Cauchy Problem for Equations with Fractional Differentiation Bessel Operator in the Space of Temperate Distributions

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Abstract. The well-posedness of the Cauchy problem, mentioned in title, is studied. The main result means that the solution of this problem is usual C^∞- function on the space argument, if the initial function is a real functional on the conjugate space to the space, containing the fundamental solution of the corresponding problem. The basic tool for the proof is the functional analysis technique.

Keywords: Cauchy problem, Bessel fractional differentiation, functional, Fourier transform.

1 Spaces of Based and Generalized Functions

Let $\mathbf{R}^n$ be a n-dimensional Euclidean space, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ are its elements, $(x, y) = x_1 y_1 + ... + x_n y_n$ is a scalar product on $\mathbf{R}^n$, $\|x\| = (x, x)^{1/2}$, and $C^\infty(\mathbf{R}^n)$ is a space of all infinite differentiable functions, which are described on $\mathbf{R}^n$. For arbitrary $\alpha > 0$ and $\beta > 0$ we assign

$$S_\alpha^\beta = \Big\{\varphi \in C^\infty(\mathbf{R}^n) | \exists\, c > 0,\ \exists\, A > 0,\ \exists\, B > 0,\ \forall\,\{k; q\} \subset \mathbf{Z}_+^n,\\ \forall\, x \in \mathbf{R}^n :\ \big|x^q D_x^k \varphi(x)\big| \le c A^{|q|} B^{|k|} k^{\beta k} q^{\alpha q}\Big\}, \tag{1}$$

where $\mathbf{Z}_+^n$ is a set, which consists of n-dimensioned vectors. Coordinates of these vectors are non-negative integer numbers; $D_x^m = \dfrac{\partial^{|m|}}{\partial x_1^{m_1} ... \partial x_n^{m_n}}$, $|m| = m_1 + ... + m_n$, $m^{\gamma m} = m_1^{\gamma m_1} \cdots m_n^{\gamma m_n}$ for $m \in \mathbf{Z}_+^n$ and $\gamma > 0$.

Space S_α^β is non-trivial for $\alpha + \beta \geq 1$ and consists of such functions $\varphi \in C^\infty(\mathbf{R}^n)$, which satisfy inequality

$$\left|D_x^k \varphi(x)\right| \leq c A^{|k|} k^{\beta k} e^{-\delta \|x\|^{1/\alpha}}, \quad k \in \mathbf{Z}_+^n,\ x \in \mathbf{R}^n, \tag{2}$$

with some positive constants c, A and δ, which depend only on φ [1].

The following additional assertions are valid.

Lemma 1. $\forall\, \{\varphi; \psi\} \subset S_\alpha^\beta$:

$$J_r(\xi) \equiv \int\limits_{\|x\| \leq r} \varphi(x)\psi(x+\xi)dx \underset{r \to +\infty}{\overset{S_\alpha^\beta}{\to}} \int\limits_{\mathbf{R}^n} \varphi(x)\psi(x+\xi)dx \equiv J(\xi), \quad \xi \in \mathbf{R}^n$$

(here $\varphi_\nu \underset{\nu \to \nu_0}{\overset{\Phi}{\to}} \varphi_{\nu_0}$ is a limit in a sense of space Φ topology).

Proof. According to definition of limit in the space S_α^β (see [1]), it is enough to verify the validity of the following conditions:

$$\text{I)}\ \forall k \in \mathbf{Z}_+^n:\ \left|D_\xi^k (J_r(\xi) - J(\xi))\right| \underset{r \to +\infty}{\overset{\xi \in \mathbf{K} \subset \mathbf{R}^n}{\rightrightarrows}} 0$$

i.e. uniformly on $\xi \in \mathbf{K}$ tends to zero on every compact $\mathbf{K}$ from $\mathbf{R}^n$;

II) $\exists\, c > 0,\ \forall\, A > 0,\ \exists\, B > 0,\ \forall\, \{q; k\} \subset \mathbf{Z}_+^n,\ \forall r > 0,\ \forall \xi \in \mathbf{R}^n$:
$\left|\xi^q D_\xi^k J_r(\xi)\right| \leq c A^{|k|} B^{|q|} k^{\beta k} q^{\alpha q}$.

Taking into consideration that $\{\varphi; \psi\} \subset S_\alpha^\beta$ and inequality

$$\left| \int\limits_{\mathbf{R}^n} \varphi(x) D_\xi^k \psi(x+\xi) dx \right| \leq c \int\limits_{\mathbf{R}^n} |\varphi(x)| dx < +\infty, \quad k \in \mathbf{Z}_+^n,\ \xi \in \mathbf{R}^n,$$

we obtain that

$$\left|D_\xi^k \big(J_r(\xi) - J(\xi)\big)\right| \leq \int\limits_{\|x\| > r} |\varphi(x)| \left|D_\xi^k \psi(x+\xi)\right| dx$$

$$\leq \int\limits_{\|x\| > r} |\varphi(x)| dx \underset{r \to +\infty}{\overset{\xi \in \mathbf{K}}{\rightrightarrows}} 0, \quad k \in \mathbf{Z}_+^n,\ \mathbf{K} \subset \mathbf{R}^n.$$

Thus, the first condition is valid.

The second condition also is valid as long as

$$\forall\{\varphi;\psi\}\subset S_{\alpha}^{\beta},\ \forall\{k;q\}\subset \mathbf{Z}_{+}^{n},\ \forall\xi\in\mathbf{R}^{n},\ \forall r>0:$$

$$\begin{aligned}\left|\xi^{q}D_{\xi}^{k}J_{r}(\xi)\right| &\leq \sum_{l=0}^{q}C_{q}^{l}\Big(\int\limits_{\mathbf{R}^{n}}|x|^{l}|\varphi(x)|\left|(x+\xi)^{q-l}D_{x+\xi}^{k}\psi(x+\xi)\right|dx\Big)\\ &\leq cc_{1}B^{|k|}k^{\beta k}\int\limits_{\mathbf{R}^{n}}e^{-\frac{\delta}{2}\|x\|^{1/\alpha}}dx\sum_{l=0}^{q}C_{q}^{l}A^{|q-l|}(q-l)^{\alpha(q-l)}\\ &\quad\times\sup_{t>0}\left\{t^{l}e^{-t^{1/\alpha}}\right\}\\ &\leq c_{2}A_{1}^{|q|}B^{|k|}k^{\beta k}q^{\alpha q},\end{aligned}$$

where C_m^n is a binomial coefficient; c_2, A_1, B are positive constants, which are independent on k, q, r and ξ; c, A, B, δ are constants from corresponding estimations of type (1), (2) for functions φ and ψ.

Lemma 1 is proved. □

Lemma 2. *If* $\{\varphi;\psi\}\subset S_{\alpha}^{\beta}$, *and* $K(r)=\{x\in\mathbf{R}^{n} \mid \|x\|\leq r\}$, $r>0$, *then for all* $\xi\in\mathbf{R}^{n}$ *function* $\varphi(\cdot)\psi(\cdot+\xi)$ *is integrated on the set* $K(r)$ *in the sense of space* S_{α}^{β} *topology.*

Proof. Let p be an arbitrary fixed partition of the set $K(r)$ and let us consider upper

$$J^{*}(p,\xi)=\sum_{i}\varphi(x_{i}^{*})\psi(x_{i}^{*}+\xi)\Delta x_{i},\quad \xi\in\mathbf{R}^{n}$$

and lower

$$J^{**}(p,\xi)=\sum_{i}\varphi(x_{i}^{**})\psi(x_{i}^{**}+\xi)\Delta x_{i},\quad \xi\in\mathbf{R}^{n}$$

integral Darboux sums. Then for proving of the Lemma 2 it is enough to show that

$$\left|J^{*}(p,\xi)-J^{**}(p,\xi)\right|\underset{d\to 0}{\overset{S_{\alpha}^{\beta}}{\longrightarrow}}0$$

(here d is a maximal diameter of the elements of p-partition of the set $K(r)$); i.e. that

$$\text{I)}\ \forall\, k \in \mathbf{Z}^n_+ : \ \left| D^k_\xi \big(J^*(p,\xi) - J^{**}(p,\xi)\big)\right| \underset{d\to 0}{\overset{\xi\in\mathbf{K}\subset\mathbf{R}^n}{\rightrightarrows}} 0;$$

$$\text{II)}\ \exists\, c>0,\ \exists\, A>0,\ \exists B>0,\ \forall\,\{k;q\}\subset \mathbf{Z}^n_+,\ \forall\, p\ \forall\,\xi\in\mathbf{R}^n :$$
$$\left|\xi^q D^k_\xi \big(J^*(p,\xi) - J^{**}(p,\xi)\big)\right| \le cA^{|q|}B^{|k|}k^{\beta k}q^{\alpha q}.$$

Condition 1 is valid. Really, according to the Lagrange intermediate value theorem (on finite increases) and $\{\varphi;\psi\}\subset S^\beta_\alpha$ we obtain that

$$\Big|\sum_i D^k_\xi\big(\varphi(x^{**}_i)\psi(x^{**}_i+\xi) - \varphi(x^*_i)\psi(x^*_i+\xi)\big)\Delta x_i\Big|$$
$$\le \sum_i c_i \|x^{**}_i - x^*_i\|\Delta x_i \underset{d\to 0}{\to} 0,$$

where c_i are positive constants, which do not depend on $\xi\in\mathbf{R}^n$, and $k\in\mathbf{Z}^n_+$.

Let us verify the validity of the condition 11. Since $\{\varphi;\psi\}\subset S^\beta_\alpha$, then for these functions condition (1) is valid with corresponding constants c_i, A_i, B_i, $i\in\{1;2\}$. Thus,

$$\forall\,\{k;q\}\subset\mathbf{Z}^n_+,\ \forall\,\xi\in\mathbf{R}^n,\ \forall\, p :$$
$$\left|\xi^q D^k_\xi J^*(p,\xi)\right|$$
$$\le \sum_i \Big(\sum_{j=0}^{q} C^j_q \big|(x^*_i)^{q-j}\varphi(x^*_i)\big|\big|(x^*_i+\xi)^j D^k_{(x^*_i+\xi)}\varphi(x^*_i+\xi)\big|\Big)\Delta x_i \qquad (3)$$
$$\le \sum_i \Big(\sum_{j=0}^{q} C^j_q c_1 A_1^{|q-j|}(q-j)^{\alpha(q-j)} c_2 A_2^{|j|}B_2^{|k|}k^{\beta k}j^{\alpha j}\Big)\Delta x_i$$
$$\le cV_r A^{|q|}B_2^{|k|}q^{\alpha q}k^{\beta k},$$

where $V_r = \mathrm{mes} K(r)$, and c, A, B_2 are positive constants, they do not depend on k, q, ξ and partition p.

Similarly it is possible to show that inequality (3) (with the same constants) is valid for $J^{**}(p,\cdot)$.

Taking into consideration that

$$\left|\xi^q D^k_\xi\big(J^*(p,\xi) - J^{**}(p,\xi)\big)\right| \le \left|\xi^q D^k_\xi J^*(p,\xi)\right| + \left|\xi^q D^k_\xi J^{**}(p,\xi)\right|,$$
$$\xi\in\mathbf{R}^n,\ \{k;q\}\subset\mathbf{Z}^n_+,$$

we obtain the second condition. Q.E.D.

From mentioned above we obtain the following assertion.

Theorem 1. *For all* φ *and* ψ *from* S_α^β, *and for* $\xi \in \mathbf{R}^n$, *function* $\varphi(\cdot)\psi(\cdot+\xi)$ *is integrable on* $\mathbf{R}^n$ *in the sense of space* S_α^β *topology.*

Further, let $(S_\alpha^\beta)'$ be a space, which is topologically conjugated with S_α^β.

Fourier transform of generalized function $f \in (S_\alpha^\beta)'$ and convolution $(\varphi * f)$ of this function with functional of function φ type from S_α^β we define by the following equalities [1]:

$$\big\langle F[f], F[\psi]\big\rangle = (2\pi)^n \langle f, \psi\rangle,$$
$$\langle \varphi * f, \psi\rangle = \langle f, \varphi * \psi\rangle, \quad \psi \in S_\alpha^\beta.$$

From abovementioned we obtain that

$$\forall\, \varphi \in S_\alpha^\beta,\ \forall\, f \in (S_\alpha^\beta)': \quad F[\varphi * f] = F[f]F[\varphi]. \tag{4}$$

Definition 1. Functional f from $(S_\alpha^\beta)'$ is called a real-valued functional, if $\overline{\langle f, \varphi\rangle} = \langle f, \varphi\rangle$ for all $\varphi \in S_\alpha^\beta$ (here $\bar{v}$ is a number, which is complex conjugated to v).

The following assertion is valid.

Theorem 2. *Let* f *be a real-valued functional from* $(S_\alpha^\beta)'$, *and* $\varphi \in S_\alpha^\beta$. *Then:*

1) $\big\langle f, \varphi(\cdot+\xi)\big\rangle \in C^\infty(\mathbf{R}^n)$;

2) $\varphi * f = \big\langle f, \varphi(\cdot+\xi)\big\rangle$.

Proof. Assertion 1) of this theorem is evident. Really, in the space S_α^β shift operation is continuous and infinite differentiable [1].

Let us prove assertion 2). Since $\big\langle f, \varphi(\cdot+\xi)\big\rangle \in C^\infty(\mathbf{R}^n)$ (see assertion 1) of this theorem), then

$$\forall\, \psi \in S_\alpha^\beta: \ \Big\langle \big\langle f, \varphi(\cdot+\xi)\big\rangle, \psi\Big\rangle = \int\limits_{\mathbf{R}^n} \big\langle f, \overline{\overline{\psi(x)}\varphi(x+\xi)}\big\rangle dx.$$

Hence, according to Theorem 1, taking linearity and continuity of functional f lnto account, we obtain, that

$$\Big\langle\langle f,\varphi(\cdot+\xi)\rangle,\psi\Big\rangle = \overline{\Big\langle f,\int\limits_{\mathbf{R}^n}\overline{\psi(x)}\varphi(x+\xi)dx\Big\rangle}$$
$$= \langle f,\varphi*\psi\rangle$$
$$= \langle\varphi*f,\psi\rangle,\quad \psi\in S_\alpha^\beta.$$

Theorem 2 is proved. □

2 Cauchy Problem

Let $a>0$, E be a unit operator, Δ is n-dimensioned Laplace operator. Note that fractional degree $\gamma>0$ of operator $(aE-\Delta)^{\frac{1}{2}}$ is called pseudo differential Bessel operator of γ-order with positive parameter a [2] (denote it as $\widehat{B}_a^\gamma$). In [3] it is obtained that $\widehat{B}_a^\gamma:(S_\alpha^\beta)'\to(S_\alpha^\beta)'$, and

$$\forall f\in(S_\alpha^\beta)':\ \widehat{B}_a^\gamma f=j_a^\gamma*f,\quad \gamma\neq 2k,\quad k\in\mathbf{N},$$

where $\mathbf{N}$ is a set of natural numbers, j_a^γ is a regularisator in the space $(S_\alpha^\beta)'$ of Bessel kernel with positive parameter and negative order (detailed see in [3]).

Note that according to the assertion of the Theorem 4 from [3]

$$\langle\widehat{B}_a^\gamma f,\varphi\rangle=\Big\langle f,F^{-1}\big[(a+\xi^2)^{\gamma/2}F[\varphi]\big]\Big\rangle,$$
$$f\in(S_\alpha^\beta)',\quad \varphi\in S_\alpha^\beta,\quad \alpha\geq 1,\quad \beta>0,$$

where $\gamma>0$ and $\gamma\neq 2k$, $k\in\mathbf{N}$, and F^{-1} is inverse Fourier transform.

Let us consider equation

$$\frac{\partial u(t,x)}{\partial t}=\big(P(t,\widehat{B}_a^\alpha)u\big)(t,x),\quad (t,x)\in\Omega\equiv(0;+\infty)\times\mathbf{R}^n,\qquad (5)$$

where $P(t,\widehat{B}_a^\alpha)u=\sum\limits_{j=0}^{m}b_j(t)\widehat{B}_{a_j}^{\alpha_j}u$, $m\in\mathbf{N}$, $a_j>0$, $\alpha_j>0$ and $\alpha_j\neq 2k$, $k\in\mathbf{N}$, $j\in\{0;1;...;m\}$, and $b_j(\cdot)$ are continuous, defined on $(0;+\infty)$, bounded on module, complex-valued functions. Let us assume that only one maximal number exists among $\alpha_j, j \in \{0; 1; ...; m\}$. Denote this number as α_l. Let function $b_l(\cdot)$ be so that

$$\exists\,\widehat{\delta} > 0,\ \forall\, t > 0: \quad \mathrm{Re}\, b_l(t) \le -\widehat{\delta}.$$

From these assumptions for polynomial $P\Big(t, \big(a + \|\cdot\|^2\big)^{\frac{\alpha}{2}}\Big)$, $t > 0$, from equation (5) (further we denote it as $P(t, \cdot)$) we obtain that the following condition is valid

$$\begin{aligned} &\exists\,\delta^* > 0,\ \exists\, d \ge 1,\ \forall\, t > 0,\ \forall\, x \in \mathbf{R}^n,\ \|x\| > d: \\ &\mathrm{Re} P(t, x) \le -\delta^* (a + x^2)^{\frac{\gamma}{2}} \end{aligned} \tag{6}$$

(here and further $a \equiv a_l, \gamma \equiv \alpha_l$).

Let $\theta_t(\cdot) = \exp\left\{ \int\limits_0^t P(\tau, \cdot) d\tau \right\}$, $t > 0$. The following additional assertions characterize the properties of this function.

Lemma 3. $\forall\, t > 0: \quad \theta_t(\cdot) \in S^1_{\frac{1}{\gamma}}$.

Proof. We analyze the proof scheme in case of $n = 2$ (to simplify the calculations). With the help of mathematical induction method this scheme can be applied for arbitrary natural $n > 2$.

Function $\theta_t(\cdot), t > 0$ is infinite differentiable. Therefore it is enough to show that

$$\begin{aligned} &\exists\,\delta_1 > 0,\ \exists\,\delta_2 > 0,\ \exists\, c > 0,\ \exists\, A > 0,\ \forall\, k \in \mathbf{Z}^2_+,\ \forall\, x \in \mathbf{R}^2,\ \forall\, t > 0: \\ &\left| D^k_x \theta_t(x) \right| \le c e^{\delta_2 t} (\widehat{t} A)^{|k|} k^k e^{-\delta_1 t \|x\|^\gamma}, \end{aligned} \tag{7}$$

where

$$\widehat{t} \equiv \begin{cases} t, & t \ge 1, \\ 1, & 0 < t < 1. \end{cases}$$

According to the Faa de Bruno formula of composite function differentiation, we obtain

$$\begin{aligned} D^{k_1}_x f\big(\varphi(x)\big) = &\sum_{p_1}^{k_1} \frac{k_1!}{q_1! j_1! ... h_1!} \frac{d^{p_1} f(\varphi)}{d\varphi^{p_1}} \left(\frac{d\varphi(x)}{1! dx} \right)^{q_1} \\ &\times \left(\frac{d^2 \varphi(x)}{2! dx^2} \right)^{j_1} ... \left(\frac{d^{L_1} \varphi(x)}{L_1! dx^{L_1}} \right)^{h_1}, \ x \in \mathbf{R},\ k_1 \in \mathbf{Z}_+ \end{aligned}$$

(here summation symbol extends to all solutions in integer non-negative numbers of the equation $k_1 = q_1 + 2j_1 + ... + L_1 h_1$, and number $p_1 = q_1 + j_1 + ... + h_1$). Hence we obtain that

$$\forall\, k \in \mathbf{Z}_+^2 :$$

$$\left|D_x^k \theta_t(x)\right| \le \sum_{p_1}^{k_1} \frac{k_1!}{q_1! j_1! ... h_1!} \sum_{j=0}^{k_2} C_{k_2}^j \left|D_{x_2}^j \widehat{P}(t,x)\right| \left|D_{x_2}^{k_2-j} \theta_t(x)\right|, \tag{8}$$

where

$$\widehat{P}(t,x) \equiv \left(\int_0^t \frac{\partial P(\tau,x)}{1! \partial x_1} d\tau \right)^{q_1} \left(\int_0^t \frac{\partial^2 P(\tau,x)}{2! \partial x_1^2} d\tau \right)^{j_1} \times \cdots$$

$$\times \left(\int_0^t \frac{\partial^{L_1} P(\tau,x)}{L_1! \partial x_1^{L_1}} d\tau \right)^{h_1}, \quad t > 0,\ x \in \mathbf{R}^2.$$

With the help of Faa de Bruno formula we obtain that

$$\forall\, r \in \mathbf{Z}_+ :$$

$$\left|D_{x_2}^r \theta_t(x)\right| \le \sum_{p_2}^{r} \frac{r!}{q_2! j_2! ... h_2!} \left|\theta_t(x)\right| \left(\int_0^t \left| \frac{\partial P(\tau,x)}{1! \partial x_2} \right| d\tau \right)^{q_2} \times \cdots$$

$$\times \left(\int_0^t \left| \frac{\partial^{L_2} P(\tau,x)}{L_2! \partial x_2^{L_2}} \right| d\tau \right)^{h_2}, \quad x \in \mathbf{R}^2,\ t > 0,$$

where $p_2 = q_2 + j_2 + ... + h_2$, and $r = q_2 + 2j_2 + ... + L_2 h_2$.

Since

$$\int_0^t \left| \frac{\partial^\nu P(\tau,x)}{\nu! \partial x_2^\nu} \right| d\tau \le tYA^\nu (a^* + x^2)^{\frac{\gamma}{2}},$$

where A is a positive constant, which does not depend on $t > 0$, $\nu \in \mathbf{Z}_+$ and $x \in \mathbf{R}^2$, $Y \equiv \sum_{j=0}^{m} \max_{t>0} \left\{ \left|b_j(t)\right| \right\}$, $a^* \equiv \max_{0 \le j \le m} \{a_j\}$.

Taking into consideration (6) and the following inequalities:

$$\frac{p_2!}{q_2! j_2! ... h_2!} \le 2^r, \quad \sum_{p_2}^{r} 1 \le (2e)^r$$

(here $p_2 = q_2 + j_2 + ... + h_2$, and $r = q_2 + 2j_2 + ... + L_2 h_2$) we obtain that

$$\left|D^{\tau}_{x_2}\theta_t(x)\right| \leq c_1 e^{\delta_1 t}(\widehat{t}A_1)^{|\tau|} r! e^{-\rho/2t\|x\|^{\gamma}}, \tag{9}$$

where c_1, δ_1, A_1 are positive constants, that do not depend on $x \in \mathbf{R}^2, t > 0$ and $r \in \mathbf{Z}_+$, $\rho = \min\{\delta^*, \widehat{\delta}\}$.

For $\bar{a} > 0$, $\{r; L\} \subset \mathbf{Z}_+$ and $x \in \mathbf{R}^2$

$$\begin{aligned}\left|\frac{\partial^{L+r}(\bar{a}+x^2)^{\alpha/2}}{L!r!\partial x_1^L \partial x_2^r}\right| &= \left|\sum_{p_1}^{L}\frac{2^{i_1}x_1^{i_1}}{i_1!j_1!}\sum_{p_2}^{r}\frac{2^{i_2}x_2^{i_2}}{i_2!j_2!}\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\times ...\right. \\ &\quad \left.\times\left(\frac{\alpha}{2}-(p_1+p_2)+1\right)(\bar{a}+x^2)^{\frac{\alpha}{2}-(p_1+p_2)}\right| \\ &\leq c_r A_2^{L+r}(\bar{a}+x^2)^{\frac{\alpha}{2}}\sum_{p_1}^{L}\frac{p_1!}{i_1!j_1!}\sum_{p_2}^{r}\frac{p_2!}{i_2!j_2!} \\ &\leq c_2 A_3^{L+r}(\bar{a}+x^2)^{\frac{\alpha}{2}},\end{aligned} \tag{10}$$

where c_2, A_3 are positive constants, that do not depend on r, L and x, and $p_k = i_k + j_k$, $k \in \{1; 2\}$, $L = i_1 + 2j_1$, $r = i_2 + 2j_2$. Therefore

$$\left|\int_0^t \frac{\partial^{L+r}P(\tau,x)}{L!r!\partial x_1^L \partial x_2^r}d\tau\right| \leq tc_4 A_4^{L+r}(a+x^2)^{\frac{\gamma}{2}}$$

(here constants $c_4 > 0$, $A_4 > 0$ independent of $t > 0, x \in \mathbf{R}^2$ and $\{L; r\} \subset \mathbf{Z}_+$), then

$$\forall\{\nu; L; h\} \subset \mathbf{Z}_+,\ \forall t > 0,\ \forall x \in \mathbf{R}^2:$$

$$\begin{aligned}&\left|D^{\nu}_{x_2}\left(\left(\int_0^t \frac{\partial^L P(\tau,x)}{L!\partial x_1^L}d\tau\right)^h\right)\right| \\ &\quad \leq \sum_{p}^{\nu}\frac{\nu!}{i!j!...\mu!}\begin{pmatrix}\frac{h!}{(h-p)!}\left|\int_0^t \frac{\partial^L P(\tau,x)}{L!\partial x_1^L}d\tau\right|^{h-p}, & h \geq p, \\ 0, & h < p\end{pmatrix} \\ &\quad \times\left|\int_0^t \frac{\partial^{L+1}P(\tau,x)}{L!\partial x_1^L \partial x_2}d\tau\right|^q \times ... \times \left|\int_0^t \frac{\partial^{L+s}P(\tau,x)}{L!s!\partial x_1^L \partial x_2^s}d\tau\right|^{\mu} \\ &\quad \leq \nu! A_5^{\nu}(tc_6 A_6^L(a^*+x^2)^{\frac{\gamma}{2}})^h,\end{aligned}$$

$$p = i + j + ... + \mu, \quad \nu = i + 2j + ... + s\mu,$$

where A_5, c_6, A_6 are positive constants, independent of t, x, ν, L and h.

Further,

$$\begin{aligned}
&\left|D^j_{x_2}\widehat{P}(t,x)\right| \\
&\quad \le \sum_{\nu_1=0}^{j} C_j^{\nu_1}\left|D_{x_2}^{j-\nu_1}\left(\left(\int_0^t \frac{\partial P(\tau,x)}{\partial x_1}d\tau\right)^{q_1}\right)\right| \\
&\qquad \times \sum_{\nu_2=0}^{\nu_1} C_{\nu_1}^{\nu_2}\left|D_{x_2}^{\nu_1-\nu_2}\left(\left(\int_0^t \frac{\partial^2 P(\tau,x)}{2!\partial x_1^2}d\tau\right)^{j_1}\right)\right| \times \cdots \\
&\qquad \times \sum_{\nu_{L_1-1}=0}^{\nu_{L_1-2}} C_{\nu_{L_1-2}}^{\nu_{L_1-1}}\left|D_{x_2}^{\nu_{L_1-2}-\nu_{L_1-1}}\left(\left(\int_0^t \frac{\partial^{L_1-1} P(\tau,x)}{(L_1-1)!\partial x_1^{L_1-1}}d\tau\right)^{\mu_1}\right)\right| \qquad (11) \\
&\qquad \times \left|D_{x_2}^{\nu_{L_1-1}}\left(\left(\int_0^t \frac{\partial^{L_1} P(\tau,x)}{L_1!\partial x_1^{L_1}}d\tau\right)^{h_1}\right)\right| \\
&\quad \le j!A_5^j A_6^{k_1}\left(tc_6(a^*+x^2)^{\gamma/2}\right)^{p_1}\left(\sum_{\nu_1=0}^{j}\sum_{\nu_2=0}^{\nu_1}\cdots\sum_{\nu_{L_1-1}=0}^{\nu_{L_1-2}} 1\right),
\end{aligned}$$

$j \in \mathbf{Z}_+,\ t > 0,\ x \in \mathbf{R}^2.$

Taking into consideration inequalities (8), (9) and the following inequality

$$\sum_{\nu_1=0}^{j}\sum_{\nu_2=0}^{\nu_1}\cdots\sum_{\nu_{L_1-1}=0}^{\nu_{L_1-2}} 1 \le \frac{(j+L_1-1)^{L_1-1}}{(L_1-1)!} \le e^{j+L_1-1},$$

we obtain estimation (7). Q.E.D.

Lemma 4. $\forall\varphi \in S^1_{\frac{1}{\gamma}}:\ \theta_t(\cdot)\varphi(\cdot) \underset{t\to+0}{\overset{S^1_{1/\gamma}}{\to}} \varphi(\cdot).$

Proof. It is enough to obtain that the following conditions are valid:

I) $\forall k \in \mathbf{Z}^n_+ : D^k_x\big(\theta_t(x)\varphi(x)\big) \underset{t\to+0}{\overset{x\in\mathbf{K}\subset\mathbf{R}^n}{\rightrightarrows}} D^k_x\varphi(x);$

II) $\exists\delta_1 > 0,\ \exists c_1 > 0,\ \exists A_1 > 0,\ \forall t \in (0;1),\ \forall k \in \mathbf{Z}^n_+,\ \forall x \in \mathbf{R}^n :$

$$\left|D^k_x\big(\theta_t(x)\varphi(x)\big)\right| \le c_1 A_1^{|k|}k^k e^{-\delta_1\|x\|^\gamma}.$$

Note that

$$D_x^k\big(\theta_t(x)\varphi(x)\big) = \theta_t(x)D_x^k\varphi(x) + \sum_{|j|=1}^{|k|} C_k^j D_x^j\theta_t(x)D_x^{k-j}\varphi(x),$$

$k \in \mathbf{Z}_+^n,\ x \in \mathbf{R}^n.$

Since for every compact set $\mathbf{K}$ from $\mathbf{R}^n$

$$D_x^j\theta_t(x)D_x^{k-j}\varphi(x) \underset{t\to+0}{\to} 0, \quad \theta_t(x) \underset{t\to+0}{\to} 1$$

uniformly on $x \in \mathbf{K}$ for all $|j| \in \{1; 2; ...; |k|\}$, then condition 1) is valid.

Let us prove the validity of condition 11) for $n = 2$. Since $\varphi \in S^1_{\frac{1}{\gamma}}$, then $\exists\,\delta_0 > 0,\ \exists\,c_0 > 0,\ \exists\,A_0 > 0,\ \forall\,k \in \mathbf{Z}_+^2,\ \forall\,x \in \mathbf{R}^2 : \big|D_x^k\varphi(x)\big| \leq c_0A_0^{|k|}k^k e^{-\delta_0\|x\|^\gamma}$.

Hence, taking inequality (7) into consideration, we obtain that

$$\Big|D_x^k\big(\theta_t(x)\varphi(x)\big)\Big| \leq \sum_{|j|=0}^{|k|} C_k^j\big|D_x^j\theta_t(x)\big|\big|D_x^{k-j}\varphi(x)\big| \leq c_2A_2^{|k|}k^k e^{-\delta_0\|x\|^\gamma},$$

where c_2, A_2, δ_0 are positive constants, that do not depend on $k \in \mathbf{Z}_+^2$, $x \in \mathbf{R}^2$ and $t \in (0; 1)$. Thus for $n = 2$ condition 11) is valid.

In case of $n > 2, n \in \mathbf{N}$ with the help of mathematical induction method the validity of condition 11) is proved analogously. Lemma 4 is proved. □

Lemma 5. *Function $\theta_t(\cdot)$ is differentiable on $t > 0$ in the sense of space $S^1_{\frac{1}{\gamma}}$ topology.*

Proof. It is enough to show that limit relation

$$\Phi_{\Delta t}(x) \equiv \frac{1}{\Delta t}\big[\theta_{(t+\Delta t)}(x) - \theta_t(x)\big] \underset{\Delta t\to 0}{\to} P(t,x)\theta_t(x)$$

is valid in the following sence:

I) $\forall\,k \in \mathbf{Z}_+^n,\ \forall\,t > 0 : D_x^k\Phi_{\Delta t}(x) \overset{x\in\mathbf{K}\subset\mathbf{R}^n}{\underset{\Delta t\to 0}{\rightrightarrows}} D_x^k\big(P(t,x)\theta_t(x)\big);$

II) $\exists c_3 > 0,\ \exists\,A_3 > 0,\ \exists\delta_3 > 0,\ \forall\,k \in \mathbf{Z}_+^n,\ \forall\,x \in \mathbf{R}^n, \forall\,t > 0,$
$\forall\,\Delta t \in (-1; 1), t + \Delta t > 0 :$
$\big|D_x^k\Phi_{\Delta t}(x)\big| \leq c_3A_3^{|k|}k^k e^{-\delta_3\|x\|^\gamma}.$

Function $\theta_t(\cdot), t > 0$ is differentiable on t in ordinary sense, therefore

$$\Phi_{\Delta t}(x) = P(t+\eta\Delta t, x)\theta_{(t+\eta\Delta t)}(x), \quad t+\eta\Delta t > 0,\ 0 < \eta < 1,\ x \in \mathbf{R}^n.$$

Thus,

$$D_x^k \Phi_{\Delta t}(x) = \sum_{|j|=0}^{|k|} C_k^j D_x^j P(t+\eta\Delta t, x) D_x^{k-j} \theta_{(t+\eta\Delta t)}(x), \tag{12}$$
$$t+\eta\Delta t > 0,\ 0 < \eta < 1,\ x \in \mathbf{R}^n,\ k \in \mathbf{Z}^n.$$

Since

$$D_x^j P(t+\eta\Delta t, x) D_x^{k-j} \theta_{(t+\eta\Delta t)}(x) \underset{\Delta t \to 0}{\overset{x\in \mathbf{K}\subset \mathbf{R}^n}{\rightrightarrows}} D_x^j P(t, x) D_x^{k-j}\theta_t(x),$$

then from (12) condition 1) is obtained.

The validity of condition 1l) follows from (12) and inequalities of type (7), (10), taking into consideration, that functions $b_j(\cdot)$, $j \in \{0; 1; ...; m\}$ are bounded by module at $(0; +\infty)$.

Lemma 5 is proved. □

The following corollary is valid, taking into account that operator F^{-1} is continuous in the space S_α^β [1].

Corollary 1. $\forall\, t > 0$:

$$F^{-1}\left[\frac{\partial}{\partial t}\theta_t(\cdot)\right] = \frac{\partial}{\partial t} F^{-1}[\theta_t(\cdot)].$$

If for equation (5) the following initial condition is given

$$u(t,\cdot)|_{t=0} = f, \quad f \in (S_\alpha^\beta)', \tag{13}$$

then the solution of Cauchy problem (5), (13) in the space $(S_\alpha^\beta)'$ is called such function u from this space, that

$$\forall\ \varphi \in S_\alpha^\beta : \quad \left\langle \frac{\partial u}{\partial t} - P(t, \widehat{B}_a^\alpha)u, \varphi \right\rangle \equiv 0,$$

i.e. this function satisfies equation (5) in weak sense and initial condition (13) in the sense that $u(t,\cdot) \underset{t\to+0}{\overset{(S_\alpha^\beta)'}{\longrightarrow}} f$.

Let $G_t(\cdot) = F^{-1}\left[\theta_t(x)\right](\cdot)$, $t > 0$. From Lemma 3 we obtain that for arbitrary fixed $t > 0$ function $G_t(\cdot)$ belongs to the space $S_1^{\frac{1}{\gamma}}$.

The following assertion is valid.

Theorem 3. *If f from $(S_1^{\frac{1}{\gamma}})'$ is a real-valued functional, then for Cauchy problem* (5), (13) *in the space $(S_1^{\frac{1}{\gamma}})'$ there exists a solution, that is unique, differentiable on t, infinite differentiable on x in ordinary sense, and it satisfies the following conditions:*

$$1)\ F\left[\frac{\partial}{\partial t}u\right] = \frac{\partial}{\partial t}F[u], t > 0;$$
$$2)\ u(t,x) = G_t(x) * f, \quad (t,x) \in \Omega.$$

Proof. Suppose, that for solutions of equation (5) in the space $(S_1^{\frac{1}{\gamma}})'$ the condition 1) of this theorem is valid. Since

$$\forall\, \varphi \in S_1^{\frac{1}{\gamma}},\ \forall t > 0 :\ \langle P(t, \widehat{B}_a^{\alpha})u, \varphi\rangle = (2\pi)^n \langle P(t,\xi)\tilde{u}, \tilde{\varphi}\rangle,$$

(here and further $\tilde{v} \equiv F[v]$), then equation (5) in the space $(S_1^{\frac{1}{\gamma}})'$ is equivalent to equation

$$\frac{\partial \tilde{u}}{\partial t} = P(t,\cdot)\tilde{u}, \quad t > 0 \tag{14}$$

in the space $(S_{\frac{1}{\gamma}}^1)'$. The initial condition (13) is valid if and only if

$$\tilde{u}(t,\cdot) \underset{t\to+0}{\overset{(S_{\frac{1}{\gamma}}^1)'}{\longrightarrow}} \tilde{f}. \tag{15}$$

Thus, question on correct solvability in the space $(S_1^{\frac{1}{\gamma}})'$ of the Cauchy problem (5), (13) is equivalent to the question on correct solvability in the space $(S_{\frac{1}{\gamma}}^1)'$ of the Cauchy problem (14), (15).

Note, that (14) is a differential equation with separable variables. Its general solution is

$$\tilde{u}(t,\cdot) = C(\cdot)\theta_t(\cdot), \quad t > 0. \tag{16}$$

Taking into consideration assertion of Lemma 4 and condition (15), we obtain from (16) that $\tilde{u}(t,\cdot) = \tilde{f}\theta_t(\cdot), t > 0$ is a solution of Cauchy problem (14), (15) in the space $(S^1_{\frac{1}{\gamma}})'$. Uniqueness of this solutions is proved by contradiction.

According to Lemma 3 and equality (4) we obtain that

$$u(t,\cdot) = G_t(\cdot) * f, \quad f \in (S_1^{\frac{1}{\gamma}})', \quad t > 0.$$

Since f is a real-valued functional, then from assertion 2) of Theorem 2 and from Lemma 5 we obtain that

$$\frac{\partial u(t,\cdot)}{\partial t} = \Big(\frac{\partial}{\partial t} G_t(\cdot)\Big) * f, \quad f \in (S_1^{\frac{1}{\gamma}})', \quad t > 0.$$

From here, taking into consideration that f is real-valued functional from $(S_1^{\frac{1}{\gamma}})'$, and from Corollary 1 and equality (4), we obtain that

$$\begin{aligned} &\forall t > 0: \\ &F\Big[\frac{\partial}{\partial t}u\Big] = F[f]F\Big[\frac{\partial}{\partial t}G_t(\cdot)\Big] = F[f]\frac{\partial}{\partial t}\Big(F\big[G_t(\cdot)\big]\Big) \\ &\qquad = \frac{\partial}{\partial t}\Big(F\big[G_t(\cdot) * f\big]\Big) = \frac{\partial}{\partial t}F[u]. \end{aligned}$$

Thus, condition 1) of this theorem is valid for the solution of the Cauchy problem (5), (13) in the space $(S_1^{\frac{1}{\gamma}})'$.

It follows from Theorem 2 and Lemma 5 that the solution of the Cauchy problem (5), (13) is differentiable on t and infinite differentiable on x in ordinary sense.

Theorem 3 is proved. □

Note that if f from $(S_1^{\frac{1}{\gamma}})'$ is a convolutor in the space $S_1^{\frac{1}{\gamma}}$, then corresponding solution of the Cauchy problem (5), (13) is an element of $S_1^{\frac{1}{\gamma}}$ for arbitrary fixed $t > 0$. This solution satisfies equation (5) in ordinary sense, if instead of operator $\widehat{B}_a^{\alpha}$ we consider its narrowing on the space S (here S is a Schwartz space [1]).

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