0^2 + \frac{1}{2}\sum_{\mathbf{k}} \{\alpha^{+}_{\mathbf{k}}, \alpha_{\mathbf{k}}\} + \frac{1}{2}\sum_{\mathbf{k}} \{\beta^{+}_{\mathbf{k}}, \beta_{\mathbf{k}}\} = H_L + H_R \ , \tag{51}$$

where $H_L(H_R)$ depends only on $\alpha^{\mu}_{\mathbf{k}}(\beta^{\mu}_{\mathbf{k}})$ variables and $\mathbf{k} \in \bar{\mathbf{N}}^p \backslash 0$.

In the case of the T^p-type p-brane, we have two different possibilities: (a) to impose a more detailed condition $H_L|\varphi\rangle = H_R|\varphi\rangle = 0$ or an equivalent $H|\varphi\rangle = H_L|\varphi\rangle = 0 \quad (H|\varphi\rangle = H_R|\varphi\rangle = 0)$; (b) using the discrete symmetry condition $X^{\mu}(\sigma,\tau) = X^{\mu}(-\sigma,\tau)$ and, consequently, the correlation between $\alpha^{\mu}_{\mathbf{k}}$ and $\beta^{\nu}_{\mathbf{k}}$ operators, we may impose only one condition $H|\varphi\rangle = 0$.

In the first case, we have the same properties for the T^p-type of p-brane as for D^n-type one:

$$H_L = \beta_0^2 + \sum_{\mathbf{k}} \beta^{+}_{\mathbf{k}} \beta_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k}} k \ ,$$

$$H_L = \alpha_0^2 + \sum_{\mathbf{k}} \alpha^{+}_{\mathbf{k}} \alpha_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k}} k \tag{52}$$

where $\mathbf{k} \in \bar{\mathbf{N}}^p \backslash 0$ and, according to the conditions (a), we obtain a tachyon in a ground state and the D_{cr} corresponding to that in the table for the D^m-type p-brane.

In the second case we may express H only in the terms of the right (left) operators $\alpha^{\mu}_{\mathbf{k}}(\beta^{\mu}_{\mathbf{k}})$; where $\mathbf{k} \in \mathbf{Z}^p \backslash 0$:

$$H_L = \alpha_0^2 + \sum_{\mathbf{k} \neq \mathbf{0}} \alpha^{+}_{\mathbf{k}} \alpha_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k} \neq \mathbf{0}} k \quad , \tag{53}$$

and $k = \sqrt{k_1^2 + ... + k_p^2}$.

Using the definition of the ordinary Riemann zeta-function [14], where we are summing over all possible integer as well as all positive and negative values of k_i contrary to the contracted Riemann zeta-function (44)

$$\zeta_p(s) = \sum_{k \neq 0} (k_1^2 + k_2^2 + ... + k_p^2)^{-s}, k \in Z^p \backslash 0, \tag{54}$$

with the same properties (45),(46), we may find the first meanings of $\zeta_p(-\frac{1}{2})$:

p	2	3	4	5	6	7	8
$-\zeta_p(-\frac{1}{2})$	0.229	0.267	0.297	0.325	0.373	0.407	0.462
D_{cr}	11.734	11.491	11.734	12.154	13.362	12.914	13.329

Table 2. The Riemann zeta-function $\zeta_p(-\frac{1}{2})$ and the values of the critical dimensions D_{cr} for p-brane homeomorphous to T^p for p from 2 to 8.

Then, substituting the quantities $\zeta_p(-\frac{1}{2})$ in (53), we find no divergencies of the Hamiltonian H. In this case, the ground state of the T^p-type p-brane is also a tachyon, and the critical dimension $D_{cr} = 1 + p - 2(\sum_{k \neq 0} k)^{-1}$.

DISCUSSION

In this article we have considered the simplest case of the bosonic and fermionic membranes, when they contain only linear terms in their equations of motion. The general situation is much more complicated.

An essential point of our consideration is imposing additional conditions like $H = 0$. But in the case of the model with the quadratic Lagrangian we can consider these conditions as a certain remnant constraint condition like $L_n = 0$.

One would remark that D_{cr} in the bosonic case is not an integer and, consequently, has no physical meaning. Indeed, in all considered cases $D_{cr} \neq \mathbf{N}$. But even in the case when $D_{cr} \in \mathbf{N}$, D_{cr} has no physical meaning. The point is that we cannot pick out physical states among all possible states in the Hilbert space, as we have not enough constraints or the conditions like those and can not obtain the physical sector. On the other hand, the discrete values of the space-time dimension D_{cr} imply the existence of the fractal properties of the extended objects. Some of the aspects of these properties are considered in [15].

In the supersymmetric case we have additional possibilities to impose condition, at which the supercurrent $J^\alpha = K\gamma^\beta\gamma^\alpha\psi^\mu\partial_\beta X_\mu$ vanishes. In this case the condition $J^\alpha = 0$ is equivalent to six conditions $\partial_\alpha X^\mu\psi^i_\mu = 0$ or their Fourier transformation $F^{\alpha i}_{\mathbf{n}} = \int_{-\pi}^{\pi} d^2\sigma e^{i\vec{n}\vec{\sigma}}\partial_\alpha X^\mu\psi^i_\mu$. The supersymmetric action contains the constraints $F^{\alpha i}_{\mathbf{n}} = 0$. We may also express this quantity in the $\alpha_{\mathbf{n}}, d^{(i)}_{\mathbf{n}}$ variables and consider the quantum case, but this will be also not enough to distinguish the physical sector. Nevertheless, due to the quadratic action we can analytically calculate the partition function and transition amplitude for this model.

The model with the quadratic Lagrangian allows us to separate linear and nonlinear effects in the general (super)p-brane. For instance, in [16], due to the restriction of the constraint condition for the bosonic p-brane, D_{cr} has been obtained, whereas the purely the model with the quadratic Lagrangian has no critical dimensions. This means that in [16] a nontrivial conformity between the model with the quadratic Lagrangian and the imposed constraint condition was obtained.

We may try to impose sufficient constraint conditions as an additional condition, but in this case a very important question arises: how to conform the solution of the equation of motion with the constraint conditions? We can make it sure that in the bosonic sector the simplest quadratic constraints $\dot{X}^2 + X^2_{;1} + ... + X^2_{;p} = 0$, $\dot{X}^\mu X_{;i\mu} = 0$, which are a natural generalization of the string constraints, cannot coexist with the solutions of the linear wave equation of motion for the bosonic p-brane. Thus, the conformity between the solution of the equation of motion in the model with the quadratic Lagrangian and the additional constraint conditions is nontrivial and of interest in itself.

On the other hand, we may not only use global supersymmetry and vanishing of the supercurrent J^α, but also the condition of local supersymmetry may be imposed. Indeed, we may use the model with the quadratic Lagrangian of the (super)p-brane with local supersymmetry and try to find the conformity between the solutions and constraints. However, (1) it is not clear how to do it even in a less complicated case without supersymmetry, and (2) this will be not enough to distinguish the physical sector, either.

Thus, we may consider the model with the quadratic Lagrangian an auxiliary model of the (super)p-brane. An important aspect of this consideration is the possibility to separate the physical properties belonging to the model with the quadratic Lagrangian from other properties characteristic of the essentially nonlinear behavior of the relativistic (super)p-brane.

APPENDIX: NOTATION AND CONVENTIONS

(i) *General conventions*

sign $h^{ij} = (-,+,+)$, $\quad h = -\mathrm{det} h_{ij}$.
sign $\eta_{ab} = (-,+,+,+,+,+,+,+,+,+)$.
$1/\sqrt{h}\varepsilon^{ijk}$, and $\sqrt{h}\varepsilon_{ijk}$ are tensors, where $\varepsilon_{012} = -\varepsilon^{012} = 1$.
The Clifford algebra of Γ^a obeys to condition $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$,
$\Gamma^{ab} = 1/2(\Gamma^a\Gamma^b - \Gamma^b\Gamma^a)$; $\quad (\Gamma^a)_{\alpha\beta}$, $(\Gamma^{ab})_{\alpha\beta}$ are symmetric,
$\Gamma^{abc} = 1/3!\,\Gamma^{[a}\Gamma^b\Gamma^{c]}$; $\quad \Gamma^{[a_1\ldots a_n]} = 1/n!\,\Gamma^{[a_1}\Gamma^{a_2}\ldots\Gamma^{a_n]}$,
$(x_1 x_2 \ldots x_n)$ means symmetric and $[x_1\, x_2 \ldots x_n]$ antisymmetric permutation of indexes x_i; $\quad (\Gamma^{\mu\nu\rho})_{\alpha\beta}$ is antisymmetric.
We use representation when $\Gamma^{a\dagger} = \Gamma^0\Gamma^a\Gamma^0$
and charge conjugation matrix $C_{ab} = -C_{ba}$, $\quad C^{ab}C_{bc} = \delta^a_c$.
Majorana spinors obey to conditions $\theta^\alpha = C^{\alpha\beta}\theta_\beta$, $\theta_\alpha = \theta^\beta C_{\beta\alpha}$, $\bar\theta = \theta^\dagger\Gamma_0$,
$(\bar\chi\Gamma^{\nu_1\ldots\nu_n}\lambda)^\dagger = -(\bar\chi\Gamma^{\nu_1\ldots\nu_n}\lambda)$ for anticommuting $\chi\,, \lambda$;
$\bar\chi\lambda = \chi_\alpha\lambda^\alpha$, $\quad \bar\chi\Gamma_\mu\lambda = \chi_\alpha(\Gamma_\mu)^\alpha_\beta\lambda^\beta$, etc.

(ii) *Superspace conventions*

Superspace coordinates are $Z^M = (X^m, \theta^{\dot\alpha})$,
supervielbein is $E^A_M (A = a, \alpha)$.
$E^A_M E^N_A = \delta^N_M$, $\quad E^M_A E^B_M = \delta^B_A$.
$V^A = V^M E^A_M$, $\quad V_A = E^M_A V_M$.
$V^M = V^A E^M_A$, $\quad V_M = E^A_M V_A$.
$E^A = dZ^M E^A_M$ $\quad (E^a E^b = -E^b E^a$ but $E^\alpha E^\beta = E^\beta E^\alpha)$.
$F = (1/p!)E^{A_1}\ldots E^{A_p} F_{A_p\ldots A_1} = (1/p!)dZ^{M_1}\ldots dZ^M_p F_{M_p\ldots M_1}$.
$d(FG) = FdG + (-1)^q dFG$ for p-form F and q-form G
$H = dB$; $\quad H_{MNPQ} = \partial_M B_N PQ + 3$ more terms.

(iii) *Light cone conventions*

$V^\pm = (1/\sqrt{2})(\pm V^0 + V^{10})$.
$V^\mu W_\mu = V^I W^I + V^+ W^- V^- W^+$, $\quad I = 1,\ldots,9$.

$\varepsilon^{0ab} = -\varepsilon^{ab}\,, \quad \varepsilon^{ab}\varepsilon^{cd} = h(h^{ac}h^{bd} - h^{bc}h^{ad}),\ a = 1,2.$
$\Gamma^{\mu} = (\Gamma^{I}, \Gamma^{+}, \Gamma^{-})$, with $\Gamma^{I} = \gamma^{I} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
$\Gamma^{+} = 1/\sqrt{2}(\Gamma^{0} + \Gamma^{10}) = I_{16} \otimes \begin{pmatrix} 0 & 0 \\ \sqrt{2}i & 0 \end{pmatrix}$,
$\Gamma^{-} = 1/\sqrt{2}(-\Gamma^{0} + \Gamma^{10}) = I_{16} \otimes \begin{pmatrix} 0 & \sqrt{2}i \\ 0 & 0 \end{pmatrix}$,
$\{\gamma^{I}, \gamma^{J}\} = 2\delta^{IJ}\,, \quad \{\Gamma^{+}, \Gamma^{-}\} = 2, \quad (\Gamma^{+})^{2} = (\Gamma^{-})^{2} = 0.$
$\theta = (i\theta_{1}, \theta_{2}), \quad \bar{\theta} = (-i\bar{\theta}_{2}, -\bar{\theta}_{1}).$
$\bar{\theta}_{1}\chi_{1} = -\bar{\chi}_{1}\theta_{1}, \quad \bar{\theta}_{1}\gamma^{I}\chi_{1} = -\bar{\chi}_{1}\gamma^{I}\theta_{1}, \quad \bar{\theta}_{1}\chi^{IJ}\chi_{1} = +\bar{\chi}_{1}\gamma^{IJ}\theta_{1}$, etc.

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Kai kurios p-branos modelio su kvadratiniu lagranžianu savybės

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Tiriamos kai kurios bendros reliatyvistinių p-mačių paviršių, panardintų į D-matį erdvėlaikį savybės ir atliktas jų nagrinėjimas paprasčiausio kvadratinio lagranžiano atveju. Pateikti tokio modelio judėjimo lygčių sprendiniai, atitinkantys bet kokios topologijos p-braną, bemasės pagrindinės būsenos bei kritiniai erdvėlaikio išmatavimai. Aptariami apibendrinimai supermembranos atveju.

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