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On Heights of Polynomials with Real Roots

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Abstract

We prove Schinzel's theorem about the lower bound of the Mahler measure of totally real polynomials. Under certain additional conditions this theorem is strengthened. We also consider certain Chebyshev polynomials in order to investigate how sharp are the lower bounds for the heights.

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1 Lower Bounds

Let

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0 = a_d(x-\alpha_1)\dots(x-\alpha_d)$$

be a polynomial with complex coefficients of degree d (so that $a_d \neq 0$). Its Mahler measure (height) is given by

$$M(P) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\}.$$

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Theorem A (Schinzel). *Suppose that all d zeros of a polynomial $P \in \mathbf{Z}[x]$ are real, and let $P(-1)P(0)P(1) \neq 0$. Then*

$$M(P) \geq \Big(\frac{1+\sqrt{5}}{2}\Big)^{d/2}.$$

Moreover, if the zeros of $P \in \mathbf{Z}[x]$ are all positive, and $P(1) \neq 0$, then

$$M(P) \geq \Big(\frac{1+\sqrt{5}}{2}\Big)^{d}.$$

Define the absolute (logarithmic) Weil height of P by

$$h(P) = \frac{\log M(P)}{d}.$$

The absolute logarithmic Weil height of an algebraic number is defined to be the Weil height of its minimal polynomial over $\mathbf{Z}[x]$.

Corollary A. *If α is a totally real algebraic number of degree $d \geq 2$, then*

$$h(\alpha) \geq \frac{1}{2}\log\Big(\frac{1+\sqrt{5}}{2}\Big).$$

Moreover, if α is totally positive, then

$$h(\alpha) \geq \log\Big(\frac{1+\sqrt{5}}{2}\Big).$$

In [4] we investigated the following quantity

$$\mathcal{R}(P) = |a_d||\alpha_1||\alpha_2|^{\frac{d-2}{d-1}}|\alpha_3|^{\frac{d-3}{d-1}}\ldots|\alpha_{d-1}|^{\frac{1}{d-1}},$$

where $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_{d-1}| \geq |\alpha_d|$. Let us write

$$h_{\mathcal{R}}(P) = \frac{\log \mathcal{R}(P)}{d},$$

resembling the definition of h. Suppose that P is a polynomial in $\mathbf{Z}[x]$ such that $a_0 \neq 0$. We proved in [4] that $\sqrt{M(P)} \leq \mathcal{R}(P) \leq M(P)$. So that

$$\frac{1}{2}h(P) \leq h_{\mathcal{R}}(P) \leq h(P).$$

Let $h_{\mathcal{R}}(\alpha)$ for an algebraic number α be defined as $h_{\mathcal{R}}(P)$, where P is the minimal polynomial of α over $\mathbf{Z}[x]$.

Corollary B. *If α is a totally real algebraic number of degree $d \geq 2$, then*

$$h_{\mathcal{R}}(\alpha) \geq \frac{1}{4}\log\Big(\frac{1+\sqrt{5}}{2}\Big).$$

Moreover, if α is totally positive, then

$$h_{\mathcal{R}}(\alpha) \geq \frac{1}{2}\log\Big(\frac{1+\sqrt{5}}{2}\Big).$$

Various proofs of Theorem A (Corollary A) were given by Schinzel [9], Smyth [10], [11], Flammang [6], Hoehn and Skoruppa [8], Everest and Ward [5]. We give two different proofs here. Firstly, we prove Corollary B following Zaimi [13]. Secondly, we give a very simple proof of Theorem A. This leads to a refined version of Schinzel's theorem if some additional information about zeros is known. For both proofs, the following lemma is crucial.

Lemma. *Let $y_1, y_2, \dots, y_n$ be nonnegative real numbers. Then*

$$1+\Bigg(\prod_{j=1}^{n} y_j\Bigg)^{1/n} \leq \Bigg(\prod_{j=1}^{n}(1+y_j)\Bigg)^{1/n}.$$

Although the lemma is well–known (see, e.g., Hardy, Littlewood and Polya [7], we give a self–contained proof.

Proof of the lemma. Assume without loss of generality that y_j are all positive. Let us fix a product of these $Y = y_1 \dots y_n$. The infimum of the product $Z = (1+y_1)\dots(1+y_n)$ with fixed Y is clearly the minimum. Suppose it is attained at $(y_1, \dots, y_n) = (z_1, \dots, z_n)$. Suppose also that for some pair i, j, we have $z_i \neq z_j$. Replacing z_i and z_j both by $\sqrt{z_i z_j}$ (which will not change Y), we get a smaller value for Z, a contradiction. So that $z_1 = \dots = z_n$, and the lemma is proved.

Proof of Corollary B via discriminant. Suppose first that α is totally positive $\alpha_1 > \alpha_2 > \dots > \alpha_d > 0$. By the lemma, we have

$$1+\prod_{i<j}\Bigg(\frac{\alpha_i}{\alpha_j}-1\Bigg)^{2/d(d-1)} \leq \Bigg(\prod_{j=1}^{d}\alpha_j^{d+1-2j}\Bigg)^{2/d(d-1)}.$$

Note that

$$\prod_{i<j}\Bigg(\frac{\alpha_i}{\alpha_j}-1\Bigg)^2 = \frac{\text{disc}\,(\alpha)}{a_d^{2d-2}\alpha_2^2\alpha_3^4\dots\alpha_d^{2(d-1)}} =$$

$$= \frac{\mathrm{disc}\,(\alpha)\big(\mathcal{R}(\alpha)/a_d\big)^{2(d-1)}}{|a_0|^{2(d-1)}} \geq \left(\frac{\mathcal{R}(\alpha)}{a_d|a_0|}\right)^{2(d-1)}.$$

Also,

$$\prod_{j=1}^{d} \alpha_j^{d+1-2j} = \frac{\big(\mathcal{R}(\alpha)/a_d\big)^{2(d-1)}}{\big(|a_0|/a_d\big)^{d-1}} = \left(\frac{\mathcal{R}(\alpha)}{\sqrt{a_d|a_0|}}\right)^{2(d-1)}.$$

We deduce that

$$1+\left(\frac{\mathcal{R}(\alpha)}{a_d|a_0|}\right)^{2/d} \leq \left(\frac{\mathcal{R}(\alpha)}{a_d|a_0|}\right)^{4/d}.$$

This implies that

$$\mathcal{R}(\alpha)^{4/d}-\mathcal{R}(\alpha)^{2/d} \geq \big(a_d|a_0|\big)^{2/d} \geq 1.$$

Therefore, $\mathcal{R}(\alpha)^{2/d} \geq (1+\sqrt{5})/2$, and the second part of Corollary B follows. The first part can be proved analogously, by replacing above each α_j by α_j^2.

Proof of Theorem A via the value of a polynomial at 1. We begin with the second part again. Suppose k is a nonnegative integer such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 1 \geq \alpha_{k+1} \geq \cdots \geq \alpha_d > 0.$$

Using the lemma twice, we obtain

$$\begin{aligned}
1 &\leq |P(1)| = |a_d| \prod_{j=k+1}^{d} \alpha_j \prod_{j=k+1}^{d} \big(\alpha_j^{-1} - 1\big) \prod_{j=1}^{d} \big(\alpha_j - 1\big) \\
&\leq \frac{|a_d a_0|}{M(P)} \left(\left(\frac{M(P)}{|a_0|}\right)^{1/(d-k)} - 1\right)^{d-k} \left(\left(\frac{M(P)}{|a_d|}\right)^{1/k} - 1\right)^{k} \\
&= M(P)^{-1}\big(M(P)^{1/(d-k)} - |a_0|^{1/(d-k)}\big)^{d-k}\big(M(P)^{1/k} - |a_d|^{1/k}\big)^{k} \\
&\leq M(P)^{-1}\big(M(P)^{1/(d-k)} - 1\big)^{d-k}\big(M(P)^{1/k} - 1\big)^{k}.
\end{aligned}$$

So that

$$M(P) \leq \big(M(P)^{1/(d-k)}-1\big)^{d-k}\big(M(P)^{1/k}-1\big)^{k}. \tag{1}$$

Applying the lemma once more we get $M(P) \leq \big(M(P)^{2/d}-1\big)^{d}$, and the second part of the theorem follows. The proof of the first part is analogous (replacing each α_j by α_j^2 and arguing as above).

Note that in order to obtain (1) we only used the inequalities

$$|P(1)|, |a_d|, |a_0| \geq 1.$$

The conditions on polynomial $P \in \mathbf{Z}[x]$, $P(1) \neq 0$ are much stronger. It is also clear that inequality (1) implies a stronger bound if the number of roots of P in intervals $(0;1)$ and $(1;+\infty)$ is dictinct. Taking the d-th root in (1) we have the following statement.

Theorem 1. *Suppose that all d zeros of a polynomial $P \in \mathbf{R}[x]$ are real, and let $|P(-1)P(1)|, |a_d|, |a_0| \geq 1$. If P has λd roots in the interval $(-1;1)$, then*

$$M(P) \geq w(\lambda)^{d/2},$$

where $w(\lambda)$ is the solution of the equation

$$\left(w^{1/(1-\lambda)} - 1\right)^{1-\lambda} \left(1 - w^{-1/\lambda}\right)^{\lambda} = 1 \tag{2}$$

in $[(1+\sqrt{5})/2; 2)$.

Moreover, if the zeros of $P \in \mathbf{R}[x]$ are all positive, exactly λd of these lie in the interval $(0;1)$, and $|P(1)|, |a_d|, |a_0| \geq 1$, then

$$M(P) \geq w(\lambda)^{d}.$$

It is clear from (2) that $w(\lambda) = w(1-\lambda)$. The maximum of the function $w(\lambda)$ is the golden ratio $w(1/2) = (1+\sqrt{5})/2 = 1.618....$ For $\lambda \in (0;1/3] \cup [2/3;1)$ we have $w(\lambda) \geq w(1/3) = 1.656....$ If λ is very small or close to 1, say, $\lambda \in (0;1/100] \cup [99/100;1)$, then $w(\lambda) \geq w(1/100) = 1.986...$ (compare with the results in [3]). In the next section we investigate some upper hounds.

2 The Heights of Chebyshev Polynomials

We consider two Chebyshev polynomials:

$$T(x) = \prod_{j=1}^{d} \left(x - 2\cos\left(\frac{\pi j}{d+1}\right)\right),$$

$$Q(x) = \prod_{j=1}^{d} \left(x - 4\cos^2\left(\frac{\pi j}{2(d+1)}\right)\right).$$

Clearly, $T(x-2) = Q(x)$. Also, $T, Q \in \mathbf{Z}[x]$, both polynomials are of degree d, the zeros of T are all in $(-2; 2)$, the zeros of Q are all in $(0; 4)$. If $d+1$ is not a multiple of 6, then $T(-1)T(0)T(1) \neq 0$. If $d+1$ is not a multiple of 3, then $Q(1) \neq 0$.

Theorem B (Boyd). *We have*

$$M(T) = \beta^{d+\mathcal{O}(1)},$$

$$M(Q) = \beta^{2d+\mathcal{O}(1)},$$

where $\beta = 1.381...$ and $\mathcal{O}(1)$ is an absolute constant.

The constant β in Theorem B is the Mahler measure of a polynomial in two variables

$$\beta = M(1 + x_1 + x_2).$$

Its logarithm can also be expressed as the integral

$$\log \beta = \frac{1}{\pi} \int_0^{2\pi/3} \log\big(2\cos(t/2)\big) dt,$$

or via the value of the L – function

$$\log \beta = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}) = \frac{3\sqrt{3}}{4\pi} \sum_{s=1}^{\infty} (\frac{-3}{s}) s^{-2}$$

(see Boyd [1], Smyth [12]). Boyd [2] also found that β is the optimal upper bound for the d-th root of the ratio of two norms $\| g \| / \| f \|$, where g divides f (with degree d) and g, f are both complex monic polynomials. See [2] for more examples involving the number β.

In [4] we showed that if

$$P(x) = \prod_{j=1}^{d} (x - \alpha_j)$$

is a monic polynomial of degree d with nonzero roots, then

$$\mathcal{R}(P) = \sqrt{|P(0)|} M(Q)^{1/2(d-1)},$$

where

$$Q(x) = \prod_{i \neq j} \big(x - \alpha_i/\alpha_j\big)$$

is the "ratio" polynomial of degree $d(d-1)$. So that $\mathcal{R}(P)$ (which we call the Remak height in [4]) is also essentialy the Mahler measure. Therefore, it is not surprising at all that for $\mathcal{R}(T)$ and $\mathcal{R}(Q)$ we get the same type of constants.

Theorem 2. *We have*

$$\mathcal{R}(T) = \gamma^{d/2+\mathcal{O}(1)},$$

$$\mathcal{R}(Q) = \gamma^{d+\mathcal{O}(1)},$$

where $\gamma = \exp\left(7\zeta(3)/2\pi^2\right) = 1.531....$

The constant γ here is the Mahler measure of a polynomial in three variables

$$\gamma = M(1+x_1+x_2+x_3)$$

(see Boyd [1], Smyth [12]).

Proof of Theorem 2. We have

$$\log \mathcal{R}(Q) = 2\sum_{j=1}^{d} \frac{d-j}{d-1} \log\left(2\cos\left(\frac{\pi j}{2(d+1)}\right)\right)$$

$$= 2d\int_0^1 (1-t)\log\left(2\cos\left(\frac{\pi t}{2}\right)\right)dt + \mathcal{O}(1).$$

Replacing t by $(1-t)/\pi$ in the integral and using the well–known representation

$$\int_0^u \log\left(2\sin\left(\frac{t}{2}\right)\right)dt = -\sum_{k=1}^{\infty}\frac{\sin(ku)}{k^2},$$

we compute the integral:

$$\int_0^1 (1-t)\log\left(2\cos\left(\frac{\pi t}{2}\right)\right)dt$$

$$= \frac{1}{\pi^2}\left(-t\sum_{k=1}^{\infty}\frac{\sin(kt)}{k^2}\Big|_0^{\pi} + \int_0^{\pi}\sum_{k=1}^{\infty}\frac{\sin(kt)}{k^2}dt\right)$$

$$= \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1-\cos(\pi k)}{k^3} = \frac{2}{\pi^2}\sum_{k-odd}\frac{1}{k^3} = \frac{7\zeta(3)}{4\pi^2}.$$

So that

$$\log \mathcal{R}(Q) = \frac{7\zeta(3)}{2\pi^2} d + \mathcal{O}(1).$$

The proof for $\mathcal{R}(T)$ is analogous. Theorem 2 is proved.

The heights of irreducible polynomials

$$T_0(x) = \prod_{(j,n)=1} \left(x - 2\cos\left(\frac{\pi j}{n}\right)\right),$$

$$Q_0(x) = \prod_{(j,n)=1} \left(x - 4\cos^2\left(\frac{\pi j}{n}\right)\right),$$

where $d = \varphi(n)$ are essentially the same, because of

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{(j,n)=1} f\left(\frac{j}{n}\right) = \int_0^1 f(x)dx$$

for a "smooth" enough function as above. Similarly as to in Theorem B and in Theorem 2 we get

$$M(T_0) = \beta^{d(1+\ o(1))},$$

$$M(Q_0) = \beta^{2d(1+\ o(1))},$$

$$\mathcal{R}(T_0) = \gamma^{(d/2)(1+\ o(1))},$$

$$\mathcal{R}(Q_0) = \gamma^{d(1+\ o(1))},$$

where $o(1) \to 0$ as $d \to \infty$. Thus, in Corollary A the constant $(1/2)\log\left((1+\sqrt{5})/2\right) = 0.240...$ cannot be replaced by a constant greater than $\log\beta = 0.323....$ In Corollary B the constant $(1/4)\log\left((1+\sqrt{5})/2\right) = 0.120...$ cannot be replaced by a constant greater than $7\zeta(3)/4\pi^2 = 0.213....$

The polynomial T_0 has $(d/3)(1+\ o(1))$ roots in the interval $(-1;1)$. So that the constant $w(1/3) = 1.656...$ in Theorem 1 cannot be replaced by a constant greater than $\beta^2 = 1.908....$

Smyth [11] showed that the set $\{h(\alpha)\}$, where α runs over the set of totally positive algebraic integers, is everywhere dense in $(0.546...;+\infty)$. The first accumulation point here is closely related with the so–called Gorshkov–Wirsing sequence $\{\beta_n\}_{n\geq 0}$, where $\beta_1 = 1$ and $\beta_{n+1} = (\beta_n + \sqrt{\beta_n^2+4})/2$. The smallest elements is this set $\{h(\alpha)\}$ are known to be of two types: of Chebyshev type $h(\alpha_n)$, where $\alpha_n = 4\cos^2(2\pi/n)$, or of Gorshkov–Wirsing type $h(\beta_n^2)$.

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