Global Observer for Homogeneous Vector Fields

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Abstract. This paper presents an algebraic approach to the problem of nonlinear observer design. We show, that an observer which converges globally and asymptotically can be designed for a class of homogeneous systems of odd degree.

Keywords: Homogeneous systems, Lyapunov function, asymptotic observer.

1 Introduction

Given an input-output nonlinear system, a state observer is a dynamic system which is expected to produce an estimation of the state of the system. The nonlinear observer has been a topic of interest in control theory [1–4]. For linear systems, it has been extensively studied, and has proven extremely useful, especially for control applications. For nonlinear systems, the theory of observers is not nearly as complete nor successful as it is for the linear case. Many authors have worked on the development of state observers. Some observers were designed for a restricted class of nonlinear systems such as bilinear systems [5–8]. A variety of methods has been developed for constructing nonlinear observers for some classes of systems [9–17]. In [12], an observer which guarantees the convergence to zero of the error has been presented, based on a Lyapunov-like sufficient condition. Also, this problem has been recently solved by [15] for nonlinear system which are uniformly observable for any input and can be transformed into a canonical form. Even if these conditions are satisfied, the construction of the observer still remains a difficult problem due to the need to solve a set of simultaneous partial differential equations to obtain the actual transformation function. In this paper we are devoted to developing a geometrical design method of continuous observers for a class of homogeneous systems of odd degree. This is possible thanks to the feedback law proposed by the authors in [18] which is required for stabilization of homogeneous nonlinear systems of odd degree. The sufficient conditions we propose is of Lyapunov type that guarantees the observation error to be globally and asymptotically stable, and it turns out to be also necessary to the linear case.

2 Conception of the observer

In this paper we consider the following system

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^l$ is the input $y \in \mathbb{R}^m$ is the output of the system and f is a smooth vector field on $\mathbb{R}^n$ such that all its components f_i are homogeneous polynomials of the same odd degree $k \geq 1$ and C (respectively B) is a $m \times n$ (respectively $n \times l$) constant matrix. Recall that homogeneous of degree k means that for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda^k f(x)$.

When the states of the system (1) are not available, the usual techniques is to build a control system whose inputs are the input and output of the initial system called observer which is designed to give an approximation of the state of (1).

Let $p \in \mathbb{N}$ be the rank of C. Without loss of generality, we can write system (1) in the following form:

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y_i = x_i, \quad i = 1, \dots, p. \end{cases} \tag{2}$$

The matrix C is such that

$$^tCC = \mathrm{diag}(\lambda_1, \dots, \lambda_p, 0, \dots, 0), \quad \lambda_i = 1, \quad i = 1, \dots, p.$$

Notice that such a change of coordinates does not affect the properties neither on the observability nor on the construction of an observer. So, throughout this paper, we consider the system (1) as in the form (2). Recall that, a global asymptotic observer for the system (1) is a dynamic system of the form

$$\dot{\hat{x}} = g(\hat{x}, y, u), \tag{3}$$

which is expected to produce the estimation $\hat{x}(t)$ of the state $x(t)$ of the system (1). More precisely, if system (1) and (3) are initialized at the same point $\big(x(0) = \hat{x}(0)\big)$, we want to have $\big(x(t) = \hat{x}(t)\big)$, $\forall t \geq 0$. It means that,

$$g(x, Cx, u) = f(x) + Bu, \quad \forall x \in \mathbb{R}^n.$$

This means that the observer and the plant have the same dynamics under the condition that the output function $C\hat{x}$ copies the output function Cx (see [19]). Also, for any initial condition $\|\hat{x}(0) - x(0)\|$ one has $\|\hat{x}(t) - x(t)\|$ tends to zero globally and asymptotically.

Letting, $e = \hat{x} - x$, the derivative is given by $\dot{e} = \dot{\hat{x}} - \dot{x}$. Thus,

$$\dot{e} = g(x + e, y, u) - f(x) - Bu.$$

We want that the error equation to be globally asymptotically stable about the origin. Therefore, it suffices to prove the existence of a Lyapunov function W positive definite on $\mathbb{R}^n$ such that its time-derivative along the trajectories of the error equation is negative definite on $\mathbb{R}^n$.

In the following, we will assume the existence of a definite positive function $V\colon \mathbb{R}^n \to \mathbb{R}$ homogeneous, proper and independent of the time which satisfies the following hypothesis:

$(\mathcal{H}_1)$ $\quad \nabla V(e)\big(f(x+e) - f(x)\big) < 0, \quad \forall e \in \operatorname{Ker} C \setminus \{0\}, \quad \forall x \in \mathbb{R}^n;$

$(\mathcal{H}_2)$ $\quad \dfrac{\partial V}{\partial e_i}(e) = 0, \quad$ for $\quad i = 1, \ldots, p, \quad \forall e \in \operatorname{Ker} C.$

Recall that for autonomous function V positive definite means that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$ and proper means that $V(x) \to +\infty$ as $\|x\| \to +\infty$.

Consider the system

$$\dot{\hat{x}} = f(\hat{x}) - \alpha\big(\|\hat{x}\|^{k-1} + \|{}^tC(C\hat{x} - y)\|^{k-1}\big)^t C(C\hat{x} - y) + Bu \tag{4}$$

with $\alpha > 0$.

Theorem 1. *If there exists a positive definite and homogeneous function V of degree $2d$ which satisfies assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, then for a certain $\alpha > 0$ the system* (4) *is a global asymptotic observer for* (1).

Proof. The error equation is given by

$$\dot{e} = f(x+e) - f(x) - \alpha\big(\|x+e\|^{k-1} + \|tCCe\|^{k-1}\big)^t CCe. \tag{5}$$

Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Consider the following function

$$W(e) = \frac{1}{2d}({}^t e^t CCe)^d + U(e),$$

where $U(e) = V(0, ..., 0, e_{p+1}, ..., e_n)$.

First, remark that W is homogeneous definite on $\mathbb{R}^n$ which will be used as a Lyapunov function candidate for the system (5).

Taking into account the form of W, we have

$$\begin{aligned}\nabla W = {} & ({}^t e^t CCe)^{d-1}.{}^t e^t CC \\ & + \Big(0, \ldots, 0, \frac{V}{\partial e_{p+1}}(0, \ldots, 0, e_{p+1}, \ldots, e_n), \\ & \qquad \ldots, \frac{\partial V}{\partial e_n}(0, \ldots, 0, e_{p+1}, \ldots, e_n)\Big).\end{aligned}$$

Since ${}^t({}^t CCe) ={}^t (e_1, \ldots, e_p, 0, \ldots, 0)$ then, the time-derivative of W along the trajectories of (5) is given by

$$\begin{aligned}\dot{W}(e) = {} & \nabla W\big(f(x+e) - f(x)\big) \\ & - \alpha({}^t e^t CCe)^{d-1}\big(\|x+e\|^{k-1} + \|{}^t CCe\|^{k-1}\big)\|{}^t CCe\|^2.\end{aligned} \tag{6}$$

$\dot{W}$ is a homogeneous function of (x, e) of even degree $2d+k-1$. Hence, its sign doesn't change along any ray issuing from the origin of $\mathbb{R}^n \times \mathbb{R}^n$ [20]. This sign can be evaluated on the sphere

$$S = \big\{(x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \|(x, e)\| = \sqrt{\|x\|^2 + \|e\|^2} = \sqrt{2}\big\}.$$

Let

$$\begin{aligned}\mathcal{D}_1 &= \big\{(x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \|x\| = 1,\ \|e\| = 1\big\}, \\ \mathcal{D}_2 &= \big\{(x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ 1 < \|x\| \leq \sqrt{2},\ \|e\| < 1\big\}, \\ \mathcal{D}_3 &= \big\{(x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \|x\| < 1,\ 1 < \|e\| \leq \sqrt{2}\big\}.\end{aligned}$$

Obviously, we have

$$S \subset \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3.$$

Let

$$C_- = \big\{(x,e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \nabla W\big(f(x+e) - f(x)\big) < 0\big\},$$
$$C_+ = \big\{(x,e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \nabla W\big(f(x+e) - f(x)\big) \geq 0\big\}.$$

On $C_- \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3)$ we have $\dot{W}(e) < 0$. Still to prove that

$$\dot{W}(e) < 0, \quad \forall (x,e) \in C_+ \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3), \quad e \neq 0.$$

Let

$$C_+^1 = \big\{(x,e) \in \mathbb{R}^n \times \mathbb{R}^n \ / \ \|(x,e)\| \leq \sqrt{3}\big\}.$$

Remark that

$$C_+ \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) \subset C_+^1 \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3).$$

So, it suffices to show that

$$\dot{W}(e) < 0, \quad \forall (x,e) \in C_+^1 \cap (\mathcal{D}_1 \cup, \mathcal{D}_2 \cup \mathcal{D}_3), \quad e \neq 0.$$

Let π_1 and π_2 be the projection defined as follow $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\pi_1(x,e) = x$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\pi_2(x,e) = e$. Denote by $\pi_i(C_+^1) = Q_i$, $i = 1,2$, the C_+^1-projections on $\mathbb{R}^n$. Since C_+^1 is a compact set and π_i are continuous functions, Q_1 and Q_2 are compacts sets. From $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and taking into account the form of W and U, where U is the second part of the Lyapunov function W, we can deduce that for all $e \in \operatorname{Ker} C \setminus \{0\}$, we obtain

$$\begin{aligned} \nabla W\big(f(x+e) - f(x)\big) &= \nabla U(e)\big(f(x+e) - f(x)\big) \\ &= \nabla V(e)\big(f(x+e) - f(x)\big) < 0. \end{aligned}$$

This implies that

$$\mathbb{R}^n \times \operatorname{Ker} C \subset C_- \cap \mathbb{R}^n \times \{0\}.$$

Since $C_- \cap C_+ = \emptyset$, we have

$$Q_2 \cap \operatorname{Ker} C = \{0\}. \tag{7}$$

On the other hand, let F be the function defined from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ by

$$F(x,e) = \nabla W(e)\big(f(x+e) - f(x)\big).$$

For all $\lambda \in \mathbb{R}$ and $(x,e) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$F\big(\lambda(x,e)\big) = F(\lambda x, \lambda e) = \lambda^{2d+k-1}\nabla W(e)\big(f(x+e) - f(x)\big) = \lambda^{2d+k-1}F(x,e),$$

which implies that F is homogeneous on $\mathbb{R}^n \times \mathbb{R}^n$. It follows that Q_2 is a cone. Indeed, let $e \in Q_2$ it implies that, there exists $x \in \mathbb{R}^n$ such that $(x,e) \in C_+$. So, $F(x,e) \geq 0$.

Now, because $2d + k - 1$ is even then for all $\lambda \in \mathbb{R}$, we have

$$F(\lambda x, \lambda e) = \lambda^{2d+k-1}F(x,e) \geq 0$$

This implies that, $(\lambda x, \lambda e) \in C_+$ and so $\lambda e \in Q_2$. Next, since Q_2 is a cone, it follows that

$$\big\{e \,/\, \|e\| = r\big\} \cap Q_2 = r\big\{e \,/\, \|e\| = 1\big\} \cap Q_2, \quad \forall r \in \mathbb{R}. \tag{8}$$

Since Q_2 is a compact set then $\big\{e \,/\, \|e\| = 1\big\} \cap Q_2$ is also a compact set. Hence, the minimum of the quadratic form $\|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1}$, which is a continuous function, exists and positive. Taking into account the equality (7), the minimum is strictly positive. Letting

$$\min_{\{e/\|e\|=1\}\cap Q_2} \|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1} = h > 0.$$

Then by (8), we have

$$\min_{\{e/\|e\|=r\}\cap Q_2} \|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1} = r^{2d+k-1}h > 0.$$

Let

$$\eta = \max_{(x,e)\in C_+^1\cap(\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3)} \big|\nabla W(e)\big(f(x+e) - f(x)\big)\big|.$$

We will study the sign of $\dot{W}(e)$ separately on $\mathcal{D}_1 \cap C^1_+$, $\mathcal{D}_2 \cap C^1_+$ and $\mathcal{D}_3 \cap C^1_+$. On $\mathcal{D}_1 \cap C^1_+$, we have

$$\dot{W}(e) \le \eta - \alpha \min_{\{e/\|e\|=1\}\cap Q_2} \|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1}$$

which gives

$$\dot{W}(e) \le \eta - \alpha h < 0 \ \text{ for } \ \alpha > \frac{\eta}{h}.$$

On $\mathcal{D}_2 \cap C^1_+$ and $\|e\| = r > 0$, we have

$$\begin{aligned}
\dot{W}(e) &= \nabla W(e)\big(f(x+e) - f(x)\big) \\
&\quad - \alpha\big(\|x+e\|^{k-1} + \|{}^tCCe\|^{k-1}\big)({}^te{}^tCCe)^{d-1}\|{}^tCCe\|^2 \\
&\le \nabla W(e)\big(f(x+e) - f(x)\big) \\
&\quad - \alpha\big((\|x\| - \|e\|)^{k-1} + \|{}^tCCe\|^{k-1}\big)({}^te{}^tCCe)^{d-1}\|{}^tCCe\|^2 \\
&\le \nabla W(e)\big(f(x+e) - f(x)\big) \\
&\quad - \alpha\big((1 - \|e\|)^{k-1} + \|{}^tCCe\|^{k-1}\big)({}^te{}^tCCe)^{d-1}\|{}^tCCe\|^2.
\end{aligned}$$

If $\|e\| = r \ge \frac{1}{2}$, then

$$\begin{aligned}
\dot{W}(e) &\le \eta - \alpha\|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1} \\
&\le \eta - \alpha \min_{\left\{e/\|e\|=1\right\}\cap Q_2} \|{}^tCCe\|^{k+1}({}^te{}^tCCe)^{d-1} \\
&\le \eta - \alpha r^{2d+k-1} h \\
&\le \eta - \alpha(\frac{1}{2})^{2d+k-1} h.
\end{aligned}$$

This last quantity is negative definite if we choose

$$\alpha > \frac{\eta 2^{2d+k-1}}{h}.$$

If $\|e\| = r < \frac{1}{2}$, then with the fact that $\|e\| < 1$ and $\|x\| > 1$, we have

$$\begin{aligned}
&\big|\nabla W(e)\big| \le \delta_1 \|e\|^{2d-1}, \\
&\big\|f(x+e) - f(x)\big\| = \Big|\sum_{\alpha^i,\beta^i} a_{\alpha^i,\beta^i} x_1^{\alpha_1{}^i} \dots x_n^{\alpha_n{}^i} e_1^{\beta_1{}^i} \dots e_n^{\beta_n{}^i}\Big|
\end{aligned}$$

with $\alpha^i = \alpha_1^i + \ldots + \alpha_n^i$, $\beta^i = \beta_1^i + \ldots + \beta_n^i$, $\alpha^i + \beta^i = k$ and $\beta^i \geq 1$ for all i. Since $\|e\| < 1, \|x\| > 1, \alpha^i + \beta^i = k$ and $\beta^i \geq 1$ for all i, we have

$$\big\|f(x+e) - f(x)\big\| \leq \delta_2 \|e\| \|x\|^{k-1}.$$

So, one gets

$$\big\|f(x+e) - f(x)\big\| \leq \lambda_1 r^{2d} \|x\|^{k-1}$$

and

$$\dot{W}(e) \leq \lambda_1 r^{2d} \|x\|^{k-1} - \alpha(1-r)^{k-1}({}^t e\, {}^t CCe)^{d-1} \|{}^t CCe\|^2.$$

Using the fact that $x \in \mathcal{D}_2$, we obtain

$$\dot{W}(e) \leq \lambda_1 r^{2d} (\sqrt{2})^{k-1} - \alpha \Big(\frac{1}{2}\Big)^{k-1} \min_{\{e/\|e\|=1\} \cap Q_2} ({}^t e\, {}^t CCe)^{d-1} \|{}^t CCe\|^2.$$

Thus

$$\dot{W}(e) \leq \lambda_1 r^{2d} (\sqrt{2})^{k-1} - \alpha \Big(\frac{1}{2}\Big)^{k-1} \lambda_2 r^{2d}.$$

This last quantity is negative definite if we choose

$$\alpha > \frac{\lambda_1 (2\sqrt{2})^{k-1}}{\lambda_2}.$$

On $\mathcal{D}_3 \cap C_+^1$ and $\|e\| = r > 1$, we have

$$\begin{aligned}
\dot{W}(e) &\leq \eta - \alpha \|{}^t CCe\|^{k+2} ({}^t e\, {}^t CCe)^{d-1} \\
&\leq \eta - \alpha \min_{\{e/\|e\|=r\} \cap Q_2} \|{}^t CCe\|^{k+2} ({}^t e\, {}^t CCe)^{d-1} \\
&\leq \eta - \alpha r^{2d+k-1} h \\
&\leq \eta - \alpha h.
\end{aligned}$$

It follows that in this case one gets

$$\dot{W}(e) < 0 \quad \text{for} \quad \alpha > \frac{\eta}{h}.$$

Therefore, if α satisfies the three conditions given above, it means that

$$\alpha > \sup \Big(\frac{\eta}{h},\ \frac{\eta 2^{2d+k-1}}{h},\ \frac{\lambda_1 (2\sqrt{2})^{k-1}}{\lambda_2} \Big),$$

we obtain

$$\dot{W}(e) < 0, \quad \forall (x,e) \in C_+ \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) \quad \text{with} \quad e \neq 0.$$

The last expression in conjunction with the fact that

$$\dot{W}(e) < 0, \quad \forall (x,e) \in C_- \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) \quad \text{with} \quad e \neq 0$$

yields

$$\dot{W}(e) < 0, \quad \forall (x,e) \in (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) \quad \text{with } e \neq 0.$$

Thus, the time-derivative of W along the trajectories of the error equation given in (6) is negative definite on the sphere S and by homogeneity on $\mathbb{R}^n$. We have

$$\dot{W}(e) < 0, \quad \forall e \in \mathbb{R}^n \setminus \{0\}.$$

It follows that, the system (4) is a global asymptotic observer for (1). □

Suppose now, that the assumption $(\mathcal{H}_2)$ hold and the following condition which can replace $(\mathcal{H}_2)$ for the construction of the observer.
$(\mathcal{H}_3) \quad \big\langle \nabla V(e), {}^tCCe \big\rangle \geq 0, \; \forall e \in \mathbb{R}^n.$

Theorem 2. *If there exists a positive definite and homogeneous function V of degree $2d$ which satisfies assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, then for a certain $\alpha > 0$ the system (4) is a global asymptotic observer for* (1)*.*

Proof. If $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold then by the same argument as in the proof of the Theorem 1, we can show using the Lyapunov function

$$W(e) = \frac{1}{2d}({}^te{}^tCCe)^d + V(e)$$

that the following estimation holds.

$$\begin{aligned}\dot{W}(e) &= \nabla W(e)\big(f(x+e) - f(x)\big)\\ &- \alpha\big(\|x+e\|^{k-1} + \|{}^tCCe\|^{k-1}\big)\big(({}^te{}^tCCe)^{d-1}\|{}^tCCe\|^2 + \langle \nabla V(e), {}^tCCe\rangle\big).\end{aligned}$$

This inequality implies that

$$\begin{aligned}\dot{W}(e) &\leq \nabla W(e)\big(f(x+e) - f(x)\big)\\ &\qquad - \alpha\big(\|x+e\|^{k-1} + \|{}^tCCe\|^{k-1}\big)({}^te{}^tCCe)^{d-1}\|{}^tCCe\|^2.\end{aligned}$$

It follows, as in the proof of the Theorem 1, that an estimation of the form

$$\dot{W}(e) < 0,\ \forall e \in \mathbb{R}^n \setminus \{0\}$$

can be obtained, and therefore an observer of the form (4) can be designed for the system (1).

Next we give an example on $\mathbb{R}^3$ to illustrate the applicability of the result of this paper.

Example. Consider the following system,

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) = x_2^3 + x_3^3, \\ \dot{x}_2 = f_2(x_1, x_2, x_3) = x_1^3, \\ \dot{x}_3 = f_3(x_1, x_2, x_3) = -x_3^3 + x_1 x_2^2 + 5x_1^3 + u, \\ y = (x_1, x_2), \end{cases} \tag{9}$$

which has the form of (1) with $u \in \mathbb{R}$, $y_1 = x_1$ and $y_2 = x_2$. The matrix C which is a (2×3) constant matrix is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A simple computation gives

$${}^tCC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, 0), \quad \lambda_i = 1, \quad i = 1, 2.$$

Notice that the system (1) and (2) are equivalent by using a change of coordinates.

Let

$$V(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

be a Lyapunov function candidate for the above system which is definite positive proper and homogeneous function which satisfies assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Indeed, in this case

$$\operatorname{Ker} C = \left\{ e \in \mathbb{R}^3 / e_1 = e_2 = 0 \right\}.$$

We can verify that

$$\frac{\partial V}{\partial e_1} = \frac{\partial V}{\partial e_2} = 0, \quad \forall e =^t (e_1, e_2, e_3) \in \operatorname{Ker} C$$

and using a simple computation we obtain

$$\nabla V(e)\big(f(x+e) - f(x)\big) = -e_3^2(e_3^2 + 3x_3e_3 + 3x_3^2) < 0,$$
$$\forall x \in \mathbb{R}^3, \quad \forall e \in \operatorname{Ker} C \setminus \{0\}.$$

According to Theorem 1, the following system

$$\dot{\hat{x}} = f(\hat{x}) + Bu - \alpha\big(\|\hat{x}\|^2 + \|^tC(C\hat{x} - y)\|^2\big)^tC(C\hat{x} - y)$$

is an observer for system (9) for a suitable value of α with

$$f(x) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This system can be written as

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2^3 + \hat{x}_3^3 - \alpha(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 + e_1^2 + e_2^2)e_1, \\ \dot{\hat{x}}_2 = \hat{x}_1^3 - \alpha(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 + e_1^2 + e_2^2)e_2, \\ \dot{\hat{x}}_3 = -\hat{x}_3^3 + \hat{x}_1\hat{x}_2^2 + 5\hat{x}_1^3 + u \end{cases}$$

with $e(t) = \hat{x}(t) - x(t)$ which tends to zero globally and asymptotically for $\alpha > 0$ taken large enough.

Note that for linear system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx. \end{cases} \tag{10}$$

The system is said detectable if there exists a matrix L such that the matrix $(A - LC)$ is globally asymptotically stable. A sufficient condition for (10) to be detectable is that if it is observable or simply the pair (A, C) is observable i.e., its observability matrix has full rank,

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

In this case for this kind of systems a Luenberger observer can be designed, it can be taken as

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y),$$

where L is the gain matrix which is chosen in such away $\mathrm{Re}\big(\lambda(A - LC)\big) < 0$ and a Lyapunov function candidate for the error equation

$$\dot{e}(t) = \dot{\hat{x}}(t) - x(t)$$

can be taken as

$$V(e) = {}^t ePe$$

with P is positive definite symmetric matrix satisfying the Lyapunov equation

$$P(A - LC) + {}^t(A - LC)P = -Q$$

with Q is positive definite symmetric matrix. By taking the time-derivative of V along the trajectories of

$$\dot{e}(t) = (A - LC)e(t)$$

one can obtain the following estimation

$$\dot{V}(e) = -{}^t eQe$$

which is negative definite.

It turns out that the condition stated in $(\mathcal{H}_1)$ is necessary for the conception of an observer of the form (4) for systems of the form (10). Indeed, the system (4) becomes with $f(x) = Ax$ and $k = 1$,

$$\dot{\hat{x}} = A\hat{x} + Bu - \alpha^t C(C\hat{x} - y).$$

The error equation is given by

$$\dot{e} = Ae - \alpha^t CCe = (A - \alpha^t CC)e.$$

If we consider the Lyapunov function $V(e) =^t ePe$, the time-derivative along the trajectories of the error equation is given by

$$\dot{V}(e) = 2^t ePAe - 2\alpha^t eP^t CCe < 0, \quad \forall e \neq 0.$$

Let now $e \in \mathrm{Ker}C$. The previous expression reduces to ${}^t ePAe < 0$ this yields $(\mathcal{H}_1)$. □

3 Conclusion

Consider a homogeneous system of the form (1) having some states not available for direct measurement. It is shown, in this paper, that an asymptotic observer can be designed under some sufficient conditions based on the stabilizing feedback law given by [18]. Moreover, an numerical example is given to illustrate the applicability of the main result.

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