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On Solvability of the Boundary Value Problems for the Multidimensional Elliptic Systems of Partial Equations

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Abstract. The investigation of a question of solvability boundary value problems for elliptic partial systems is presented. Attention is paid to differences in case of two independent variables and of multidimensional one.

Keywords: boundary value problem, elliptic partial system, solvability.

AMS classifications:

1 Introduction

The fact that the classic boundary value problems which are well-posed for the Laplace equation also satisfy the Fredholm alternative for a single elliptic partial differential equation with smooth enough coefficients is well known [1]. In general the character of solvability of boundary value problems for elliptic partial differential systems is essentially different from the case of a single elliptic equation.

A.Bicadze [2] has demonstrated the first example of such a system:

$$\begin{aligned} u_{xx} - u_{yy} + 2v_{xy} = 0, \\ 2u_{xy} - v_{xx} + v_{yy} = 0. \end{aligned} \tag{1}$$

This system is an elliptic one, however, the Dirichlet problem for it does not satisfy the Noether conditions.

Later the elliptic partial systems with two independent variables were divided into two classes, called weakly-connected and strongly-connected elliptic systems [3]. The Dirichlet problem for the first-class elliptic systems is always well-stated but not for the second class systems. The definition of these two classes of elliptic partial systems with two independent variables is based on certain expressions of their solutions by analytic functions of a complex variable, hence its extension into the case of multidimensional elliptic systems fails.

However, it was noticed that for each strongly-connected elliptic system there exists a half-plane in which the Dirichlet problem does not satisfy the Noether conditions [4]. Based on this property the multidimensional elliptic partial system is called a multidimensional analog of strongly-connected elliptic systems with two independent variables when there exists a half-space in which the Dirichlet problem does not satisfy the Noether conditions.

2 The Multidimensional Elliptic Systems

Let us consider an elliptic system of the first order

$$\sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i} U(X) = 0, \qquad (2)$$

where P_i are the $m \times m$ -dimensional real constant matrixes satisfying

$$P_i^* P_j + P_j^* P_i = 2 p_{ij} E, \quad i, j = \overline{1, n}$$

with certain real constant matrixes P_i^* of the same dimension, E is the $m \times m$ identity and $P = \|p_{ij}\|, i, j = \overline{1, n}$ is positively definite real constant matrix, hence

$$\sum_{i,j=1}^{n} p_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

is an elliptic operator, $U(X) = (u_1, \ldots, u_m)$, $X = (x_1, \ldots, x_n)$, $X \in \mathbf{R}^n$.

The maximal number of such $m \times m$ -dimensional real matrixes P_i is

$$R(m) = 2^c + 8d,$$

where $m = (2a+1)\,2^b$, $b = c + 4d$, a, b, c, d – nonnegative integers and $0 < c < 4$, see [5].

Let $R_j R_j^*,\ j = \overline{1,n}$ is the other collection of analogous matrixes and the corresponding elliptic operator is

$$\sum_{i,j=1}^{n} r_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Let us consider the Dirichlet problem in a half-space $\mathbf{R}_+^n : \left\{ X \in \mathbf{R}^n \big|_{x_n > 0} \right\}$ for the following elliptic system of the second order:

$$\sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i} \sum_{j=1}^{n} R_j \frac{\partial}{\partial x_j} U(X) = 0 \qquad (3)$$

with boundary conditions

$$U(X)\big|_{x_n=0} = f(X'),$$

where $f(X')$ is the given vector function, $X' = (x_1, \ldots, x_{n-1})$. This system is not strongly-elliptic [6] in general. Let us denote by

$$\mathsf{R}_{\xi} v(\xi) = \left(\sum_{j=1}^{n-1} \mathrm{i}\,\xi_j R_j - r(\xi) E \right) v(\xi),$$

where

$$r(\xi) = \mathrm{i} \sum_{j=1}^{n-1} r_{jn} \xi_j + \left[-\left(\sum_{j=1}^{n-1} r_{jn} \xi_j \right)^2 + \sum_{i,j=1}^{n-1} r_{ij} \xi_i \xi_j \right]^{\frac{1}{2}},$$

$$\xi \in \mathbf{R}^{\text{n-1}}, \quad \xi = (\xi_1, \ldots, \xi_{n-1}), \quad v(\xi) \in \mathbf{C}^m,$$

i is the imaginary unit. Let us define by analogy the operator P_{ξ} and let us denote by $\overset{\circ}{\mathsf{R}}_{\xi}$ and $\overset{\circ}{\mathsf{P}}_{\xi}$ the kernels of the operators R_{ξ} and P_{ξ} respectively.

U(*X*) is a regular solution of the system (3) if it satisfies the system and belongs the Hoelder classes $U \in H_2(\Omega)$ and $DvU \in H_1(\mathbf{R}_+^n)$, $|v| = 1$, where Ω

is an arbitrary infinite layer in $\mathbf{R}_{+}^{n}$. Boundary value function is supposed $f(X') \in H_{3/2}(\mathbf{R}^{n-1})$.

In [7] the following statement was proved:

Theorem 1. *Let* $R_{\xi}^{\circ} \cap P_{\xi}^{\circ} = \varnothing$ *for each* $\xi \in \mathbf{R}^{n-1}, |\xi| \neq 0$. *Then there exist a unique regular solution of the Dirichlet problem in* $\mathbf{R}_{+}^{n}$ *for the system* (3). *If there exists* $v_0(\xi) \in R_{\xi}^{\circ} \cap P_{\xi}^{\circ}$ *for each* $\xi \in \mathbf{R}^{n}$ *then the homogenous boundary value problem has an infinite number of lineary independent solutions.*

There we can observe three essentially different situations: R_{ξ}° and P_{ξ}° intersect only at point $\xi = 0$, they intersect for each $\xi \in \mathbf{R}^{n-1}$, and the third, when those ξ compose some manifold of lower dimension to compare with $n - 1$, including $\xi = 0$.

We have a unique solution of the Dirichlet problem in the given half-space in the first situation, our system is the multidimensional analog of strongly-connected elliptic systems with two independent variables in the second situation, and we need additional studies in the third one.

We can observe only the first two situations for the systems with two independent variables and the third one is impossible.

By a linear change of variables system (2) can be reduced to the system

$$L_{m,n}U(X) = \left(E\frac{\partial}{\partial x_n} + \sum_{k=1}^{n-1} M_k \frac{\partial}{\partial x_k} \right) U(X) = 0, \tag{4}$$

where $m \times m$-dimensional real matrixes M_i satisfy the following equalities:

$$\begin{aligned} M_i M_j &= -M_j M_i, \quad i \neq j, \\ M_i^2 &= -E, \end{aligned} \tag{5}$$

$i, j = \overline{1, n-1}$, see [8]. As we remember, the operator $L_{m,n}$ exists under condition $n \le R(m)$. The elliptic system of the second order

$$\left(hE\frac{\partial}{\partial x_n} + \sum_{k=1}^{n-1} M_k \frac{\partial}{\partial x_k} \right)\left(lE\frac{\partial}{\partial x_n} + \sum_{k=1}^{n-1} M_k \frac{\partial}{\partial x_k} \right) U(X) = 0, \tag{6}$$

where *h, l* are nonzero real constants, and the Dirichlet problem for it was studied in [9]. In particular, under the same assumptions on boundary value function the following was proved:

Theorem 2. *If* $\operatorname{sgn} h \neq \operatorname{sgn} l$, *then there exists a unique regular solution of the Dirichlet problem in a half-space* $\mathbf{R}_{+}^{n}$ *for the system* (6), *otherwise this problem does not satisfy Fredholm alternative and Noether conditions.*

Hence in the case of $\operatorname{sgn} h = \operatorname{sgn} l$ system (6) is a multidimensional analog of strongly-connected elliptic systems. Particularly, when $h = l = 1$ the system (6) transforms into the system

$$L_{m,n}^{2} U(X) = 0. \tag{7}$$

Suppose $n < R(m)$ (i.e. we have used "incomplete collection" of matrixes in (4). By multiplying both sides of (4) by M_n we obtain the operator $\widetilde{L}_{m,n}$:

$$\widetilde{L}_{m,n} U(x) = \sum_{k=1}^{n} \overline{M}_k \frac{\partial}{\partial x_k} U(X), \tag{8}$$

where $\overline{M_n} = M_n, \overline{M_k} = M_n M_k, k = \overline{1, n-1}$. Matrixes $\overline{M}_k, k = \overline{1,n}$ also satisfy (5).

The analog of system (7)

$$\widetilde{L}_{m,n}^{2} U(X) = 0$$

is the multidimensional analog of weakly-connected elliptic systems with two independent variables [10].

Let us consider an analog of the system (6)

$$\left(h M_n \frac{\partial}{\partial x_n} + \sum_{k=1}^{n-1} M_k \frac{\partial}{\partial x_k} \right) \left(l M_n \frac{\partial}{\partial x_n} + \sum_{k=1}^{n-1} M_k \frac{\partial}{\partial x_k} \right) U(X) = 0. \tag{9}$$

By transforming this system into the form of (6) and by the use of the statement of Theorem 2 it is proved that this system is the multidimensional analog of a strongly-connected elliptic systems when $\operatorname{sgn} h \neq \operatorname{sgn} l$, contrary to the situation with the system (6):

Theorem 3. *If* $\operatorname{sgn} h = \operatorname{sgn} l$ *then there exists a unique regular solution of the Dirichlet problem in a half-space* $\mathbf{R}_+^n$ *for the system* (9)*, otherwise this problem does not satisfy Fredholm alternative and Noether conditions.*

The only restrictions on matrixes $\overline{M}_k, k = \overline{1,n}$ are (5) and on boundary value function are the same as above.

We cannot observe this phenomenon in the case of elliptic systems with two independent variables and the number of unknown functions $m = 2t$, where t is a positive odd integer. Indeed, in this case $R(m) = 2$, so the "complete collection" of matrixes consists of a single matrix M satisfying $M^2 = -E$.

Let us notice that the A. Bicadze system (1) can be written as

$$L_{2,2}^2 U(X) = 0$$

with a single matrix $M\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $L_{2,2}$ now is the Cauchy-Riemann operator. The classes of operators $L_{m,n}$ and $\widetilde{L}_{m,n}$ include few generalizations of the Cauchy-Riemann operator to higher dimensions investigated by different authors [10].

Below we demonstrate one way of construction of analogs of the strongly-connected elliptic systems of higher dimensions. Let us consider the elliptic partial system of the first order with $2m$ unknown functions

$$LU(X) = 0, \tag{10}$$

where $U(X) = (u_1, \ldots, u_{2m})$, $X = (x_1, \ldots, x_{2n})$, and the hypermatrix of the operator L is

$$\begin{Vmatrix} L_{m,n} & -L_{m,n}^* \\ \overline{L}_{m,n}^* & \overline{L}_{m,n} \end{Vmatrix},$$

where $\overline{L}_{m,n}$ and $\overline{L}_{m,n}^*$ are conjugate operators to $L_{m,n}$ and L_{mn}^* respectively, $L_{m,n}$ is defined by (4) and

$$L_{m,n}^* = E\frac{\partial}{\partial x_{2n}} + \sum_{k=1}^{n-1} M_k^* \frac{\partial}{\partial x_{n+k}}$$

with real constant $m \times m$ matrixes M_k^*, $k = \overline{1, n-1}$ satisfying (5) and additionally

$$M_k M_j^* = M_j^* M_k, \quad k, j = \overline{1, n-1}.$$

Regarding the existence of M_j^* let us notice that in case of $2n < R(m) - 1$ we can choose M_j^* as

$$M_j^* = M_{n+j} M_{2n+1}, \quad j = \overline{1, n-1}.$$

Direct calculations show us that the operator L can be represented in the form

$$L_{2m,2n} U(X) = E \frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} M_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} M_{n+i-1} \frac{\partial}{\partial x_{n+i}}$$

with certain $2m \times 2m$ constant real matrixes M_i, $i = \overline{1, 2n-1}$, satisfying (5). Hence from the Theorem 2 follows the statement regarding the multidimensional elliptic system

$$L^2 U(X) = 0. \qquad (11)$$

Theorem 4. *System* (11) *is a multidimensional analog of strongly-connected elliptic systems with two independent variables.*

The question of solvability of the Dirichlet problem for second-order multidimensional systems is in connection with the one of the Riemann-Hilbert problem for the first-order elliptic partial systems [11]. This problem is well-posed for the Cauchy-Riemann system. System (10) becomes the Cauchy-Riemann system when $m = n = 1$, $L_{1,1} = \frac{\partial}{\partial x_1}, L_{1,1}^* = \frac{\partial}{\partial x_2}$.

However, the Riemann-Hilbert problem for the system (10) with the given boundary value of the component $(u_1, \ldots, u_m)$ of solution $U(X)$ of the system does not satisfy the Noether conditions [12].

Let us notice, that if (5) is satisfied, then the matrixes E, M_i, $i = \overline{1, n-1}$ form the basis for the Clifford algebra associated with the quadratic form. Remembering the idea of the definition of weakly-connected and strongly-connected elliptic systems with two independent variables it would be worth observing that the Clifford algebras have provided one of settings for the extension of analytic function theory to higher dimensions, see [8].

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