A Nonlinear Model for Topsoil Erosion Caused by Heavy Rain

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Abstract. In this paper, a nonlinear mathematical model is proposed and analysed to study the effect of heavy rain on the topsoil erosion and crop-yield. It is shown that as the velocity of rain water along the soil surface increases, the fertile topsoil depth decreases and this depth may be very small if soil is exposed continuously to the stresses generated by heavy rain. A model to conserve the fertile topsoil is also proposed. By analyzing the conservation model it is shown that the economy would follow a sustainable path if suitable efforts are adopted in time.

Keywords: rain, erosion, soil, conservation, stability.

1 Introduction

Soil is a valuable natural resource. Probably the most important use of soil is to grow world's food and fibre. In developing countries like India, where more than 60% people are involved in agricultural related activities, soil erosion is a major cause of concern. The agents of soil erosion are water and wind, each contributing a significant amount of soil loss [1–3]. The loss of soil due to heavy rain from farmland may be reflected in reduced crop production potential, lower surface water quality and damaged drainage networks [4–10].

Some investigations have been conducted to study the causes and consequences of topsoil erosion and the need of afforestation [1, 11, 12], but a little attention has been paid to study these problems using mathematical models

[13–15]. Shukla *et al.* [14] considered a single-sector economic growth model and they investigated the effect of environmental factors such as acid rain and wind on the depth of fertile topsoil and crop yiled. Recently, Dubey [13] proposed a mathematical model to study the effect of high speed wind on the depletion of depth of fertile topsoil by considering a Cobb-Douglas production function which depend upon depreciating capital stock, a labor force and depth of fertile topsoil. But in these investigations effect of heavy rain on the depletion of fertile topsoil depth has not been considered. Keeping these in view, in this paper, a mathematical model is proposed and analysed to study the effect of heavy rain on the depletion of fertile topsoil. A conservation model is also proposed to reduce the erosion of soil.

2 Mathematical model

Consider an agricultural field where we wish to model the erosion of fertile topsoil depth caused by heavy rain. We consider the Cobb-Douglas production function for the crop yield which is governed by the combination of capital stock, the labor force, and environmentally degraded topsoil depth. In such a case, the production process is governed by following factors [16, 17].

Production function. Let $Y(t)$ is the total output or net crop-yield, $K(t)$ the capital stock, $L(t)$ the size of labor force, $S(t)$ the depth of fertile topsoil at time t. Then the crop yield is assumed to follow the Cobb-Douglas production function,

$$Y = K^{\alpha_1} L^{\alpha_2} S^{\alpha_3}, \quad \sum \alpha_i = 1,\ \alpha_i > 0. \tag{1}$$

Capital stock. The dynamics of depreciating capital stock is governed by the following differential equation [17]:

$$\frac{dK}{dt} = aY - bK,$$

where a denotes the fraction of the output used for capital growth and b denotes the depreciating rate coefficient of the capital stock.

Labor force and population growth relation. The supply of labor force in the production process depends on per capita capital stock [17, 18] and its dynamics is governed by

$$L = \frac{P^{\mu+1}}{K^{\mu}}, \quad \mu > 0. \tag{2a}$$

Here $P(t)$ is the size of population at time t. In the case of highly developed economy, the rate of population growth may be constant. However, in the case of less developed economy where population control measures are not very effective, the growth rate of population may be taken as exponential. But, in general, population growth is constrained by various schemes and measures. In such a case dynamics of population would follow the logistic growth:

$$\frac{dP}{dt} = n_0\Big(1 - \frac{P}{P_0}\Big)P. \tag{2b}$$

In the above equation, n_0 is the intrinsic growth rate of population and P_0 is the maximum sustainable population size under the given environmental, ecological and economic constraints.

Depletion of fertile topsoil due to heavy rain. Let $S(t)$ be the depth of fertile topsoil, and $R(t)$ be the density of rain at time t causing the erosion of soil. It is assumed that the growth rate of topsoil depth decreases as the density of rain increases. The decrease is assumed to be proportional to the products $R(t)S(t)$ and $R^2(t)S(t)$, the former interaction being due to laminar flow and the later due to turbulent flow caused by corresponding shearing stresses on the soil surface which are assumed to be proportional to $R(t)$ and $R^2(t)$, respectively [19]. The growth rate of rain is caused by hydraulic pressure gradient $\phi(t)$, which may decrease due to natural factors and due to interaction with soil surface. Then the dynamics of S and R may be governed by the following differential equations:

$$\begin{aligned}
&\frac{dS}{dt} = q - r_1 S - r_2 RS - r_3 R^2 S,\\
&\frac{dR}{dt} = \phi(t) - k_1 R - k_2 RS - k_3 R^2 S,\\
&S(0) = S_0 > 0, \quad R(0) = R_0 > 0.
\end{aligned}$$

Here q is the natural growth rate coefficient of fertile topsoil assumed to be constant, r_1 is the depletion rate coefficient of fertile topsoil depth due to natural factors such as gravitational forces on the slope, r_2 and r_3 are its depletion rate coefficients due to stresses of heavy rain on the soil surface assumed to be proportional to R and R^2 for laminar and turbulent flow respectively. $\phi(t)$ is the hydraulic pressure gradient causing rain which is assumed to be either constant or periodic, k_1 is the natural depletion rate coefficient of rain caused by various resistances, k_2 and k_3 are the depletion rate coefficients of rain velocity due to interaction with soil.

Output capital ratio. The output capital ratio denoted by β is defined as

$$\beta = \frac{Y}{K},$$

which yields

$$\frac{\dot{\beta}}{\beta} = \frac{\dot{Y}}{Y} - \frac{\dot{K}}{K}.$$

From (1) and (2), we get respectively

$$\frac{\dot{Y}}{Y} = \alpha_1 \frac{\dot{K}}{K} + \alpha_2 \frac{\dot{L}}{L} + \alpha_3 \frac{\dot{S}}{S},$$
$$\frac{\dot{L}}{L} = (\mu + 1)\frac{\dot{P}}{P} - \mu \frac{\dot{K}}{K}.$$

A little algebraic manipulation yields

$$\frac{d\beta}{dt} = \beta\left[-a_0 a\beta + \alpha_2 n_0(1+\mu)\left(1 - \frac{P}{P_0}\right) + \alpha_3\left(\frac{q}{S} - r_1 - r_2 R - r_3 R^2\right) + a_0 b\right],$$

where $a_0 = 1 - \alpha_1 + \alpha_2\mu > 0$.

Now we are in a position to write all the equations governing the model system as follows.

$$\begin{aligned}
\frac{d\beta}{dt} &= \beta\Big[- a_0 a\beta + \alpha_2 n_0(1+\mu)\Big(1 - \frac{P}{P_0}\Big) \\
&\qquad + \alpha_3\Big(\frac{q}{S} - r_1 - r_2 R - r_3 R^2\Big) + a_0 b\Big], \\
\frac{dP}{dt} &= n_0\Big(1 - \frac{P}{P_0}\Big)P, \\
\frac{dS}{dt} &= q - r_1 S - r_2 RS - r_3 R^2 S, \\
\frac{dR}{dt} &= \phi(t) - k_1 R - k_2 RS - k_3 R^2 S, \\
&\beta(0) > 0,\ P(0) > 0,\ S(0) > 0,\ R(0) > 0.
\end{aligned} \tag{3}$$

3 Two-dimensional dynamical behavior

We consider two-dimensional subsystem of model (3), and show that there is no closed directory in either of $\beta-P,\ \beta-S,\ \beta-R,\ P-S,\ P-R,\ S-R$ planes. First of all, we consider the following two-dimensional subsystem:

$$\frac{dS}{dt}=q-r_1S-r_2RS-r_3R^2S\equiv h_1(S,R),$$
$$\frac{dR}{dt}=\phi(t)-k_1R-k_2RS-k_3R^2S\equiv h_2(S,R).$$

Let $H(S,R)=\frac{1}{SR}$. We note that $H(S,R)$ is positive in the interior of the $S-R$ plane. Then we have,

$$\Delta(S,R)=\frac{\partial}{\partial S}(h_1H)+\frac{\partial}{\partial R}(h_2H)=-\frac{q}{SR^2}-\frac{1}{S}\Big(\frac{\phi}{R^2}+k_3S\Big)<0.$$

This shows that Δ is non zero and does not change sign in the interior of the positive quadrant of $S-R$ plane. Hence using the Dulac-Bendixon criteria, it follows that there is no closed trajectory in the interior of the positive quadrant of $S-R$ plane. Thus, we can state the following theorem.

Theorem 1. *There is no periodic solution in the interior of the positive quadrant of the $S-R$ plane.*

Similary, one can prove the following theorem.

Theorem 2. *There is no periodic solution in the interior of the positive quadrant of the either of $\beta-P,\ \beta-S,\ \beta-R,\ P-S,\ P-R$ planes.*

4 Mathematical analysis

In this section, we analyse the complete model (3) in two cases, namely, $\phi(t)=\phi_0>0$, and $\phi(t)$ is periodic.

Case 1. $\phi(t)=\phi_0>0$.

In this case, model (3) has four nonnegative equilibria, namely, $E_0(0,0,S^*,R^*)$, $E_1(0,P^*,S^*,R^*)$, $E_2(\beta_1^*,0,S^*,R^*)$, $E^*(\beta^*,P^*,S^*,R^*)$. Here, we have

$P^* = P_0$, $\beta_1^* = \frac{\alpha_2 n_0(1+\mu)+a_0 b}{a_0 a}$, $\beta^* = \frac{b}{a}$, and S^*, R^* are the positive solutions of the following algebraic equations:

$$S = \frac{q}{r_1 + r_2 R + r_3 R^2}, \tag{4a}$$

$$S = \frac{\phi_0 - k_1 R}{k_2 R + k_3 R^2}. \tag{4b}$$

It can easily be checked that the above isoclines (4a) and (4b) intersect at a unique point (S^*, R^*). From the last equation of model (3), it is natural to assume that $\phi_0 > k_1 R$, otherwise dR/dt would be negative.

By computing the variational matrices [20] corresponding to each equilibrium, we note the following results.

1. E_0 is always unstable in the $\beta - P$ plane. It can be checked that E_0 is locally stable in the $S - R$ plane if the following inequality holds:

$$\frac{r_2}{r_3} > \frac{k_2}{k_3}. \tag{5}$$

2. E_1 is a saddle point with stable manifold locally in the P direction and unstable manifold locally in the β direction. If (5) holds, then E_1 has a stable manifold locally in the $S - R$ plane.

3. E_2 is also a saddle point with stable manifold locally in the β direction and with unstable manifold locally in the P direction. If (5) holds, then E_2 has a stable manifold locally in the $S - R$ plane.

The following theorem characterizes the stability behavior of E^*. The proof of this theorem follows from the Routh-Hurwitz criteria and hence omitted.

Theorem 3. *If inequality* (5) *holds, then E^* is locally asymptotically stable in the $\beta - P - S - R$ space.*

It may be pointed out here that inequality (5) is just a sufficient condition for E^* to be locally asymptotically stable. A stronger condition (in fact, a necessary and sufficient condition) is stated in the following theorem.

Theorem 4. *The equilibrium E^* is locally asymptotically stable if and only if $A > 0$, where*

$$\begin{aligned}A &= (r_1 + r_2R^* + r_3R^{*2})(k_1 + k_2S^* + 2k_3R^*S^*)\\ &\quad - (r_2S^* + 2r_3R^*S^*)(k_2R^* + k_3R^{*2})\\ &= r_1k_1 + r_1k_2S^* + 2r_1k_3R^*S^* + r_2k_1R^* + r_3k_1R^{*2}\\ &\quad + (r_2k_3 - r_3k_2)R^{*2}S^*.\end{aligned} \tag{6}$$

The above theorems imply that under certain parametric conditions, the capital output ratio and the fertile topsoil depth settle down at its equilibrium level.

Remark. *It may be noted that Theorem* 3 *is a particular case of Theorem* 4*.*

To study the global stability behavior of the positive equilibrium E^* we need the following lemma whose proof is easy and hence omitted.

Lemma 1. *The set*

$$\Omega = \big\{(\beta, P, S, R) \colon 0 < \beta \le \beta_m,\ 0 < P \le P_0,\ 0 < S \le q/r_1,\ 0 < R \le \phi_0/k_1\big\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where $\beta_m = \frac{1}{a_0 a}\left[\alpha_2 n_0(1+\mu) + a_0 b + \frac{\alpha_3 q}{S_m}\right]$, and S_m is the minimum of S in Ω.

Theorem 5. *Let the following inequality holds:*

$$\begin{aligned}&\left[\frac{c_2 q}{r_1}\left(r_2 + r_3\Big(\frac{\phi_0}{k_1} + R^*\Big)\right) + \frac{c_3\phi_0}{k_1}\Big(k_2 + \frac{k_3\phi_0}{k_1}\Big)\right]^2\\ &\qquad\qquad < c_2c_3(r_1 + r_2R^* + r_3R^{*2})(k_1 + k_2S^*).\end{aligned} \tag{7}$$

Then the positive equilibrium E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant, where c_2 and c_3 are some positive constants chosen suitably as mentioned in the proof of the theorem.

Proof. Consider the following positive definite function about E^*,

$$\begin{aligned}v &= \big(\beta - \beta^* - \beta^*\ln(\beta/\beta^*)\big) + c_1\big(P - P^* - P^*\ln(P/P^*)\big)\\ &\quad + \frac{1}{2}c_2(S - S^*)^2 + \frac{1}{2}c_3(R - R^*)^2,\end{aligned}$$

where $c_i's$ are some positive constants to be chosen suitably.

Now differentiating V with respect to time t along the solutions of model (3), a little algebraic manipulation yields:

$$
\begin{aligned}
\frac{dV}{dt} =& -a_0 a(\beta-\beta^*)^2 - \frac{c_1 n_0}{P_0}(P-P^*)^2 \\
& - c_2(r_1 + r_2 R^* + r_3 R^{*2})(S-S^*)^2 - c_3(k_1 + k_2 S^*)(R-R^*)^2 \\
& + (\beta-\beta^*)(P-P^*)\Big[-\frac{\alpha_2 n_0(1+\mu)}{P_0}\Big] + (\beta-\beta^*)(S-S^*)\Big[-\frac{\alpha_3 q}{SS^*}\Big] \\
& + (\beta-\beta^*)(R-R^*)\big[-\alpha_3 r_2 - \alpha_3 r_3(R+R^*)\big] \\
& + (S-S^*)(R-R^*)\big[-c_2 r_2 S - c_2 r_3 S(R+R^*) - c_3 k_2 R - c_3 k_3 R^2\big] \\
& - c_3 k_3 S^*(R+R^*)(R-R^*)^2.
\end{aligned}
$$

The above expression can further be written as sum of the quadratics

$$
\begin{aligned}
\frac{dV}{dt} =& -\frac{1}{2}a_{11}(\beta-\beta^*)^2 + a_{12}(\beta-\beta^*)(P-P^*) - \frac{1}{2}a_{22}(P-P^*)^2 \\
& - \frac{1}{2}a_{11}(\beta-\beta^*)^2 + a_{13}(\beta-\beta^*)(S-S^*) - \frac{1}{2}a_{33}(S-S^*)^2 \\
& - \frac{1}{2}a_{11}(\beta-\beta^*)^2 + a_{14}(\beta-\beta^*)(R-R^*) - \frac{1}{2}a_{44}(R-R^*)^2 \\
& - \frac{1}{2}a_{33}(S-S^*)^2 + a_{34}(S-S^*)(R-R^*) - \frac{1}{2}a_{44}(R-R^*)^2 \\
& - c_3 k_3 S^*(R+R^*)(R-R^*)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
& a_{11} = \frac{2}{3}a_0 a, \quad a_{22} = \frac{2c_1 n_0}{P_0}, \\
& a_{33} = c_2(r_1 + r_2 R^* + r_3 R^{*2}), \quad a_{44} = c_3(k_1 + k_2 S^*), \\
& a_{12} = -\frac{\alpha_2 n_0(1+\mu)}{P_0}, \quad a_{13} = -\frac{\alpha_3 q}{SS^*}, \\
& a_{14} = -\alpha_3 r_2 - \alpha_3 r_3(R+R^*), \\
& a_{34} = -c_2 r_2 S - c_2 r_3 S(R+R^*) - c_3 k_2 R - c_3 k_3 R^2.
\end{aligned}
$$

Sufficient conditions for dV/dt to be negative definite are that the following inequalities hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{8a}$$
$$a_{13}^2 < a_{11}a_{33}, \tag{8b}$$
$$a_{14}^2 < a_{11}a_{44}, \tag{8c}$$
$$a_{34}^2 < a_{33}a_{44}. \tag{8d}$$

By choosing

$$c_1 > \frac{3n_0\alpha_2^2(1+\mu)^2}{4aa_0P_0},$$
$$c_2 > \frac{3}{2aa_0(r_1+r_2R^*+r_3R^{*2})}\left(\frac{\alpha_3 q}{S_m S^*}\right)^2,$$
$$c_3 > \frac{3}{2aa_0(k_1+k_2S^*)}\left(\alpha_3 r_2 + \alpha_3 r_3\left(\frac{\phi_0}{k_1}+R^*\right)\right)^2,$$

we note that conditions (8a), (8b) and (8c) are automatically satisfied. Further, (7) implies (8d). This shows that V is Liapunov's function [21] with respect to E^*, whose domain contains the region of attraction Ω, proving the theorem. □

This shows that the capital output ratio, population, depth of fertile topsoil and density of rain settle down at steady state under certain parametric conditions. It is also noted that the depth of fertile topsoil decreases as the velocity of rain increases along the surface of soil.

Case 2. $\phi(t) = \phi_0 + \varepsilon\phi_1(t) \quad \phi_1(t+w) = \phi_1(t).$

In this case, the model system (3) can be written as

$$\dot{X} = A(X) + \varepsilon B(t), \quad X(0) = X_0,$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \beta \\ P \\ S \\ R \end{bmatrix}, \quad X_0 = \begin{bmatrix} \beta(0) \\ P(0) \\ S(0) \\ R(0) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \phi_1(t) \end{bmatrix},$$

$$A(X)=\begin{bmatrix} x_1\big[-a_0ax_1+\alpha_2n_0(1+\mu)\big(1-\frac{x_2}{P_0}\big)+\alpha_3\big(\frac{q}{x_3}-r_1-r_2x_4-r_3x_4^2\big)+a_0b\big] \\ n_0(1-\frac{x_2}{P_0})x_2 \\ q-r_1x_3-r_2x_3x_4-r_3x_3x_4^2 \\ \phi_0-k_1x_4-k_2x_3x_4-k_3x_3x_4^2 \end{bmatrix}.$$

Let M^* be the variational matrix corresponding to the positive equilibrium E^*. Then under an analysis similar to [13, 22], one can state the following results.

Theorem 6. *If M^* has no eigenvalue with zero real parts, then system* (3) *with $\phi(t)=\phi_0+\varepsilon\phi_1(t)$, $\phi_1(t+\omega)=\phi_1(t)$ has a periodic solution, $\big(\beta(t,\varepsilon),P(t,\varepsilon),S(t,\varepsilon),R(t,\varepsilon)\big)$, with period ω, such that $\big(\beta(t,0),P(t,0),S(t,0),R(t,0)\big)=(\beta^*,P^*,S^*,R^*)$.*

Theorem 7. *If M^* has no eigenvalue with zero real parts, then for sufficiently small ε the stability behavior of the periodic solution of the system* (3) *is same as that of E^*.*

The above two theorems show that if the hydraulic pressure gradient is periodic, then it causes a periodic behavior in the system.

5 Conservation model

It is well known that water is about 800 times heavier than air, half to one third the weight of rock and about equal in weight to loose the topsoil. When it flows, it can move loose substances. The energy of a moving object is equal to its mass multiplied by its speed squared. As water droplets grow in size, both their mass and speed increase. Thus, the destructive power of rain increases dramatically as the rainstorm produces larger drops. The larger drops of rain are not very common. But when it occurs, its effect is profoundly destructive. Thus, the heavy rain is one of the important natural factors that affects the fertility of soil making it less or non productive and consequently decreasing the crop yield. In order to fight this kind of erosion, it is necessary to take appropriate control measures such as keeping the soil covered after harvesting (stubble on the field), not overgrazing pastures, spacing tree planting, providing shelter belts, etc. Fertilization may also play an important role here in making foliage denser and in producing more leaf litter [23–29].

Keeping these in view, in this section a mathematical model is proposed to minimize the effect of soil erosion due to heavy rain. Let $F(t)$ be the density of effort applied to conserve the depth of fertile topsoil. It is assumed that $F(t)$ is proportional to the depleted level of topsoil and the increase in the depth of fertile topsoil is proportional to the effort applied. Then the dynamics of the system can be governed by the following system of differential equations:

$$
\begin{aligned}
\frac{d\beta}{dt} &= \beta\Big[- a_0 a\beta + \alpha_2 n_0 (1+\mu)\Big(1 - \frac{P}{P_0}\Big) \\
&\qquad + \alpha_3\Big(\frac{q}{S} - r_1 - r_2 R - r_3 R^2 + \frac{rF}{S}\Big) + a_0 b\Big], \\
\frac{dP}{dt} &= n_0\Big(1 - \frac{P}{P_0}\Big)P, \\
\frac{dS}{dt} &= q - r_1 S - r_2 RS - r_3 R^2 S + rF, \\
\frac{dR}{dt} &= \phi(t) - k_1 R - k_2 RS - k_3 R^2 S, \\
\frac{dF}{dt} &= \mu_1 (S_0 - S) H(S_0 - S) - \mu_0 F, \\
&\beta(0) > 0,\ P(0) > 0,\ S(0) > 0,\ R(0) > 0,\ F(0) > 0.
\end{aligned} \tag{9}
$$

In model (9), r is the growth rate coefficient of S due to the effort F, μ_1 is the growth rate coefficient of F and μ_0 is its depreciation rate coefficient. S_0 is the density of fertile topsoil depth that one wish to maintain. $H(t)$ is the unit step function.

It can be checked that model (9) has only one positive equilibrium, namely, $\bar{E}(\bar{\beta}, \bar{P}, \bar{S}, \bar{R}, \bar{F})$, where

$$
\begin{aligned}
&\bar{\beta} = b/a, \quad \bar{P} = P_0, \\
&\bar{F} = (\mu_1/\mu_0)(S_0 - \bar{S})H(S_0 - \bar{S}) = \begin{cases} (\mu_1/\mu_0)(S_0 - \bar{S}), & S_0 > \bar{S}, \\ 0, & S_0 \le \bar{S}, \end{cases}
\end{aligned}
$$

and $\bar{S}, \bar{R}$ are the positive solutions of the following equations:

$$
S = \frac{q\mu_0 + r\mu_1 S_0}{r\mu_1 + r_1\mu_0 + r_2\mu_0 R + r_3\mu_0 R^2}, \tag{10a}
$$

$$
S = \frac{\phi_0 - k_1 R}{k_2 R + k_3 R^2}. \tag{10b}
$$

It can easily be checked that the isoclines (10a) and (10b) intersect at a unique point in the positive quadrant.

In the following theorem, local stability behavior of the positive equilibrium $\bar{E}$ is studied.

Theorem 8. *Let the following inequalities hold:*

$$\left[m_2(r_2\bar{S} + 2r_3\bar{R}\bar{S}) + m_3(k_2\bar{R} + k_3\bar{R}^2)\right]^2 < \frac{3}{2}m_2m_3(r_1 + r_2\bar{R} + r_3\bar{R}^2)(k_1 + k_2\bar{S} + 2k_3\bar{R}\bar{S}), \tag{11}$$

$$(\alpha_3 r)^2 < \frac{1}{2}m_4 a a_0 \mu_0 \bar{S}, \tag{12}$$

where $m_i's$ are positive constants chosen suitably as mentioned in the proof. Then $\bar{E}$ is locally asymptotically stable.

Proof. In order to prove the above theorem, first we linearize the model system (3) by taking the following transformations:

$$\beta = \bar{\beta} + \beta_1,\ P = \bar{P} + P_1,\ S = \bar{S} + S_1,\ R = \bar{R} + R_1,\ F = \bar{F} + F_1,$$

where $\beta_1, P_1, S_1, R_1, F_1$ are small perturbations about $\bar{E}$. Then we consider the following positive definite function:

$$V_1 = \frac{1}{2}\beta_1^2 + \frac{1}{2}m_1P_1^2 + \frac{1}{2}m_2S_1^2 + \frac{1}{2}m_3R_1^2 + \frac{1}{2}m_4F_1^2.$$

Now differentiating V_1 with respect to time t along the linear version of the model system (3), and by choosing

$$m_1 > \frac{n_0\alpha_2^2(1+\mu)^2}{aa_0P_0^2},$$

$$m_2 > \frac{4\alpha_3}{3aa_0(r_1 + r_2\bar{R} + r_3\bar{R}^2)\bar{S}^2}(q + r\bar{F}),$$

$$m_3 > \frac{2}{aa_0(k_1 + k_2\bar{S} + 2k_3\bar{R}\bar{S})}(\alpha_3 r_2 + 2\alpha_3 r_3\bar{R})^2,$$

$$m_4 = \frac{m_2 r}{\mu_1},$$

one can see that dV_1/dt is negative definite under conditions (11) and (12). This proves the theorem (details of the proof are similar to that of Theorem 5). □

In order to study the global stability behavior of the positive equilibrium $\bar{E}$, we need the following lemma whose proof is easy and hence omitted.

Lemma 2. *The set*

$$\begin{aligned}\Omega_1 = \big\{&(\beta, P, S, R, F)\colon 0 < \beta \le \beta_c,\ 0 < P \le P_0,\ 0 < S \le S_c,\\ &0 < R \le \phi_0/k_1,\ 0 < F \le \mu_1 S_0/\mu_0\big\}\end{aligned}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\begin{aligned}\beta_c &= \frac{1}{a_0 a}\left[\alpha_2 n_0(1+\mu) + a_0 b + \frac{\alpha_3}{S_m}\Big(q + \frac{r\mu_1 S_0}{\mu_0}\Big)\right],\\ S_c &= \frac{1}{r_1}\Big(q + \frac{r\mu_1 S_0}{\mu_0}\Big),\end{aligned}$$

and S_m is the minimum of S in Ω_1.

Theorem 9. *Let the following inequalities hold in Ω_1:*

$$\begin{aligned}&\left[m_2 S_c\bigg(r_2 + r_3\Big(\frac{\phi_0}{k_1} + \bar{R}\Big)\bigg) + \frac{m_3\phi_0}{k_1}\Big(k_2 + k_3\frac{\phi_0}{k_1}\Big)\right]^2\\ &\qquad < \frac{3}{2} m_2 m_3 (r_1 + r_2\bar{R} + r_3\bar{R}^2)(k_1 + k_2\bar{S}),\end{aligned} \tag{13}$$

$$(\alpha_3 r)^2 < \frac{1}{2} m_4 a a_0 \mu_0 \bar{S}, \tag{14}$$

where

$$\begin{aligned}m_1 &> \frac{n_0\alpha_2^2(1+\mu)^2}{a a_0 P_0},\\ m_2 &> \frac{4\alpha_3}{3 a a_0 (r_1 + r_2\bar{R} + r_3\bar{R}^2)\bar{S} S_m}\Big(q + \frac{r\mu_1 S_0}{\mu_0}\Big),\\ m_3 &> \frac{2}{a a_0 (k_1 + k_2\bar{S})}\bigg(\alpha_3 r_2 + \alpha_3 r_3\Big(\frac{\phi_0}{k_1} + \bar{R}\Big)\bigg)^2,\\ m_4 &= \frac{m_2 r}{\mu_1}.\end{aligned}$$

Then the positive equilibrium $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

The proof of this theorem is similar to that of Theorem 5, and hence omitted. This theorem implies that if suitable efforts are adopted to minimize the erosion of topsoil, then the depth of fertile topsoil can be maintained at an appropriate level.

6 Numerical simulation

In this section, computer simulation is presented to illustrate the results obtained in previous sections. For this purpose, we choose the following values of parameters:

$$\begin{array}{llllll} q = 1.8, & r_1 = 1.5, & r_2 = 2.01, & r_3 = 0.002, & \phi_0 = 10, & \\ k_1 = 8, & k_2 = 1, & k_3 = 0.001, & n_0 = 4.5, & P_0 = 30, & \\ a_0 = 2.5, & a = 3.5, & b = 14, & \alpha_2 = 0.5, & \alpha_3 = 0.2, & \mu = 0.6. \end{array} \tag{15}$$

With the above values of parameters, our computer simulation shows that the positive equilibrium E^* of model (3) exists, and it is given by

$$B^* = 4.0,\ P^* = 30.0,\ S^* = 0.4642,\ R^* = 1.1814. \tag{16}$$

It is found that condition (6) is satisfied for the values of parameters given in (15). This shows that the positive equilibrium is locally asymptotically stable. It can also be checked that condition (7) is satisfied for the set of parameters given in (15), which shows that E^* is globally asymptotically stable.

To see the effect of conservation, we choose the following values of parameters in addition to the values given in (15):

$$\mu_0 = 0.6,\ \mu_1 = 1.0,\ r = 0.5,\ S_0 = 5.0.$$

Then we note that the positive equilibrium $\bar{E}$ of model (9) exists, and it is given by

$$\bar{B} = 4.0,\ \bar{P} = 30.0,\ \bar{S} = 4.8215,\ \bar{R} = 0.9481,\ \bar{F} = 0.2975. \tag{17}$$

It can be verified that conditions (11) and (12) in Theorem 8 are satisfied, which shows that $\bar{E}$ is locally asymptotically stable. It can also be checked that conditions (13) and (14) in Theorem 9 are satisfied showing the global stability character of $\bar{E}$.

From (16) and (17), it may be noted that due to the effort F, depth of fertile topsoil has increased where as the velocity of rain water along the surface of the soil has decreased.

To see the effect of various parameters on S and R, computer simulations are performed using MATLAB. Figs. 1–4 correspond to model (3), and Figs. 5–9 correspond to model (9). Fig. 1 and Fig. 2 show the effect of r_2 and r_3 on S, respectively. These figures show that depth of fertile topsoil decreases as r_2 and r_3 increase, and tend to its steady state. From Fig. 3 and Fig. 4, it is noted that R decreases as k_2 and k_3 increase. In Fig. 5 and Fig. 6, effects of r_2 and r_3 on S in model system (9) are shown, and in Fig. 7 and Fig. 8, effects of k_2 and k_3 are illustrated. Fig. 9 shows that the effect of r on S. It is seen that as the density of effort (either in terms of tree plantation or shelter belts or covering the soil after harvesting or fertilization) increases, the depth of fertile topsoil also increases. However, an excess amount of effort will lead a decrease in the depth of fertile topsoil. It is also noted that in all cases, S and R tend to their steady state levels.

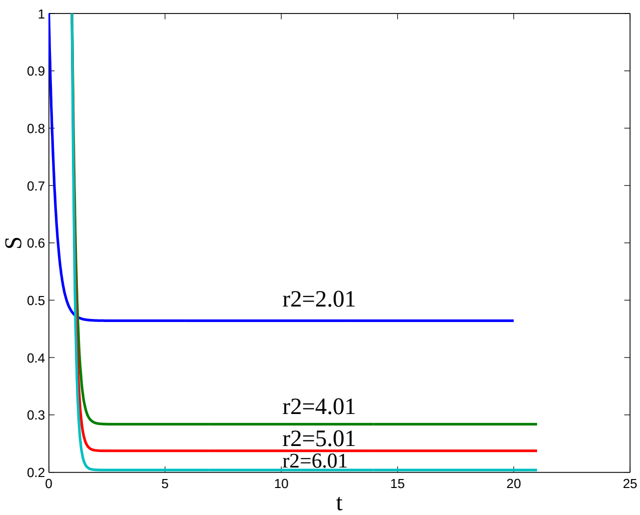

Fig. 1. Model (3): plot of S versus t for different values of r_2 obtained using the parameters: $q=1.8,\ r_1=1.5,\ r_3=0.002,\ \phi_0=10,\ k_1=8,\ k_2=1,\ k_3=0.001,$ $n_0=4.5,\ P_0=30,\ a_0=2.5,\ a=3.5,\ b=14,\ \alpha_2=0.5,\ \alpha_3=0.2,\ \mu=0.6.$

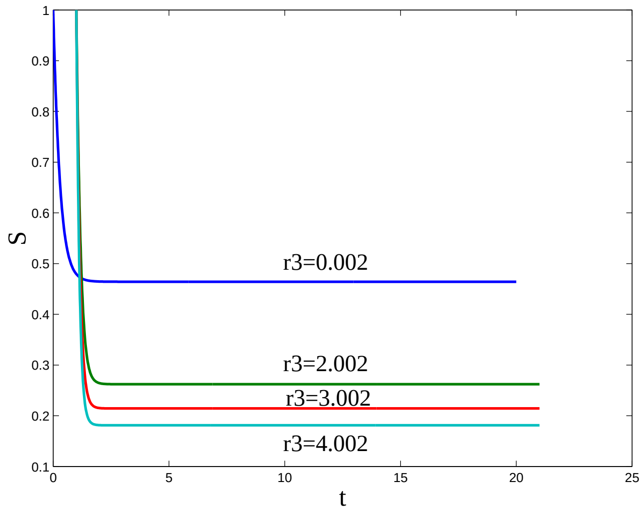

Fig. 2. Model (3): plot of S versus t for different values of r_3 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 1.

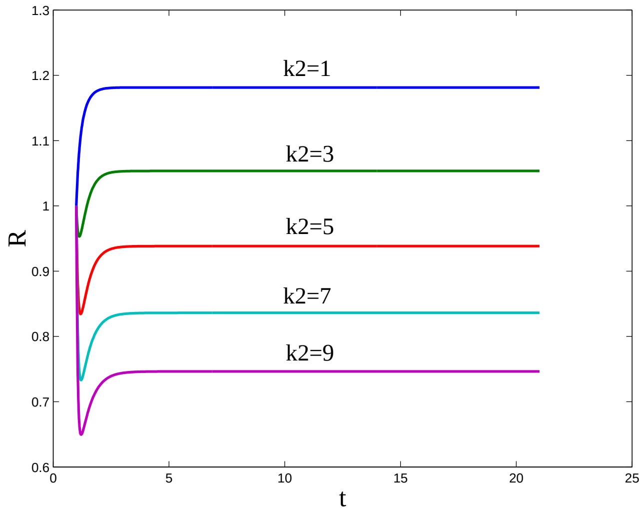

Fig. 3. Model (3): plot of R versus t for different values of k_2 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 1.

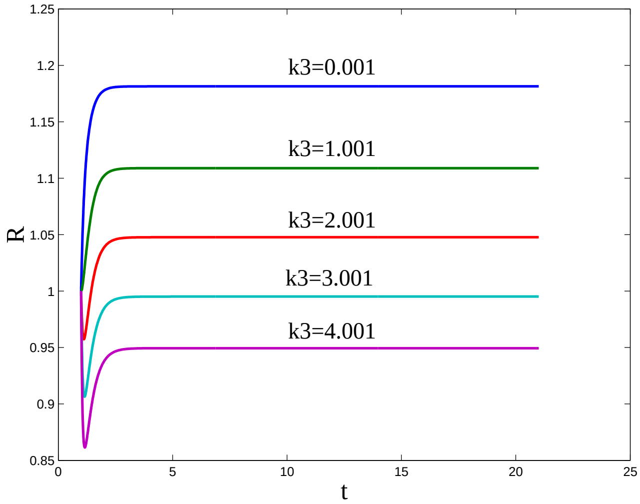

Fig. 4. Model (3): plot of R versus t for different values of k_3 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 1.

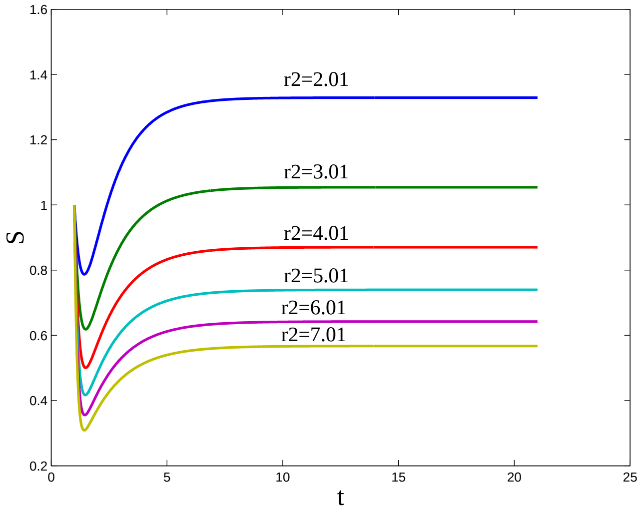

Fig. 5. Model (9): plot of S versus t for different values of r_2 obtained using the parameters: $q = 1.8,\ r_1 = 1.5,\ r_3 = 0.002,\ \phi_0 = 10,\ k_1 = 8,\ k_2 = 1,\ k_3 = 0.001,\ n_0 = 4.5,\ P_0 = 30,\ a_0 = 2.5,\ a = 3.5,\ b = 14,\ \alpha_2 = 0.5,\ \alpha_3 = 0.2,\ \mu = 0.6,\ \mu_0 = 0.6,\ \mu_1 = 1.0,\ r = 0.5,\ S_0 = 5.0.$

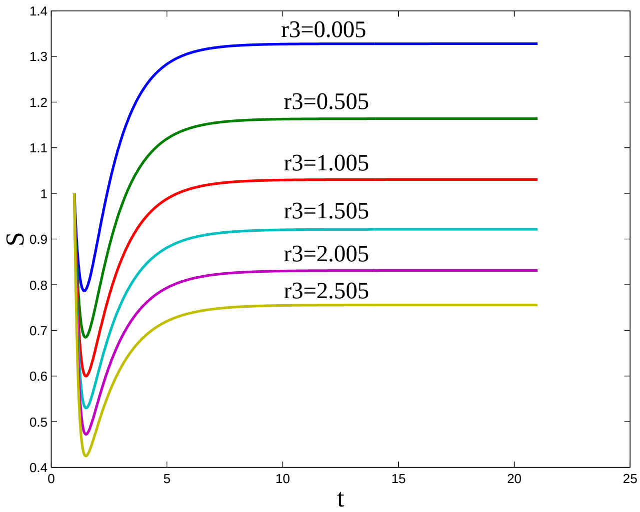

Fig. 6. Model (9): plot of S versus t for different values of r_3 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 5.

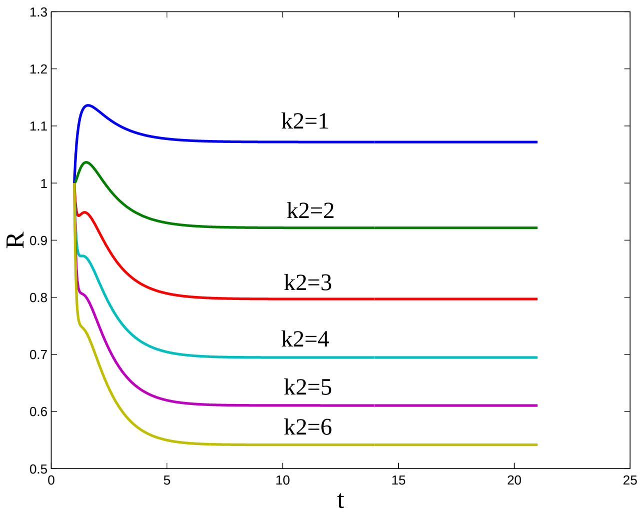

Fig. 7. Model (9): plot of R versus t for different values of k_2 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 5.

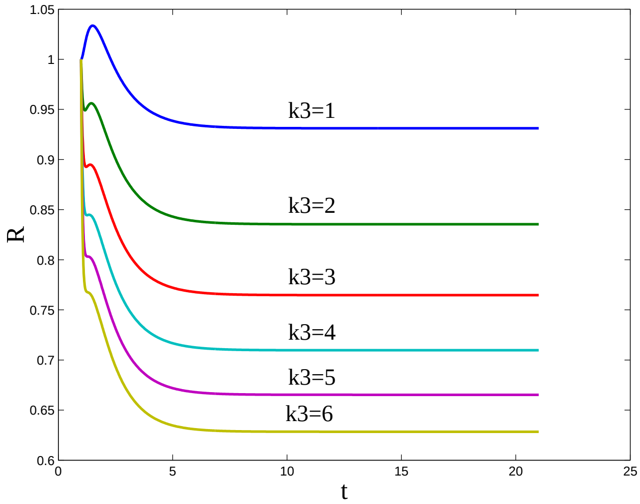

Fig. 8. Model (9): plot of R versus t for different values of k_3 obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 5.

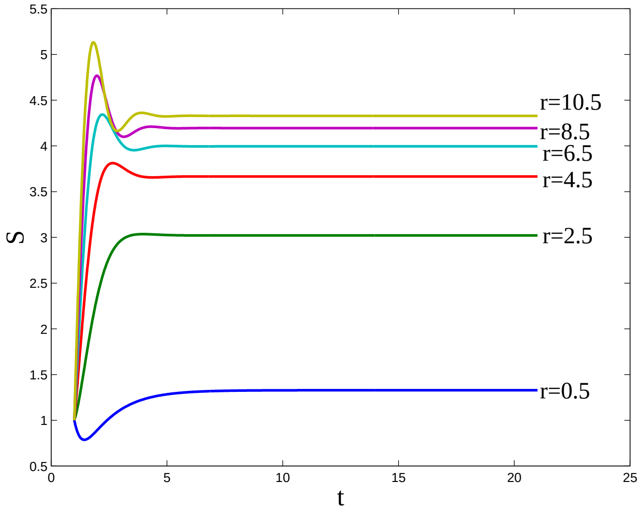

Fig. 9. Model (9): plot of S versus t for different values of r obtained with $r_2 = 2.01$, and other values of parameters are same as in Fig. 5.

7 Conclusions

In this paper, a mathematical model has been proposed to study the topsoil erosion caused by heavy rain. The model has been analysed when the hydraulic pressure gradient is constant or periodic.

When the hydraulic pressure gradient is constant, it has been shown that the depth of fertile topsoil decreases due to natural factors and this decrease becomes faster when the soil is exposed to heavy rain. When the hydraulic pressure gradient is periodic with small amplitude, it has been found that a periodic behavior occurs in the system and its stability behavior is same as that of the case of constant pressure gradient.

A model to minimize the effect of rain on the erosion of topsoil has also been proposed and analysed. By analyzing the model it has been noted that if suitable efforts are applied to conserve the topsoil, an appropriate level of fertile topsoil depth and crop yield can be maintained.

Computer simulation has been carried out to see the effect of various parameters on the depth of fertile topsoil and velocity of rail flowing along the surface of soil. In particular, it has been noted that an appropriate amount of effort would increase the depth of fertile topsoil, and if the density of effort increases beyond the threshold level, then it may cause a decrease in the depth of fertile topsoil.

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