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Geometric Zeta-Function and Euclidean Action

A. Javtokas

Department of Number Theory and Probability Theory, Vilnius University
Naugarduko 24, Vilnius LT-2600, Lithuania
ajavtokas@math.com

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Abstract. This paper provides analysis on successful combining Euclidean action and geometric zeta-function, investigating distribution of the zeta-function poles (spectrum).

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Euclidean action plays a fundamental role in physics. There is a method to compute it by using the "universality" property of the Riemann zeta-function [1]. But we will try to apply some different approach which is based on geometric zeta-functions. We will use geometric zeta-function to find information about Euclidean action, not calculating it, but finding distribution of geometric zeta poles. First, we will create a fractal string with scaling factors r_j which will generate fields $\phi(n)$ at lattice points n. Secondly, we will take scaling factors to geometric zeta-function and investigate its distribution of poles (spectrum) by defining a suitable test function φ (in our case we will use some analogy with partition functions). This method is indirect, and the information we get is much more complicated, but it is a powerful tool extending Euclidean action to new horizons.

We begin with some definitions. Let, as usual, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of all real numbers, integer numbers, positive integer numbers and complex numbers, respectively.

A fractal string L is a bounded open subset of $\mathbb{R}$, which consists of countably many open intervals, the lengths of which are denoted by $l_1 \geq l_2 \geq \ldots > 0$, and $l_j^{-1} \in \mathbb{N}$, $j = 1, 2, \ldots$.

The dimension $D = D_L$ of the fractal string L is defined by

$$D_L = \inf \left\{ \sigma > 0 \colon \sum_{j=1}^{\infty} l_j^{\sigma} < \infty \right\}.$$

Let $s = \sigma + it$ be a complex variable. The *geometric zeta*-function $\zeta_L(s)$ of the fractal string L is given by

$$\zeta_L(s) = \sum_{j=1}^{\infty} l_j^{s}.$$

The *screen* S is the contour

$$S(t) = r(t) + it, \quad t \in \mathbb{R}, \quad i = \sqrt{-1},$$

with some continuous function $r \colon \mathbb{R} \to [-\infty, D_L]$.

The set

$$W = \big\{ s \in \mathbb{C} \colon \sigma \geq r(t) \big\},$$

is called the *window*. We assume that the function $\zeta_L(s)$ has a meromorphic continuation to a neighborhood of W with set of poles

$$D_L(W) = \big\{ \omega \in \mathbb{C} \colon \ \zeta_L(s) \text{ has a pole at } \omega \big\},$$

called the *visible complex dimensions* of the fractal string L.

The total length L^* of the fractal string L is

$$L^* = \zeta_L(1) = \sum_{j=1}^{\infty} l_j.$$

Note that L^* is a finite number and equals to the Lebesgue measure of L [2].

Let $N \geq 2$, and let given positive numbers $r_1, r_2, ..., r_N$ satisfy $r_1 \geq r_2 \geq ... \geq r_N$. Assume that

$$R := \sum_{j=1}^{N} r_j < 1.$$

Then $r_1, r_2, ..., r_N$ are called *scaling factors*.

Given an open interval of length L^* we construct a *self-similar string* L with scaling factors $r_1, r_2, ..., r_N$ by procedure reminiscent of the construction of the Cantor string. Subdivide an interval I into intervals of length $r_1L, r_2L, ..., r_NL$. The remaining peace of length $L(1-R)$ is the first member of the string. Repeat this process with the remaining intervals.

Theorem 1. *Let L be a self-similar string, constructed as above with scaling factors $r_1, r_2, ..., r_N$. Then the geometric zeta-function of this string has a meromorphic continuation to the whole complex plane given by*

$$\zeta_L(s) = \frac{\big(L(1-R)\big)^s}{1 - \sum_{j=1}^{N} r_j^s}, \quad s \in \mathbb{C}.$$

Proof can be found in [2].

Now we will define the Euclidean action and relate it with geometric zeta-function. The Euclidean action on a lattice of step size a, for a finite time interval $(0, L^*)$, is given by

$$S(\phi) = \frac{1}{2a}\sum_{v=1}^{\nu-1}\big(\phi(x_v) - \phi(x_{v+1})\big)^2 + \frac{m^2a}{2}\sum_{v=1}^{\nu}\phi^2(x_v) + a\sum_{v=1}^{\nu}V\big(\phi(x_v)\big),$$

where $\nu = L^*/a$ is a number of lattice points. $\phi(x)$ is a physical field (set of continuous real functions), $\phi(x_v)$ is its value at the v-th lattice point $x_v = va$, $v = 1, 2, .., \nu$. Here V is a continuous real function, and m denotes the mass. Assume that a set of fields defines a (functional) fractal string $L_S = \phi(x_1), \phi(x_2), ..., \phi(x_\nu)$ which lengths are continuous real functions. We can create a set of fractal strings $L_\Sigma = \{L_1, L_2, ...\}$ which are generated by scaling factors $\{\widehat{r}_j\}_{j=1}^M$. L_Σ consists of fractal string lengths $\big\{\{l_{1j}\}_{j=1}^\infty, \{l_{2j}\}_{j=1}^\infty, \dots\big\}$. Then for every function $\phi(x_v)$, $v = 1, 2, ..., \nu$ and $\epsilon > 0$ we can find $|L_\Sigma(x_v) - \phi(x_v)| < \epsilon$, where $L_\Sigma(x_v)$ means that we are taking fractal string length l_{vj}. Perform this operation for all the functions $\phi(x_v)$ at all the points x_v. Finally, we will get a set of fractal strings which were used, so we will obtain scaling factors $\{\widehat{r}_j\}_{j=1}^N$, $N \le M$, which were applied to approximate $\phi(x_v)$.

Denoting $\widehat{\mathbf{r}} = (\widehat{r}_1, \widehat{r}_2, ..., \widehat{r}_N)$ we can write the *scaled Euclidean action*

$$S(\widehat{\mathbf{r}}) = c_1 \sum_{j=1}^{N-1} (\widehat{r}_j - \widehat{r}_{j-1})^2 + c_2 \sum_{j=1}^{N} \widehat{r}_j^2 + c_3 \sum_{j=1}^{N} V(\widehat{r}_j)$$
$$= c_1 R_1 + c_2 R_2 + c_3 f(\widehat{\mathbf{r}}),$$

where we can interpret $c_1 R_1$, $c_2 R_2$ and $c_3 f(\widehat{\mathbf{r}}) := c_3 R_3$ as new scaling factors. We can normalize this sum by choosing constants c_1, c_2 and c_3 such that

$$\sum_{k=1}^{3} c_k R_k < 1.$$

The latter condition is not necessary. Then we can write geometric zeta-function for Euclidean action

$$\zeta_L(s) = \frac{1}{1 - \sum_{k=1}^{3} (c_k R_k)^s}.$$

From now we turn from the direct Euclidean action investigation to the indirect investigation through the zeta-function. We will examine the complex dimensions of $\zeta_L(s)$. Complex dimensions of the geometric zeta-function with scaling factors $c_k R_k$, $k = 1, 2, 3$), is the set of solutions of the equation

$$\sum_{k=1}^{3} (c_k R_k)^\omega = 1, \quad \omega \in \mathbb{C}.$$

The next logical step is to study the distribution of these complex dimensions. To do that we will extend previously analyzed geometric partition function [2] defining two new partition functions.

The geometric partition function $\theta_L(\tau)$ of an ordinary fractal string $L = (l_j)_{j=1}^{\infty}$ is

$$\theta_L(\tau) = \sum_{j=1}^{\infty} e^{-\tau l_j^{-1}}, \quad \text{for} \quad \tau > 0.$$

Let

$$p(x) = \frac{e^{\pi\sqrt{2x/3}}}{4\sqrt{3}x}, \tag{1}$$

$$q(x) = \frac{e^{\pi\sqrt{x/3}}}{4 \cdot 3^{1/4} x^{3/4}}. \tag{2}$$

Define q and p geometric partition functions of an ordinary fractal string L

$$\theta_{L,p}(\tau) = \sum_{j=1}^{\infty} p(l_j^{-1}) e^{-\tau l_j^{-1}}, \quad \tau > 0,$$
$$\theta_{L,q}(\tau) = \sum_{j=1}^{\infty} q(l_j^{-1}) e^{-\tau l_j^{-1}}, \quad \tau > 0,$$

Note that $p(m)$ and $q(m)$, $m \in \mathbb{N}$, are the main terms of partition functions of decomposition of m. We can choose p partition function when repetition of elements is important, and q when repetition must be ignored.

To investigate the distribution of poles of geometric zeta-function we must generalize a concept of a fractal string and geometric zeta-function.

Given a complex measure η there exists a positive measure denoted $|\eta|$ which measures the total variation of η

$$|\eta|(J) = \sup \sum_i \big|\eta(J_i)\big|,$$

where the supremum is taken over all partitions $\cup J_i$ of J into measurable subsets J_i. The measure $|\eta|$ is called the *total variation measure* associated with η. Recall that $|\eta| = \eta$ if η is positive.

A *local positive measure* is just a standart positive Borel measure on $(0, \infty)$ which satisfies the following local boundedness condition:

$$\eta(J) < \infty, \text{ for all bounded subintervals } J \text{ of } (0, \infty).$$

More generally, we will say, that a set function η on $(0, \infty)$ is a *local complex measure* on $(0, \infty)$ if, the following conditions are satisfied: (i) $\eta(A)$ is well defined for any Borel subset A of $[a, b]$, and (ii) the restriction of η to the Borel subsets of $[a, b]$ is a complex measure on $[a, b]$ in the traditional sense. We will use the following notions.

1. A generalized fractal string is either a local complex or a local positive measure η on $(0, \infty)$ such that

   $$|\eta|(0, x_0) = 0$$

   for some positive number x_0.

2. The dimension of η, denoted $D = D_\eta$, is

$$D = D_\eta = \inf\left\{\sigma \in \mathbb{R}\colon \int\limits_0^\infty x^{-\sigma}|\eta|(dx) < \infty\right\}.$$

3. The geometric zeta-function $\zeta_\eta(s)$ of η is given, for $\sigma > D_\eta$, by

$$\zeta_\eta(s) = \int\limits_0^\infty x^{-s}\eta(dx).$$

To introduce the generalized geometric p and q partition functions for a generalized fractal string η , we need some notation.

Let us denote by $N_\eta^{[k]}$ the k-th primitive (or k-th antiderivative) of N_η vanishing at 0. Thus

$$N_\eta^{[k]}(x) = \int\limits_0^x \frac{(x-y)^{k-1}}{(k-1)!}\eta(dy),$$

for $x > 0$ and $k = 1, 2, ...$. In particular, $N_\eta^{[0]} = \eta$. The distributional formula describes η as a distribution. On a test function φ, η acts by

$$\langle \eta, \varphi\rangle = \int\limits_0^\infty \varphi(x)\,\eta(dx).$$

The k-th primitive of this distribution will be denoted by $P^{[k]}\eta$. More precisely, $P^{[k]}\eta$ is the distribution given for all test functions φ by

$$\langle P^{[k]}\eta, \varphi\rangle = (-1)^k\langle \eta, P^{[k]}\varphi\rangle = (-1)^{k+\mu}\langle P^{[k+\mu]}\eta, \varphi^{(\mu)}\rangle,$$

where $\varphi^{(\mu)}$ is the μ-th derivative, so a test function φ must be μ times continuously differentiable on $(0, \infty)$. We can write

$$\langle P^{[k]}\eta, \varphi\rangle = \int\limits_0^\infty\int\limits_y^\infty \frac{(x-y)^{k-1}}{(k-1)!}\varphi(x)\,dx\,\eta(dy).$$

For a generalized fractal string η, geometric p and q partition functions $\theta_{\eta,p}(t)$ and $\theta_{\eta,q}(t)$ will be defined as

$$\theta_{\eta,p}(t) = \int_0^\infty \varphi_{t,p}(x)\,\eta(dx) = \langle P^{[0]}\eta, \varphi_{t,p}\rangle, \tag{3}$$

$$\theta_{\eta,q}(t) = \int_0^\infty \varphi_{t,q}(x)\,\eta(dx) = \langle P^{[0]}\eta, \varphi_{t,q}\rangle, \tag{4}$$

where, for $x \in \mathbb{R}$, and $t > 0$,

$$\varphi_{t,p}(x) = p(x)e^{-tx}, \tag{5}$$

$$\varphi_{t,q}(x) = q(x)e^{-tx}. \tag{6}$$

Assume that ζ_η satisfies the following growth conditions [2]: there exists real constants $\kappa > 0$ and $C > 0$, and a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of real numbers tending to $\pm\infty$ as $n \to \pm\infty$, with $T_{-n} < 0 < T_n$ for $n \geq 1$ and $\lim_{n\to+\infty} T_n/|T_{-n}| = 1$, such that

($\mathbf{H}_1$) For all $n \in \mathbb{Z}$ and all $\sigma \geq r(T_n)$,
$$\big|\zeta_\eta(\sigma + iT_n)\big| \leq C\,|T_n|^\kappa;$$
($\mathbf{H}_2$) For all $t \in \mathbb{R},\ |t| \geq 1$,
$$\big|\zeta_\eta(r(t) + it)\big| \leq C\,|t|^\kappa,$$

where r is the Lipschitz continuous function i.e., there exists a nonnegative real number $\|r\|_{Lip}$ such that $\big|r(x) - r(y)\big| \leq \|r\|_{\mathrm{Lip}}|x - y|$ for all $x, y \in \mathbb{R}$, which bounds the screen S.

Hypothesis ($\mathbf{H}_1$) is a polynomial growth condition along horizontal lines (necessary avoiding the poles of ζ_η), while hypothesis ($\mathbf{H}_2$) is a polynomial growth condition along the vertical direction of the screen.

We shall denote by $\widetilde{\varphi}$ the *Mellin* transform of a (suitable) function φ on $(0, \infty)$, it is defined by

$$\widetilde{\varphi}(s) = \int_0^\infty \varphi(s)x^{s-1}\,dx, \quad s \in \mathbb{C}.$$

Henceforth, we denote by $\operatorname{res}\big(g(s);\omega\big)$ the residue of a meromorphic function $g = g(s)$ at $s = \omega$. For $k \geq 1$ we shall define the symbol $(s)_k$ by $(s)_k = s(s+1)...(s+k-1)$.

Assume that a, b are complex numbers independent on the variable z. Then the differential equation

$$z(1-z)\frac{d^2u}{dz^2} + \big(b-(a+1)z\big)\frac{du}{dz} - au = 0$$

is called a *hypergeometric equation*.

If $b \neq -m$, $m \in \mathbb{N} \cup 0$, then the function

$$u = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b)}{\Gamma(a)\Gamma(b+m)m!} z^m \overset{def}{=} {}_1F_1(a;b;z)$$

is a regular solution of the hypergeometric equation at the point $z = 0$, and the function ${}_1F_1(a;b;z)$ is called the *hypergeometric function* with parameters a, b.

Now we can state a modified version of Theorem 4.20 which will be useful for our aim.

Theorem 2. *Let η be a generalized fractal string satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. Let $k \in \mathbb{Z}$, and let $\mu \in \mathbb{N}$ be such that $k + \mu \geq \kappa + 1$. Further, let φ be a test function μ times continuously differentiable on $(0,\infty)$. Then the action of $P^{[k]}\eta$ on a test function φ is given by*

$$\begin{aligned}\langle P^{[k]}\eta, \varphi\rangle = &\sum_{\omega \in D_\eta(W)} \text{res}\left(\frac{\zeta_\eta(s)\widetilde{\varphi}(s+k)}{(s)_k}; \omega\right)\\ &+ \frac{1}{(k-1)!}\sum_{\substack{j=0\\ -j\in W\setminus D_\eta}}^{k-1} \binom{k-1}{j}(-1)^j\zeta_\eta(-j)\widetilde{\varphi}(k-j)\\ &+ \sum_{\substack{\alpha\in W\setminus D_\eta\\ \alpha\notin\{0,\dots,k-1\}}} \text{res}\left(\frac{\zeta_\eta(s)\widetilde{\varphi}(s+k)}{(s)_k}; \alpha\right) + \langle R_\eta^{[k]}, \varphi\rangle,\end{aligned}$$

where $R_\eta^{[k]}$ is the distribution given by

$$\langle R_\eta^{[k]}, \varphi\rangle = \frac{1}{2\pi i}\int\limits_S \zeta_\eta(s)\widetilde{\varphi}(s+k)\,\frac{ds}{(s)_k}.$$

Proof is analogous that of Theorems 4.12 and 4.20 from [2].

After stating this theorem we can formulate two new theorems, where we will find explicit formulas for p and q geometric partition functions (for a generalized fractal string η), (3) and (4). These results can be considered as an extension of results given in [2] for geometric partition function. They give the distributions $\left\langle P^{[0]}\eta, q(x)e^{-\tau x}\right\rangle$ and $\left\langle P^{[0]}\eta, p(x)e^{-\tau x}\right\rangle$ of an action $P^{[0]}\eta$, on test functions $\varphi_{\tau,q}$ and $\varphi_{\tau,p}$

We begin with the following statement.

Theorem 3. *Let η be a generalized fractal string satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and let $\varphi_{\tau,q}$ be a test function given by (6). Then q geometric partition function $\theta_{\eta,q}(\tau)$ is given by*

$$\begin{aligned}\theta_{\eta,q}(\tau) = &\sum_{\omega\in D_\eta(W)} \operatorname{res}\big(\zeta_\eta(s)\widetilde{\varphi}_{\tau,q}(s);\omega\big)\\ &+\frac{1}{4\cdot 3^{1/4}}\sum_{\substack{k=1\\ 3/4-k\in W\setminus D_\eta}}^{\infty}\frac{(-1)^k\tau^k}{k!}\,\zeta_\eta\left(\frac{3}{4}-k\right){}_1F_1\left(-k;\frac{1}{2};\frac{\pi^2}{12\tau}\right)\\ &+\frac{\pi}{4\cdot 3^{3/4}}\sum_{\substack{l=1\\ 1/4-l\in W\setminus D_\eta}}^{\infty}\frac{(-1)^l\tau^l}{l!}\,\zeta_\eta\left(\frac{1}{4}-l\right){}_1F_1\left(-l;\frac{3}{2};\frac{\pi^2}{12\tau}\right)\\ &+\langle R_\eta^{[0]},\varphi_{\tau,q}\rangle,\end{aligned}\tag{7}$$

where, for $\tau>0$,

$$\begin{aligned}\widetilde{\varphi}_{\tau,q}(s) = \frac{\tau^{1/4-s}}{4\cdot 3^{1/4}}\Bigg\{&\sqrt{\tau}\,\Gamma\left(s-\frac{3}{4}\right){}_1F_1\left(s-\frac{3}{4};\frac{1}{2};\frac{\pi^2}{12\tau}\right)\\ &+\frac{\pi}{\sqrt{3}}\,\Gamma\left(s-\frac{1}{4}\right){}_1F_1\left(s-\frac{1}{4};\frac{3}{2};\frac{\pi^2}{12\tau}\right)\Bigg\}\end{aligned}\tag{8}$$

and

$$\langle R_\eta^{[0]},\varphi_{\tau,q}\rangle = \frac{1}{2\pi i}\int_S \zeta_\eta(s)\widetilde{\varphi}_{\tau,q}(s)\,ds.$$

Proof. Let us begin with the first term of (7). By Theorem 2 we must calculate the Mellin transform of our test function $\varphi_{\tau,q}$, and sum over residues of poles of zeta-function. To obtain the second and the third terms in (7) we will use (4) and Theorem 2. First we calculate the Mellin transform of the function $\varphi_{\tau,q}$

$$\widetilde{\varphi}_{\tau,q}(s) = \int_0^\infty \varphi_{\tau,q}(x) x^{s-1}\, dx.$$

For this purpose we will take expression of $q(x)$ given by (1) and insert it into (5), so we need to calculate such an integral

$$\widetilde{\varphi}_{\tau,q}(s) = \frac{1}{4 \cdot 3^{1/4}} \int_0^\infty e^{\pi\sqrt{x/3} - \tau x} x^{s-7/4}\, dx.$$

After integration we obtain the expression (8) with conditions $\sigma > 3/4$, $\tau > 0$. By Theorem 2 we must calculate residues of that function, but we must take only those poles which are not dimensions of zeta-function. It is easily seen that the first term in (8) has poles at $3/4 - 1$, $3/4 - 2$, ... and the second at $1/4 - 1$, $1/4 - 2$, $\ldots$. It is well known that gamma function has poles with residues, for $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\operatorname*{Res}_{s=-m} \Gamma(s) &= \operatorname*{Res}_{s=3/4-m} \Gamma\left(s - \frac{3}{4}\right) \\ &= \operatorname*{Res}_{s=1/4-m} \Gamma\left(s - \frac{1}{4}\right) = \frac{(-1)^m}{m!}.\end{aligned}$$

Now we can decompose (8) into two terms and calculate residues for each of them. After that we just sum the residues and obtain the second and the third terms in (7). The last term is the same form as in Theorem 2. This term is called the error term, and for it the growth conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are required, because only then we can choose such a window W, for which the error term is absolutely convergent. □

Theorem 4. *Let η be a generalized fractal string satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and let $\varphi_{\tau,p}$ be a test function given by (5). Then p geometric partition function*

$\theta_{\eta,p}(\tau)$ is given by

$$\begin{aligned}\theta_{\eta,p}(\tau) &= \sum_{\omega\in D_\eta(W)} \operatorname{res}\big(\zeta_\eta(s)\widetilde{\varphi}_{\tau,p}(s);\omega\big)\\ &+ \frac{1}{4\sqrt{3}} \sum_{\substack{k=1\\ 1-k\in W\setminus D_\eta}}^{\infty} \frac{(-1)^k\tau^k}{k!}\,\zeta_\eta(1-k)\,{}_1F_1\left(-k;\frac{1}{2};\frac{\pi^2}{6\tau}\right)\\ &+ \frac{\pi}{2\sqrt{6}} \sum_{\substack{l=1\\ 1/2-l\in W\setminus D_\eta}}^{\infty} \frac{(-1)^l\tau^l}{l!}\,\zeta_\eta\left(\frac{1}{2}-l\right){}_1F_1\left(-l;\frac{3}{2};\frac{\pi^2}{6\tau}\right)\\ &+\langle R_\eta^{[0]},\varphi_{\tau,p}\rangle,\end{aligned} \tag{9}$$

where, for $\tau>0$,

$$\begin{aligned}\widetilde{\varphi}_{t,p}(s) = \frac{\tau^{1/2-s}}{4\sqrt{3}}\Bigg\{&\sqrt{t}\,\Gamma(s-1)\,{}_1F_1\left(s-1;\frac{1}{2};\frac{\pi^2}{6t}\right)\\ &+\sqrt{\frac{2}{3}}\pi\,\Gamma\left(s-\frac{1}{2}\right){}_1F_1\left(s-\frac{1}{2};\frac{3}{2};\frac{\pi^2}{6\tau}\right)\Bigg\}\end{aligned}$$

and

$$\langle R_\eta^{[0]},\varphi_{\tau,p}\rangle = \frac{1}{2\pi i}\int\limits_S \zeta_\eta(s)\widetilde{\varphi}_{\tau,p}(s)\,ds.$$

Proof. The proof is similar to the previous theorem. We must calculate the integral

$$\widetilde{\varphi}_{t,p}(s) = \frac{1}{4\sqrt{3}}\int\limits_0^\infty e^{\pi\sqrt{2x/3}-tx}x^{s-2}\,dx.$$

and repeat the same steps as in the proof of Theorem 3 we obtain a result. □

The last unanswered question is can we relate obtained distributions for different test functions. The answer is positive, and now we will find these relationships.

Theorem 5. *Let η be a generalized fractal string. Let $\tau > 0$ and $k \in \mathbb{N}$, then there exist the following relations between test functions $\varphi_\tau(x)$, $\varphi_{\tau,q}(x)$ and $\varphi_{\tau,p}(x)$:*

$$1^\circ\ \left\langle P^{[k]}\eta, \varphi_{\tau,q}(x)\right\rangle = \frac{1}{4\cdot 3^{1/4}}\left\langle P^{[k]}\eta, \varphi_\tau(x)e^{\pi\sqrt{x/3}}\right\rangle;$$
$$2^\circ\ \left\langle P^{[k]}\eta, \varphi_{\tau,p}(x)\right\rangle = \frac{1}{4\sqrt{3}}\left\langle P^{[k]}\eta, \varphi_\tau(x)e^{\pi\sqrt{2x/3}}\right\rangle;$$
$$3^\circ\ \left\langle P^{[k]}\eta, \varphi_{\tau,q}(x)\right\rangle = \left\langle P^{[k]}\eta, \varphi_{\tau,p}(x)(3x)^{1/4}e^{\pi\sqrt{x/3}(1-\sqrt{2})}\right\rangle;$$
$$4^\circ\ \left\langle P^{[k]}\eta, \varphi_{\tau,p}(x)\right\rangle = \left\langle P^{[k]}\eta, \varphi_{\tau,q}(x)(3x)^{-1/4}e^{-\pi\sqrt{x/3}(1-\sqrt{2})}\right\rangle.$$

Proof. First of all we will find relations between test functions, and later we will fit it to the distribution formulas.

We have that $\varphi_\tau(x) = e^{-\tau x}$. First two equalities are trivial, so we will give details for the third and the fourth, only.

From (5) we find

$$\begin{aligned}\ln\varphi_{\tau,p}(x) &= \ln\left(\frac{e^{\pi\sqrt{2x/3}-\tau x}}{4\sqrt{3}x}\right) = \pi\sqrt{\frac{2x}{3}} - \tau x - \ln\left(4\sqrt{3}x\right)\\ &= \pi\sqrt{\frac{2x}{3}} - \tau x - \ln 4 - \frac{1}{2}\ln 3 - \ln x.\end{aligned}$$

Similarly, from (6) we find

$$\begin{aligned}\ln\varphi_{\tau,q}(x) &= \ln\left(\frac{e^{\pi\sqrt{x/3}-\tau x}}{4\cdot 3^{1/4}x^{3/4}}\right) = \pi\sqrt{\frac{x}{3}} - \tau x - \ln 4 - \frac{1}{4}\ln 3 - \frac{3}{4}\ln x\\ &= \left(\pi\sqrt{\frac{2x}{3}} - \tau x - \ln 4 - \frac{1}{2}\ln 3 - \ln x\right)\\ &\quad - \pi\sqrt{\frac{2x}{3}} + \pi\sqrt{\frac{x}{3}} + \frac{1}{4}\ln 3 + \frac{1}{4}\ln x\\ &= \ln\varphi_{\tau,p} + \pi\sqrt{\frac{x}{3}}\left(1-\sqrt{2}\right) + \ln(3x)^{1/4}.\end{aligned}$$

Finally, we can relate $\varphi_{\tau,q}(x)$ and $\varphi_{\tau,p}(x)$ test functions

$$\varphi_{\tau,q}(x) = \varphi_{\tau,p}(x)(3x)^{1/4}e^{\pi\sqrt{x/3}(1-\sqrt{2})}.$$

Analogically

$$
\begin{aligned}
\ln \varphi_{\tau,p}(x) = & \left(\pi\sqrt{\frac{x}{3}} - \tau x - \ln 4 - \frac{1}{4}\ln 3 - \frac{3}{4}\ln x\right) \\
& - \pi\sqrt{\frac{x}{3}} + \pi\sqrt{\frac{2x}{3}} - \frac{1}{4}\ln 3 - \frac{1}{4}\ln x \\
& = \ln \varphi_{\tau,q} - \pi\sqrt{\frac{x}{3}}\left(1-\sqrt{2}\right) - \ln(3x)^{1/4},
\end{aligned}
$$

after what we get

$$
\varphi_{\tau,p}(x) = \varphi_{\tau,q}(x)(3x)^{-1/4}e^{-\pi\sqrt{x/3}(1-\sqrt{2})}.
$$

This completes the proof. $\square$

Alternatively we can define modified p and q geometric partition functions

$$
\begin{aligned}
\theta^*_{L,q} &= \sum_{j=1}^{\infty} e^{-\tau q(l_j^{-1})}, \quad \tau > 0, \\
\theta^*_{L,p} &= \sum_{j=1}^{\infty} e^{-\tau p(l_j^{-1})}, \quad \tau > 0,
\end{aligned}
$$

and investigate its distribution of poles, but in this case calculations are becoming more and more tricky, as we must calculate integral on $(0,\infty)$ of the function

$$
x^{s-1}\exp\left(-\frac{\tau}{4\sqrt{3}x}\exp\left(\pi\sqrt{2x/3}\right)\right).
$$

We will leave it to the future.

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