Nonlinear Analysis: Modelling and Control, 2001, v. 6, No. 1, 3-8

Asymptotic of the Joint Distribution of Multivariate Extrema

A. Aksomaitis and A. Jokimaitis
Department of Applied Mathematics,
Kaunas University of Technology
Studentu 50, 3028, Kaunas, Lithuania
e-mail: aksoma@fmf.ktu.lt

Received: 16.01.2001
Accepted: 14.03.2001

Abstract

Let W_{n} and Z_n be a bivariate extrema of independent identically distributed bivariate random variables with a distribution function *F*. in this paper the nonuniform estimate of convergence rate of the joint distribution of the normalized and centralized minima and maxima is obtained.

Keywords: multivariate extreme value, limit distributions, convergence rate

1 Introduction

Let $\{ X_j = (X_{1,j}, X_{2,j}), j \geq 1 \}$ be independent and identically distributed bivariate random variables with a distribution function

$$F(x_1, x_2) = P(X_{1,j} < x_1, X_{2,j} < x_2) \quad \forall j \geq 1,$$

and let $x = (x_1, x_2)$ be an arbitrary point of the Euclidean space. We define by

$$x + y = (x_1 + y_1, x_2 + y_2),$$

$$xy = (x_1 y_1, x_2 y_2),$$

$$\frac{x}{y} = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2} \right)$$

the arithmetical operations on vectors. By the inequality $x < y$ we mean the system of inequalities $x_i < y_i \quad (i = 1,2)$.

We form bivariate maxima and minima

$$Z_n = \left(\max\left(X_{1,1}, \ldots, X_{1,n}\right), \max\left(X_{2,1}, \ldots, X_{2,n}\right)\right),$$
$$W_n = \left(\min\left(X_{1,1}, \ldots, X_{1,n}\right), \min\left(X_{2,1}, \ldots, X_{2,n}\right)\right).$$

Let

$$\{a_n = \left(a_{1,n}, a_{2,n}\right), n \geq 1\}, \quad \{b_n = \left(b_{1,n}, b_{2,n}\right) > (0,0), n \geq 1\}$$

be sequences of centralizing and normalizing vectors. If the distribution function *F* and sequences of centralizing and normalizing vectors are such that the following limit exits:

$$\lim_{n\to\infty} n\left(1 - F\left(a_n + b_n y\right)\right) = u\left(y_1, y_2\right) > 0,$$

then

$$\lim_{n\to\infty} P\left(Z_n < a_n + b_n y\right) = H(y) = e^{-u\left(y_1, y_2\right)}, \tag{1}$$

where $H(y)$ is a nondegenerate distribution function of two variables.

Let

$$\{c_n = \left(c_{1,n}, c_{2,n}\right), n \geq 1\}, \quad \{d_n = \left(d_{1,n}, d_{2,n}\right) > (0,0), n \geq 1\}$$

be sequences of centralizing and normalizing vectors. If the distribution function *F* and sequences of centralizing and normalizing vectors are such that the following limits exist

$$\lim_{n\to\infty} nF_1\left(c_{1,n} + d_{1,n}x_1\right) = z_1\left(x_1\right) > 0,$$
$$\lim_{n\to\infty} nF_2\left(c_{2,n} + d_{2,n}x_2\right) = z_2\left(x_2\right) > 0,$$
$$\lim_{n\to\infty} n\left(F_1\left(c_{1,n} + d_{1,n}x_1\right) + F_2\left(c_{2,n} + d_{2,n}x_2\right) - F\left(c_n + d_n x\right)\right) = z\left(x_1, x_2\right) > 0,$$

then

$$\lim_{n\to\infty} P\left(W_n < c_n + d_n x\right) = L(x) = 1 - e^{-z_1(x)} - e^{-z_2\left(x_2\right)} + e^{-z\left(x_1, x_2\right)}, \tag{2}$$

where $L(x)$ is a nondegenerate distribution function of two variables, and $F_1\left(x_1\right)$, $F_2\left(x_2\right)$ are marginal distribution functions of the distribution function $F(x)$.

The necessary and sufficient conditions for the convergence of normalized maxima and normalized minima are formulated in [3].

Suppose that conditions (1) and (2) hold true. Since the bivariate maxima and bivariate minima are asymptotically independent (see [2]), then

$$\lim_{n\to\infty} P\left(Z_n < a_n + b_n y, W_n < c_n + d_n x\right) = H(y)L(x). \tag{3}$$

In this paper we are going to obtain a nonuniform estimate of the convergence rate in (3). It will generalize the result obtained in [1].

2 Main result

Let us denote

$$\Delta_{[c_n+d_nx,a_n+b_ny)}F = F(a_n+b_ny)+F(c_n+d_nx)-$$
$$-F(c_{1,n}+d_{1,n}x_1, a_{2,n}+b_{2,n}y_2)-F(a_{1,n}+b_{1,n}y_1, c_{2,n}+d_{2,n}x_2),$$
$$u_{1,n}(y)=n(1-F(a_n+b_ny)),$$
$$v_{1,n}(y)=u_{1,n}(y)-u(y),$$
$$u_{2,n}(y,x_1)=n(1-F(a_n+b_ny)+F(c_{1,n}+d_{1,n}x_1, a_{2,n}+b_{2,n}y_2)),$$
$$v_{2,n}(y,x_1)=u_{2,n}(y,x_1)-u(y)-z_1(x_1),$$
$$u_{3,n}(y,x_2)=n(1-F(a_n+b_ny)+F(a_{1,n}+b_{1,n}y_1, c_{2,n}+d_{2,n}x_2)),$$
$$v_{3,n}(y,x_2)=u_{3,n}(y,x_2)-u(y)-z_2(x_2),$$
$$u_{4,n}(y,x)=n(1-\Delta_{[c_n+d_nx,a_n+b_ny)}F)$$
$$v_{4,n}(y,x)=u_{4,n}(y,x)-u(y)-z(x).$$

Theorem. Suppose that (3) holds, and

$$a_n+b_ny>c_n+d_nx .$$

For all *x, y* , for which

$$|v_{1,n}(y)|\le \log 2, \qquad |v_{2,n}(y,x_1)|\le \log 2,$$
$$|v_{3,n}(y,x_2)|\le \log 2, \quad |v_{4,n}(y,x)|\le \log 2,$$

the following estimate holds true:

$$|P(Z_n<a_n+b_ny, W_n<c_n+d_nx)-H(y)L(x)|\le$$
$$\le \Delta_{1,n}(y)+\Delta_{2,n}(y,x_1)+\Delta_{3,n}(y,x_2)+\Delta_{4,n}(y,x).$$

Here

$$\Delta_{1,n}(y)=\frac{u_{1,n}^2(y)e^{-u_{1,n}(y)}}{2(n-1)}+e^{-u(y)}\left(|v_{1,n}(y)|+v_{1,n}^2(y)\right),$$

$$\Delta_{2,n}(y,x_1)=\frac{u_{2,n}^2(y,x_1)e^{-u_{2,n}(y,x_1)}}{2(n-1)}+e^{-u(y)-z_1(x_1)}\left(|v_{2,n}(y,x_1)|+v_{2,n}^2(y,x_1)\right),$$

$$\Delta_{3,n}(y,x_2)=\frac{u_{3,n}^2(y,x_2)e^{-u_{3,n}(y,x_2)}}{2(n-1)}+e^{-u(y)-z_2(x_2)}\left(|v_{3,n}(y,x_2)|+v_{3,n}^2(y,x_2)\right),$$

$$\Delta_{4,n}(y,x)=\frac{u_{4,n}^2(y,x)e^{-u_{4,n}(y,x)}}{2(n-1)}+e^{-u(y)-z(x)}\left(|v_{4,n}(y,x)|+v_{4,n}^2(y,x)\right).$$

Proof. We have

$$P(Z_n < y, W_n < x) = P(Z_n < y) - P(Z_n < y, W_{1,n} \geq x_1) -$$
$$- P(Z_n < y, W_{2,n} \geq x_2) + P(Z_n < y, W_n \geq x) =$$
$$= F^n(y) - (F(y) - F(x_1, y_2))^n - (F(y) - F(y_1, x_2))^n + (\Delta_{[x,y)} F)^n,$$

if $y > x$, and

$$P(Z_n < y, W_n < x) = P(Z_n < y) = F^n(y),$$

if $y \leq x$. Here $W_{1,n}, W_{2,n}$ are components of the bivariate minima W_n.

Let $a_n + b_n y > c_n + d_n x$. We have

$$\left|P(Z_n < a_n + b_n y, W_n < c_n + d_n x) - H(y)L(x)\right| \leq$$
$$\leq \left|F^n(a_n + b_n y) - e^{-u(y)}\right| +$$
$$+ \left|\left(F(a_n + b_n y) - F(c_{1,n} + d_{1,n} x_1, a_{2,n} + b_{2,n} y_2)\right)^n - e^{-u(y) - z_1(x_1)}\right| + \tag{4}$$
$$+ \left|\left(F(a_n + b_n y) - F(a_{1,n} + b_{1,n} y_1, c_{2,n} + d_{2,n} x_2)\right)^n - e^{-u(y) - z_2(x_2)}\right| +$$
$$+ \left|\left(\Delta_{[c_n + d_n x, a_n + b_n y)} F\right)^n - e^{-u(y) - z(x)}\right|.$$

Let us estimate the first summand of the right – hand side of the inequality (4). We have

$$\left|F^n(a_n + b_n y) - e^{-u(y)}\right| \leq \tag{5}$$
$$\leq \left|F^n(a_n + b_n y) - e^{-u_{1,n}(y)}\right| + \left|e^{-u_{1,n}(y)} - e^{-u(y)}\right|.$$

Let us estimate the second summand of the right – hand side of the inequality (5). Applying the inequality

$$0 \leq e^{-x} - \left(1 - \frac{x}{n}\right)^n \leq \frac{x^2 e^{-x}}{2(n-1)} \quad (0 \leq x \leq n),$$

we get

$$\left|F^n(a_n + b_n y) - e^{-u_{1,n}(y)}\right| =$$
$$= \left|\left(1 - \frac{u_{1,n}(y)}{n}\right)^n - e^{-u_{1,n}(y)}\right| \leq \tag{6}$$
$$\leq \frac{u_{1,n}^2(y) e^{-u_{1,n}(y)}}{2(n-1)}.$$

Let us estimate the first summand of the right – hand side of the inequality (5). Applying the inequality

$$\left|e^{-y} - e^{-x}\right| \leq e^{-x}\left(|x - y| + (x - y)^2\right) \quad (|x - y| \leq \log 2),$$

we get

$$\left|e^{-u_{1,n}(y)} - e^{-u(y)}\right| \le e^{-u(y)}\left(\left|v_{1,n}(y)\right| + v_{1,n}^2(y)\right), \tag{7}$$

if $\left|v_{1,n}(y)\right| \le \log 2$.

Taking into account inequalities (6) and (7), from the inequality (5) we get

$$\left|F^n(a_n + b_n y) - e^{-u(y)}\right| \le \Delta_{1,n}(y), \tag{8}$$

if $\left|v_{1,n}(y)\right| \le \log 2$.

Analogously, we get

$$\left|\left(F(a_n + b_n y) - F(c_{1,n} + d_{1,n}x_1, a_{2,n} + b_{2,n}y_2)\right)^n - e^{-u(y)-z_1(x_1)}\right| \le$$
$$\le \Delta_{2,n}(y, x_1), \tag{9}$$

if $\left|v_{2,n}(y, x_1)\right| \le \log 2$;

$$\left|\left(F(a_n + b_n y) - F(a_{1,n} + b_{1,n}y_1, c_{2,n} + d_{2,n}x_2)\right)^n - e^{-u(y)-z_2(x_2)}\right| \le$$
$$\le \Delta_{3,n}(y, x_2), \tag{10}$$

if $\left|v_{3,n}(y, x_2)\right| \le \log 2$;

$$\left|\left(\Delta_{[c_n+d_n x, a_n+b_n y)}F\right)^n - e^{-u(y)-z(x)}\right| \le$$
$$\le \Delta_{4,n}(y, x_2), \tag{11}$$

if $\left|v_{4,n}(y, x_2)\right| \le \log 2$.

Taking into account inequalities (8) – (11), from the inequality (4) we get the estimate in (3).

Theorem is proved.

3 The Example

Let $\{X_j, j \ge 1\}$ be independent identically distributed bivariate random variables with a distribution function

$$F(x) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1-x_2}, \quad x_1 > 0, x_2 > 0.$$

For the chosen centralizing and normalizing vectors

$$a_n = (\log n, \log n), \qquad b_n = (1,1),$$
$$c_n = (0,0), \qquad d_n = (1/n, 1/n),$$

we have

$$\lim_{n\to\infty} P(Z_n < a_n + b_n y, W_n < c_n + d_n x) = H(y)L(x); \tag{12}$$

here

$$H(y) = \exp\left(-e^{-y_1} - e^{-y_2}\right), \quad y_1, y_2 \in \boldsymbol{R},$$

$L(x) = 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2}, \quad x_1 > 0, x_2 > 0.$

We shall estimate the convergate rate in (12). We have

$$u_{1,n}(y) = e^{-y_1} + e^{-y_2} - \frac{e^{-y_1-y_2}}{n},$$

$$v_{1,n}(y) = -\frac{e^{-y_1-y_2}}{n},$$

$$u_{2,n}(y, x_1) = e^{-y_1} + e^{-y_2 - x_1/n} - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-x_1/n}\right),$$

$$v_{2,n}(y, x_1) = e^{-y_2}\left(e^{-x_1/n} - 1\right) - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-x_1/n}\right) - x_1,$$

$$u_{3,n}(y, x_2) = e^{-y_2} + e^{-y_1 - x_2/n} - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-x_2/n}\right),$$

$$v_{3,n}(y, x_2) = e^{-y_1}\left(e^{-x_2/n} - 1\right) - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-x_2/n}\right) - x_2,$$

$$u_{4,n}(y, x) = e^{-y_1 - x_2/n} + e^{-y_2 - x_1/n} - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-(x_1 - x_2)/n}\right),$$

$$v_{4,n}(y, x) = e^{-y_1}\left(e^{-x_2/n} - 1\right) + e^{-y_2}\left(e^{-x_1/n} - 1\right) - \frac{e^{-y_1-y_2}}{n} + n\left(1 - e^{-(x_1 + x_2)/n}\right) - x_1 - x_2.$$

Evidently, the order of the convergence rate with respect to *n* equals $1/n$.

4 References

1. Aksomaitis A., Jokimaitis A., “An asymptotic of the distribution of extrema of independent random variables”, *Liet. Mat. Rinkinys*, **35**, p.1-13, 1995
2. Galambos J., *The Asymptotis Theory of Extreme Order Statistics*, Wiley, New York, 1978
3. Marshall A., Olkin I., “Domains of attraction of multivariate extreme value distributions”, *Ann. Probab.*, **11**, p. 168 - 177, 1983