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On Some Properties of the Omega-Operator, Defined on Class of Analytic in the Half-Plane Functions

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Abstract. In present paper the properties of the operator introduced by authors, which is defined on special class of n-normalized analytic in the half-plane functions are investigated. This operator is closely related with automorphism of a half-plane. Close connection of this operator with divided difference of n-th order is shown. The fixed points of the operator were found. Some other invariants, related with operator are under consideration.

Keywords: analytic functions, automorphism, divided difference.

1 Major notational conventions, definitions and auxiliary statements

Let Π is a half-plane $\operatorname{Re} z > 0$, $A_n(\Pi)$ – class of analytical in Π functions $F(z)$ with condition $F^{(n)}(z) \neq 0,\ \forall z \in \Pi$. $\widetilde{A}_n(\Pi)$ – class of analytical in Π functions $F(z)$ from $A_n(\Pi)$, which are normalized by conditions:

$$F(1) = F'(1) = \ldots = F^{(n-1)}(1) = 0, \quad F^{(n)}(1) = n!\,. \tag{1}$$

Obviously, that for any fixed $m \geq 2$ every function $F(z)$ of $\widetilde{A}_n(\Pi)$ can be represented in form

$$F(z) = (z-1)^n + \sum_{k=2}^{m} a_{k,n}(z-1)^{n+k-1} + \Psi_m(z),$$

where $\Psi_m(z)$ – dependent on $F(z)$ analytical in Π function. Number $a_{k,n} = \dfrac{F^{(n+k-1)}(1)}{(n+k-1)!}$ we call by k-th coefficient of function $F(z)$.

Let us introduce the operator

$$N_n[F]=\frac{F(z)-F(1)-F'(1)(z-1)-\ldots-\dfrac{1}{(n-1)!}F^{(n-1)}(1)(z-1)^{n-1}}{\dfrac{1}{n!}F^{(n)}(1)}$$

which we call by normalizing operator. This operator transfers any function from $A_n(\Pi)$ to a function of class $\widetilde{A}_n(\Pi)$. For $n = 0$ we set that $N_0[F] = F(z)/F(1)$. It is obviously that:

1. $N_n[cF + P] = N_n[F]$, where $c \neq 0$ and P is a polynomial of the degree no higher than $n-1$;
2. $N_n\big[N_n[F]\big] = N_n[F]$.

Denote by $A(D)$ class of analytical in D functions. The n-th divided difference of function $F(z) \in A(D)$ define (see [1, 2]) by formula

$$\big[F(z); z_0, \ldots, z_n\big] = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{F(\xi)d\xi}{(\xi-z_0)\ldots(\xi-z_n)}, \tag{2}$$

where Γ is a simple closed contour, located in D and covering all the points $z_0, \ldots, z_n \in D$. In formula (2) among the points $z_0, \ldots, z_n \in D$ may occur coincident. For pairwise different points $z_0, \ldots, z_n \in D$ for n-th order divided difference, the formula

$$\big[F(z); z_0, \ldots, z_n\big] = \sum_{m=0}^{n}\frac{F(z_m)}{\eta_n'(z_m)}, \tag{3}$$

where

$$\eta_n(z) = \prod_{p=0}^{n}(z-z_p)$$

is valid ([1, 2]).

For arbitrarily fixed points $z_0, \dots, z_n \in D$ divided difference $[F(z); z_0, \dots, z_n]$ represents a linear functional, defined on class $A(D)$. Note, that

$$\big[P(z); z_0, \dots, z_n\big] = 0, \quad \forall z_0, \dots, z_n \in D,$$

if $P(z)$ is a polynomial of the degree no higher than $n-1$. The following statement is valid ([3]):

Lemma 1. *If* $\big[F(z); z_0, \dots, z_n\big] \neq 0$, *for pairwise different* $z_0, \dots, z_n \in D$, *then* $\big[F(z); z_0, \dots, z_n\big] \neq 0$ *for all* $z_0, \dots, z_n \in D$ *(i.e among points* $z_0, \dots,$ $z_n \in D$ *may occur coincident). In particular, for* $z_0 = z_1 = \dots = z_n = \zeta$ *relationship*

$$[F(z); z_0, \dots, z_n] = \frac{1}{n!} F^{(n)}(\zeta) \neq 0, \quad \forall \zeta \in D$$

holds true.

Note, that if $n = 1$ and $[F(z); z_0, z_1] \neq 0$ for all distinct $z_0, z_1 \in D$, then

$$[F(z); z_0, z_1] = \frac{F(z_0) - F(z_1)}{z_0 - z_1} \neq 0, \quad \forall z_0, z_1 \in D,$$

hence $F(z_0) \neq F(z_1)$, $\forall z_0, z_1 \in D$, so we get a class of univalent in the domain D functions ([4, 5]).

Denote by $K_n(D)$ class of analytical in D functions $F(z)$, such that $\big[F(z); z_0, \dots, z_n\big] \neq 0$ for all pairwise different $z_0, \dots, z_n \in D$. For $n = 1$, as it was shown above, class $K_1(D)$ is a class of univalent in D functions, which play a large role in conformal mapping theory and in geometrical theory of analytical functions ([4, 5]). From Lemma 1 and definition of class $K_n(D)$ we get

Lemma 2. *If* $F(z) \in K_n(D)$, *then* $F^{(n)}(z) \neq 0$ *for all* $z \in D$. *Taking into account elementary properties* [1] *and definition of class* $K_n(D)$, *we will get the following*

Lemma 3. *If* $F(z) \in K_n(D)$, *then*

$$cF(z) + P(z) \in K_n(D),$$

where $c \neq 0$, $P(z)$ *is a polynomial of the degree no higher than* $n-1$.

It is obvious, that statement formulated above stay true in case, when domain D is a half-plane Π. Using Lemma 3 we arrive at conclusion, that in class $K_n(\Pi)$ we can evolve a subclass $\widetilde{K}_n(\Pi)$ of normalized functions, which satisfy the condition (1).

We need a theorem, having also an independent interest.

Theorem 1. *Let $F(\xi) \in A(D_0)$ and linear-fractional function*

$$\xi = \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

maps domain D onto domain D_0. Let also

$$\xi_k = \frac{az_k+b}{cz_k+d}, \quad z_k \in D, \quad \xi_k \in D_0.$$

Then

$$\begin{aligned}&\big[F(\xi);\xi_0,\ldots,\xi_n\big]\\ &\qquad= (ad-bc)^{-n}\prod_{k=0}^{n}(cz_k+d)\left[(cz_k+d)^{n-1}F\left(\frac{az+b}{cz+d}\right);z_0,\ldots,z_n\right].\end{aligned}$$

Proof. For derivative of function $\eta_n(z)$ we have an expression

$$\eta_n'(z) = (ad-bc)^{-n}\prod_{k=0}^{n}(cz_k+d)(cz_m+d)^{n-1}\eta_n'(\xi_m). \tag{4}$$

Using formula (3) for pairwise different $\xi_0,\ldots,\xi_n \in D_0$ we will get

$$\begin{aligned}\big[F(\xi);\xi_0,\ldots,\xi_n\big] &= \sum_{m=0}^{n}\frac{F(\xi_m)}{\eta_n'(\xi_m)}\\ &= (ad-bc)^{-n}\prod_{k=0}^{n}(cz_k+d)\sum_{m=0}^{n}\frac{(cz_m+d)^{n-1}F\left(\dfrac{az_m+b}{cz_m+d}\right)}{\eta_n'(z_m)}\\ &= (ad-bc)^{-n}\prod_{k=0}^{n}(cz_k+d)\left[(cz+d)^{n-1}F\left(\frac{az+b}{cz+d}\right);z_0,\ldots,z_n\right].\end{aligned}$$

Having realized limit process we ascertain, that Theorem 1 holds true in case, when there exists coincident points among $\xi_0,\ldots,\xi_n \in D_0$. $\square$

Let L is a set of all linear functions of shape $w = tz,\ t > 0$. For any fixed $t > 0$ function $w = tz$ univalently maps half-plane Π onto itself. Theorem 1 implies

Corollary 1. *Let $w = tz \in L$ and $w_k = tz_k$, $k = 0, 1, \dots, z_n$. Then*

$$\big[F(w); w_0, \dots, w_n\big] = t^{-n}\big[F(tz); z_0, \dots, z_n\big].$$

Let us arbitrarily choose $w \in L$ and introduce omega-operator of n-th order by formula

$$\Omega_n^w[F] = \frac{(z-1)^n\big[F(z); w(z), \overbrace{t, \dots, t}^{n}\,\big]}{\dfrac{1}{n!}F^{(n)}(t)}.$$

This operator for any fixed $w = tz \in L$ is defined on class $A_n(\Pi)$ and transfers every function of class $A_n(\Pi)$ to the normalized function of class $\widetilde{A}_n(\Pi)$.

Remark 1. *Let $w = tz \in L$ is arbitrarily fixed. Then operator Ω_n^w transfers function $F(z)$ of class $\widetilde{A}_n(\Pi)$ onto function of the same class, moreover this is one-to-one transfer.*

Remark 2. *Basing on the Lemmas 2 and 3 we conclude, that $K_n(\Pi) \subset A_n(\Pi)$. Applying Corollary 1 we will get, that operator Ω_n^w is defined on class $K_n(\Pi)$ and for any fixed $w = tz \in L$ transfers every function of $K_n(\Pi)$ to the normalized function of $\widetilde{K}_n(\Pi)$.*

On the ground of Remarks 1 and 2 we will call the classes $\widetilde{A}_n(\Pi)$ and $\widetilde{K}_n(\Pi)$ by linearly invariant classes. In case when domain is unit disk similar linearly invariant classes were considered in papers [6]–[10].

2 The following theorem demonstrates the close connection between operators Ω_n^w and N_n

Theorem 2. *For arbitrarily fixed $w = tz \in L$ the equality*

$$\Omega_n^w\big[F(z)\big] = N_n\big[F(tz)\big]$$

holds true.

Proof. Let us represent normalization operator for function $F(tz)$ in the form

$$N_n\big[F(tz)\big] = \frac{F(tz) - P(z;t)}{\dfrac{1}{n!}F^{(n)}(t)t^n},$$

where $P(z;t)$ is a polynomial:

$$\begin{aligned} P(z;t) = F(t) + \frac{1}{1!}F(t)t(z-1) + \ldots \\ + \frac{1}{(n-1)!}F^{(n-1)}(t)t^{n-1}(z-1)^{n-1}. \end{aligned} \tag{5}$$

Let $w_k = tz_k$, and $z_k = 1$ for any integer k from the interval $[1, n]$. Then $w_k = t$ for any integer k from the interval $[1, n]$. Using Corollary 1, we will get

$$\big[F(w); w, \overbrace{t, \ldots, t}^{n}\,\big] = t^{-1}\big[F(tz); z, \overbrace{1, \ldots, 1}^{n}\,\big].$$

Note, that

$$\big[F(w); w, \overbrace{t, \ldots, t}^{n}\,\big] = \big[F(z); w, \overbrace{t, \ldots, t}^{n}\,\big].$$

By using properties of normalization operator and taking into account elementary properties of n-th divided difference ([1, 2]), we obtain

$$\big[F(z); w, \overbrace{t, \ldots, t}^{n}\,\big] = \frac{1}{n!}F^{(n)}(t)\Big[N_n\big[F(tz)\big]; z, \overbrace{1, \ldots, 1}^{n}\,\Big],$$

and so

$$\frac{\big[F(z); w, \overbrace{t, \ldots, t}^{n}\,\big]}{\dfrac{1}{n!}f^{(n)}(1)} = \Big[N_n\big[F(tz)\big]; z, \overbrace{1, \ldots, 1}^{n}\,\Big] = (z-1)^{-n}N_n\big[F(tz)\big].$$

Multiplying both sides of last equation by $(z-1)^n$, we will get statement of Theorem 2. $\square$

3 Let us find the fixed functions of operator Ω_n^w

It is clear, that function $F(tz)$ is representable in the form

$$F(tz) = P(z;t) + \frac{F^{(n)}(t)}{n!} t^n (z-1)^n + \frac{F^{(n+1)}(t)}{(n+1)!} t^{n+1} (z-1)^{n+1} + \Psi(z;t),$$

where $\Psi(z;t)$ is analytical in Π function with parameter t, and $P(z;t)$ is a polynomial (5). If $F(z) \in \widetilde{A}_n(\Pi)$, then

$$\Omega_n^w\big[F(z)\big] = N_n\big[F(tz)\big] = (z-1)^n + a_{2,n}(t)(z-1)^{n+1} + \varphi(z;t) \in \widetilde{A}_n(\Pi),$$

where

$$a_{2,n}(t) = \frac{tF^{(n+1)}(t)}{(n+1)F^{(n)}(t)} \quad \text{and} \quad \varphi(z;t) = \frac{\psi(z;t)}{\frac{1}{n!}F^{(n)}(t)t^n}. \tag{6}$$

Let s be certain complex number and $s \notin \{0, 1, 2, \dots, n-1\}$. Represent function $z^s \in A_n(\Pi)$ (assume $1^s = 1$ for any complex s) in the form

$$z^s = P_s(z) + \sum_{k=1}^{m} \frac{1}{(n+k-1)!} s(s-1)\dots \\ \big(s-(n+k-2)\big)(z-1)^{n+k-1} + \Psi_{s,m}(z),$$

where

$$P_s(z) = 1 + \frac{1}{1!}s(z-1) + \frac{1}{2!}s(s-1)(z-1)^2 + \dots \\ + \frac{1}{(n-1)!}s(s-1)\dots\big(s-(n-2)\big)(z-1)^{n-1}$$

and $\Psi_{s,m}(z)$ is an analytical in Π function.

Theorem 3. *Only function $\Phi_{n,a}(z) = N_n[z^s]$, where*

$$s = (n+1)a + n, \quad a = \frac{\Phi_{n,a}^{(n+1)}(1)}{(n+1)!},$$

is a fixed point of operator Ω_n^w for any $w \in L$, i.e.

$$\Omega_n^w[\Phi_{n,a}] = \Phi_{n,a}, \quad \forall w \in L. \tag{7}$$

This function belongs to class $\widetilde{A}_n(\Pi)$.

Proof. Assume, that there exists function $F_*(z)$, satisfying condition

$$\Omega_n^w[F_*] = F_*, \quad \forall w \in L. \tag{8}$$

Represent function $F_*(z)$ in the form

$$F_*(z) = (z-1)^n + a(z-1)^{n+1} + \Psi(z),$$

where $\Psi(z)$ is analytical in Π function. Further

$$\Omega_n^w\big[F_*(z)\big] = (z-1)^n + a_{2,n}(t)(z-1)^{n+1} + \Psi(z;t).$$

Requirement (8) implies that $a_{2n}(t) = a, \ \forall t > 0$. Using formula (6), we obtain

$$\frac{tF_*^{(n+1)}(t)}{(n+1)F_*^{(n)}(t)} = a.$$

Consequently $F_*^{(n)}(t) = n!t^{(n+1)a}, \ \forall t > 0$. Since $F_*^{(n)}(t)$ is an analytical in Π function, then, using principle of analytical extension, we will get

$$F_*^{(n)}(z) = n!z^{(n+1)a}. \tag{9}$$

Integrating (9) n times, we obtain

$$F_*(z) = cz^s + P(z),$$

where c – nonzero coefficient, and $P(z)$ is a polynomial of the degree no higher than $n-1$. Moreover, function $F_*(z)$ must be normalized by conditions (1). Thus, if certain function is satisfying condition (8), it must be of form

$$N_n[F_*] = N_n[cz^s + P] = N_n[z^s].$$

Prove now, that any function $\Phi_{n,a}(z) = N_n[z^s]$, which we will call by *main*, is satisfying condition (7). Basing on properties of normalization operator, we will get

$$\begin{aligned}\Omega_n^w\big[\Phi_{n,a}(z)\big] &= N_n\big[\Phi_{n,a}(tz)\big] = N_n\Big[N_n\big[(tz)^s\big]\Big] \\ &= N_n\big[(tz)^s\big] = N_n[t^s z^s] = N_n[z^s] = \Phi_{n,a}(z).\end{aligned}$$

Since $\Phi^{(n)}(z) = n!z^{(n+1)a} \neq 0$ for any $z \in \Pi$, then $\Phi_{n,a} \in \widetilde{A}_n(\Pi)$. The theorem is proved. $\square$

Remark 3. *Main function $\Phi_{n,a}(z)$ can be represented in form*

$$\Phi_{n,a}(z) = (z-1)^n + \sum_{k=2}^{m} c_{k,n}(z-1)^{n+k-1} + \Psi_{s,m}(z),$$

where

$$c_{k,n} = \frac{n!}{(n+k-1)!}(n+1)a\big((n+1)a-1\big)\dots\big((n+1)a-(k-2)\big)$$

and $\Psi_{s,m}(z)$ – analytical in Π function.

4 Let us raise a question on invariant coefficients

Let k-th coefficient of certain function $F(z)$ of class $\widetilde{A}_n(\Pi)$ is equal to number b_k, i.e. $a_{k,n} = b_k$, where $k \geq 2$. If number b_k is the k-th coefficient of function $F(z;t) = \Omega_n^w\big[F(z)\big]$ for any $w \in L$, then we will call this coefficient by invariant (fixed) coefficient of function $F(z)$. It follows from Theorem 3 that every coefficient of main function $\Phi_{n,a}(z)$ is an invariant one. The below theorem allows to establish all of the functions with a previously fixed k-th coefficient.

Theorem 4. *Let equation*

$$\frac{n!}{(n+k-1)!}\prod_{m=0}^{k-2}\big((n+1)a-m\big) = b_k \tag{10}$$

with respect to a has $k-1$ pairwise different roots $a_1,\dots,a_{k-1}$. Then only function of form

$$F(z) = \sum_{m=1}^{k-1} c_k\Phi_{n,a_k}(z), \quad c_1+\dots+c_{k-1} = 1, \tag{11}$$

has number b_k as its k-th invariant coefficient.

Proof. With a purpose of finding functions with fixed k-th coefficient we need to solve differential equation

$$\frac{n!t^{k-1}F^{(n+k-1)}(t)}{(n+k-1)!F^{(n)}(t)} = b_k.$$

Replacing real variable t by complex one z, convert last equation to linear homogeneous differential equation

$$n!z^{k-1}F^{(n+k-1)}(z) - (n+k-1)!b_kF^{(n)}(z) = 0 \tag{12}$$

of $(n+k-1)$-th order with respect to $F(z)$. Equation (12) is easily converted to well-known Euler's differential equation with respect to function $\varphi(z) = F^{(n)}(z)$. Since numbers $a_1, \ldots, a_{k-1}$ are roots of equation (10). Then it easily to check up that every function

$$1, z, \ldots, z^{n-1}, \Phi_{n,a_1}(z), \ldots, \Phi_{n,a_{k-1}}(z) \tag{13}$$

satisfies the equation (12). These functions make up the fundamental system of solutions of equation (12). Any linear combination of functions (13) satisfies the equation (12). Since functions representable by such linear combination satisfies the normalization condition (1), then functions $1, z, \ldots, z^{n-1}$ are omitted, so only functions $\Phi_{n,a_1}(z), \ldots, \Phi_{n,a_{k-1}}(z)$ will remain. Taking into account normalization conditions, we will get only linear combinations (11). The proof is complete. □

Corollary 2. *If in Theorem 4 we assume that $b_k = 0$, then only for function of form*

$$F(z) = \sum_{m=0}^{k-1} c_m\Phi_{n,a_m}(z), \quad \textit{where} \quad \sum_{m=0}^{k-1} c_m = 1 \quad \textit{and} \quad a_m = \frac{m}{n+1},$$

number $b_k = 0$ is its invariant coefficient.

Remark 4. *Roots $a_i,\ i = 1, \ldots, k-1$ of equation* (10) *are the second coefficients of expansion of corresponding functions $\Phi_{n,a_i}(z),\ i=1, \ldots, k-1$, about the point $z = 1$, i.e.*

$$\Phi_{n,a_i}(z) = (z-1)^n + a_i(z-1)^{n+1} + \ldots, \quad i = 1, \ldots, k-1.$$

Remark 5. *Let us investigate separately equation* (10)*, which we can rewrite in the form*

$$\chi(\lambda) - c = 0, \tag{14}$$

where

$$c = \frac{b_k(n+k-1)!}{n!}, \quad \lambda = (n+1)a, \quad \chi(\lambda) = \prod_{m=0}^{k-2} (\lambda - m).$$

Let us denote $\chi_c(\lambda) = \chi(\lambda) - c$. *It is obvious that* $\chi_c(\lambda)$ *is a polynomial of* $(k-1)$*-th order. We can state that any root of polynomial* $\chi_c(\lambda)$ *has the multiplicity no higher than two. Indeed, polynomial* $\chi_0(\lambda)$ *has* $k-1$ *pairwise different roots. According Rolle's theorem derivative* $\chi_0'(\lambda)$ *of such polynomial has* $k-2$ *pairwise different roots, among which there are no roots of polynomial* $\chi_0(\lambda)$. *In other words, every root of polynomial* $\chi_0'(a)$ *is of multiplicity one. Assume that* λ_c *is a root of polynomial* $\chi_c(\lambda)$ *of the multiplicity three. Then polynomial* $\chi_c'(\lambda) \equiv \chi_0'(\lambda)$ *has the root* λ_c *of the multiplicity two that is impossible. Thus, equation* (14) *and then, equation* (10) *have no root with multiplicity greater than two.*

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