Fixed Point in Minimal Spaces

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Abstract. This paper deals with fixed point theory and fixed point property in minimal spaces. We will prove that under some conditions $f\colon (X,\mathcal{M}) \to (X,\mathcal{M})$ has a fixed point if and only if for each m-open cover $\{B_\alpha\}$ for X there is at least one $x \in X$ such that both x and $f(x)$ belong to a common B_α. Further, it is shown that if $(X,\mathcal{M})$ has the fixed point property, then its minimal retract subset enjoys this property.

Keywords: fixed point, orbits, multifunction, minimal space.

1 Introduction and preliminaries

Fixed point theory is a very attractive subject, which has recently drawn much attention from the communities of physics, engineering, mathematics etc. In this field, there have been many representative approaches method by orbits [1]. In [2], authors used fixed point theory to find a method to estimate the optimum neighborhood with the chosen gain matrix.

In this paper, we prove some results too stunning not be in the spotlight. These results are typical of the most attractive aspects of the fixed point theory in minimal spaces in that they are proved. We show any retract subset of a space with fixed point property would have the fixed point property.

In 1950, H. Maki, J. Umehara and T. Noiri [3] introduced the notions of minimal structure and minimal space. They achieved many important results compatible by the general topology case. Some other results about minimal spaces can be found in [4–9].

For easy understanding of the material incorporated in this paper we recall some basic definitions. For details on the following notions we refer to [4], [3] and [7].

Definition 1. [3] *A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be minimal structure on X if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a minimal space. Throughout this paper (X,M) or $(Y,\mathcal{N})$ means minimal space.*

Example 1. [3] Let (X, τ) be a topological space. Then $\mathcal{M} = \tau, SO(X)$, $PO(X), \alpha O(X)$ and $\beta O(X)$ are examples of minimal structures on X.

Definition 2. [3] *A set $A \in \mathcal{P}(X)$ is said to be an m-open set if $A \in \mathcal{M}$. $B \in \mathcal{P}(X)$ is an m-closed set if $B^c \in \mathcal{M}$. We set*

$$m - Int(A) = \bigcup\{U\colon U \subseteq A,\ U \in \mathcal{M}\},$$
$$m - Cl(A) = \bigcap\{F\colon A \subseteq F,\ F^c \in \mathcal{M}\}.$$

Remark 1. *Choosing one of the $\tau, SO(X), PO(X), \alpha O(X)$ and $\beta O(X)$ instead of $\mathcal{M}$, then $m - Int(A)$ would be $Int(A)$, $sInt(A), pInt(A), \alpha Int(A)$ and $\beta Int(A)$ respectively. Similarly, $m - Cl(A)$ is equal to $Cl(A), sCl(A), pCl(A)$, $\alpha Cl(A)$ and $\beta Cl(A)$ respectively.*

Proposition 1. [3] *For any two sets A and B,*

(i) $m - Int(A) \subseteq A$ *and* $m - Int(A) = A$ *if* A *is an m-open set;*

(ii) $A \subseteq m - Cl(A)$ *and* $A = m - Cl(A)$ *if* A *is an m-closed set;*

(iii) $m - Int(A) \subseteq m - Int(B)$ *and* $m - Cl(A) \subseteq m - Cl(B)$ *if* $A \subseteq B$ *;*

(iv) $m - Int(A \cap B) = (m - Int(A)) \cap (m - Int(B))$ *and* $(m - Int(A)) \cup (m - Int(B)) \subseteq m - Int(A \cup B)$*;*

(v) $m - Cl(A \cup B) = (m - Cl(A)) \cup (m - Cl(B))$ *and* $m - Cl(A \cap B) \subseteq (m - Cl(A)) \cap (m - Cl(B))$*;*

(vi) $m - Int(m - Int(A)) = m - Int(A)$ *and* $m - Cl(m - Cl(B)) = m - Cl(B)$*;*

(vii) $(m - Cl(A))^c = m - Int(A^c)$ *and* $(m - Int(A))^c = m - Cl(A^c)$*.*

Definition 3. [7] *A minimal space $(X, \mathcal{M})$ enjoys the property U if the arbitrary union of m-open sets is an m-open set.*

Proposition 2. [7] *For a minimal structure $\mathcal{M}$ on a set X, the following are equivalent.*

(i) *$\mathcal{M}$ has the property U.*

(ii) *If $m - Int(A) = A$, then $A \in \mathcal{M}$.*

(iii) *If $m - Cl(B) = B$, then $B^c \in \mathcal{M}$.*

Definition 4. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. We say that a function $f\colon (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a minimal continuous (briefly m-continuous) if $f^{-1}(B) \in \mathcal{M}$, for any $B \in \mathcal{N}$.*

The following results are the immediate consequences of Definition 4.

Proposition 3. *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are minimal spaces. Then*

(i) *the identity map $id_X\colon (X, \mathcal{M}) \to (X, \mathcal{M})$ is m-continuous;*

(ii) *$id_X\colon (X, \mathcal{M}) \to (X, \mathcal{N})$ is m-continuous where $(X, \mathcal{M})$ and $(X, \mathcal{N})$ are minimal spaces and $\mathcal{N} \leq \mathcal{M}$;*

(iii) *any constant function $f\colon (X, \mathcal{M}) \to (Y, \mathcal{N})$ is m-continuous.*

Theorem 1. *The composition of two m-continuous functions is an m-continuous function.*

2 Orbits and fixed point

For two sets X and Y and each element x of X we associate a nonempty subset $F(x)$ of Y and this correspondence $x \mapsto F(x)$ is called a *multi-valued mapping* or a *multifunction* from X into Y; i.e., F is a function from X to $2^Y \setminus \{\emptyset\}$ and is denoted by $F\colon X \to 2^Y$. The *lower inverse* of a multi-valued mapping F is the multi-valued mapping F^l of Y into X defined by

$$F^l(y) = \{x \in X\colon\ y \in F(x)\},$$

also for any nonempty subset B of Y we have,

$$F^l(B) = \{x \in X\colon\ F(x) \cap B \neq \emptyset\},$$

finally it is understood that $F^l(\emptyset) = \emptyset$. The set $\{x \in X\colon F(x) \subseteq B\}$ is the *upper inverse* of B and is denoted by $F^u(B)$. f is *minimal lower semicontinuous* ($m.l.s.c.$), if for every $U \subseteq Y$ m-open, $f^l(U)$ is m-open in X.

Let $f\colon X \to 2^Y$ and $g\colon Y \to 2^Z$ be two multifunctions. The composition $gof\colon X \to 2^Z$ is defined by

$$gof(x) = \bigcup_{y \in f(x)} g(y).$$

Definition 5. [7] *For a minimal space $(X, \mathcal{M})$,*

(i) *a family of m-open sets $\mathcal{A} = \{A_j\colon j \in J\}$ in X is called an m-open cover of K if $K \subseteq \bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also m-open cover of K is called a subcover of $\mathcal{A}$ for K;*

(ii) *a subset K of X is m-compact whenever given any m-open cover of K has a finite subcover;*

(iii) *$(X, \mathcal{M})$ is said to be $m - T_2$ space if for each distinct points $x, y \in X$, there exists $U, V \in \mathcal{M}$ containing x and y respectively, such that $U \cap V = \emptyset$.*

In the following lemma we show the equivalence of point-wise m-continuity and m-continuity which has a key role for our result.

Lemma 1. *Suppose $f\colon X \to Y$ is a function, where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are two minimal spaces. Then f is point-wise m-continuous if f is m-continuous.*

Proof. Assume f is point-wise m-continuous, $x \in X$, $V \in \mathcal{N}$ and $f(x) \in V$. Then $x \in W = f^{-1}(V) \in \mathcal{M}$. Therefore, $f(W) \subseteq V$. $\square$

Definition 6. *A minimal space $(X, \mathcal{M})$ enjoys the property I if the finite intersection of m-open sets is an m-open set.*

Theorem 2. *Suppose $\mathcal{M}$ is a minimal structure with property I on X, $(X, \mathcal{M})$ is $m - T_2$ space and $f\colon (X, \mathcal{M}) \to (X, \mathcal{M})$ is m-continuous. Then f has a fixed point if and only if for each m-open cover $\{B_\alpha\colon \alpha \in \mathcal{A}\}$ of X there is $x \in X$ and $\alpha \in \mathcal{A}$ such that both x and $f(x)$ lie in B_α.*

Proof. One direction is straightforward. For the converse, suppose that f has no fixed point. Then $x \neq f(x)$ for each $x \in X$. Since f is m-continuous, X is $m-T_2$ and $\mathcal{M}$ has property I, so there is W_x and U_x in $\mathcal{M}$ containing respectively x and $f(x)$ such that $U_x \cap W_x \neq \emptyset$ and $f(W_x) \subseteq U_x$. Now $\{W_x\colon x \in X\}$ is an m-cover of X, so there is $z \in X$ and $x_0 \in X$ such that W_{x_0} contains both z and $f(z)$. Since $z \in W_{x_0}$, so $f(z) \in f(W_{x_0}) \subseteq U_{x_0}$. On the other hand $f(z) \in W_{x_0}$, so $f(z) \in W_{x_0} \cap U_{x_0}$ which is impossible. $\square$

Extending one direction of Theorem 2 is our next task.

Theorem 3. *Suppose $F\colon X \to 2^X$ is a multifunction such that:*

(i) *for each m-open cover $\{W_\alpha\colon \alpha \in \mathcal{A}\}$ for X there are $z \in X$ and $\alpha_0 \in \mathcal{A}$ for which W_{α_0} contain both z and $F(z)$ (i.e., $z \in W_{\alpha_0}$ and $F(z) \subseteq W_{\alpha_0}$);*

(ii) *if $z \notin F(z)$ then there are W_z and U_z of $\mathcal{M}$ with $\{z\} \subseteq W_z$ and $f(z) \subseteq U_z$ such that $U_z \cap W_z = \emptyset$.*

Then F has a fixed point.

Proof. On the contrary, if $z \notin F(z)$ for each $z \in X$ then from (ii) there are W_z and U_z with mentioned properties. Then $\{W_z\colon z \in X\}$ is an m-open cover for X so from (i) there are two elements $x_0, z_0 \in X$ such that both $\{x_0\}$ and $F(x_0)$ are contained in W_{z_0}. On the other hand, $x_0 \in W_{z_0}$ implies that $F(x_0) \subseteq f(W_{z_0}) \subseteq U_{z_0}$. Since $F(x_0) \subseteq W_{z_0}$, so $F(x_0) \subseteq U_{z_0} \cap W_{z_0} = \emptyset$ which is a contradiction. $\square$

Definition 7. *A family $\{A_j\colon j \in J\}$ in $\mathcal{P}(X)$ has the finite intersection property if any its finite subfamily has nonempty intersection.*

Our next result indicates the relation of fixed point and orbits of a multifunction.

Proposition 4. *Suppose $F\colon X \to 2^X$ is a multi-valued map and there is $x_0 \in X$ such that $O(x_0)$ has finite intersection property. Then F has a fixed point if $O(F^2(x)) \subseteq F(x)$ for all $x \in X$.*

Proof. It is easy to see that $F(O(x_0)) \subseteq O(x_0)$, so

$$\mathcal{K} = \big\{A \subseteq O(x_0)\colon A \neq \emptyset,\ F(A) \subseteq A\big\}$$

is a nonempty set. Partially ordered $\mathcal{K}$ by inclusion. Since $O(x_0)$ has finite intersection property, so from Zorn's lemma $\mathcal{K}$ has minimal element, say C. $F(C) \subseteq C$ and $F(F(C)) \subseteq F(C)$ imply that $F(C) = C$. Now, if $u \notin F(u)$ for each $u \in C$, then $u \notin O(F^2(u))$. $F(u) \subseteq F(C) = C$ follows from the fact that $u \in C$, therefore $F^k(u) \subseteq C$ for any nonnegative integer k. $O(F^2(u)) = C$ can be derived from minimality of C. Consequently, $u \in O(F^2(u))$ which is a contradiction. ☐

We are ready to extend a result due to Ciric [10].

Definition 8. *Suppose $(X, \mathcal{M})$ is a minimal space. A subset A of X is said to be have minimal closure finite intersection property if the intersection of elements of any family $\mathcal{A} = \{m - Cl(A_\alpha) \subseteq A\colon \alpha \in I\}$ is nonempty, where any its finite intersection of elements of $\mathcal{A}$ is nonempty.*

Theorem 4. *Suppose $(X, \mathcal{M})$ is a minimal space, $F\colon X \to 2^X$ is a multifunction and*

(i) *there is $x_0 \in X$ such that $m - Cl(O(x_0))$ has minimal closure finite intersection property,*

(ii) $m - Cl(O(F^2(x))) \subseteq F(x)$ *for all* $x \in X$,

(iii) $F(m - Cl(O(x_0))) \subseteq m - Cl(O(x_0))$.

Then F has fixed point.

Proof. Set $\mathcal{K} = \{m - Cl(A) \colon A \subseteq O(x_0),\ F(m - Cl(A)) \subseteq m - Cl(A)\}$ which is a nonempty set by (iii). $\mathcal{K}$ is a partially ordered set by inclusion. Then $\mathcal{K}$ has a maximal element by Zorn's lemma. We denote this maximal element by $m - Cl(B)$. Consequently, $F(m - Cl(B)) \subseteq m - Cl(B)$ and so $F(m - Cl(B)) = m - Cl(B)$. Now if $x \notin F(x)$ for all $x \in B$, then (ii) implies that $x \notin m - Cl(O(F^2(x)))$. But $x \in B$, so $O(F^2(x)) \subseteq B$, thus $m - Cl(O(F^2(x))) \subseteq m - Cl(B)$. Then

$$F\big(m - Cl\big(O\big(F^2(x)\big)\big)\big) \subseteq F^2(x) \subseteq m - Cl\big(O\big(F^2(x)\big)\big).$$

Therefore, $O(F^2(x)) = B$ concludes from maximality of B, so $x \in O(F^2(x))$ which is a contradiction. $\square$

An immediate consequence of Theorem 4 can be state in the following.

Corollary 1. *Suppose (X, τ) is a topological space, $F \colon X \to 2^X$ is a set valued map and*

(i) *there is $x_0 \in X$ such that $\overline{O(x_0)}$ is compact,*

(ii) $\overline{O(F^2(x))} \subseteq F(x)$ *for all* $x \in X$,

(iii) $F(\overline{O(x_0)})) \subseteq \overline{O(x_0)}$.

Then F has fixed point.

Proof. It should be noticed that in topological space minimal closure finite intersection property is equivalent to the compactness. Applying Theorem 4 completes the proof. $\square$

Definition 9. *A function $f \colon X \to X$ is called strongly non-periodic if for every $x \in X$, $x \neq f(x)$ implies $x \notin \bar{O}(f^2(x))$. A function f is said to be orbitally continuous if for each $x, y \in X$, $y = \lim_i f^{n_i}(x)$ implies $f(y) = \lim_i f^{n_i+1}(x)$.*

Corollary 2. [10] *Let X be a topological space and $f \colon X \to X$ be a strongly non-periodic and orbitally continuous mapping. If for some $x_0 \in X$ the set $\bar{O}(x_0)$ is compact, then there exist a cluster point $x \in O(x_0)$ such that $f(x) = x$. Furthermore, if for every $(x, y) \in X \times X$, $x \neq y$ implies $(fx, fy) \neq (x, y)$, then x is a unique fixed point of f in X.*

Proof. Apply Corollary 1 but for single valued map f. $\square$

3 Fixed point property

Definition 10. *$(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are called m-homeomorphic if there exists a bijective function $f\colon X \to Y$ for which f and f^{-1} are m-continuous. In this case, f is called an m-homeomorphism and X and Y are said to be m-homeomorphic.*

Definition 11. *$(X, \mathcal{M})$ is said to have the fixed point property if every m-continuous function $f\colon X \to X$ has a fixed point.*

Example 2. Suppose $X = \{x_1, x_2, x_3\}$ and $\mathcal{M} = \{\emptyset, \{x_1\}, \{x_2\}, X\}$ is a minimal structure on X. In order to show that X has the fixed point property it is enough to show that any function $f\colon (X, \mathcal{M}) \to (X, \mathcal{M})$ which has not fixed point is not m-continuous. If f has not fixed point, then $f(x_3) \neq x_3$ and then $f(x_3) = x_1$ or $f(x_3) = x_2$. Therefore, $x_3 \in f^{-1}(\{x_1\})$ and since $f^{-1}(\{x_1\}) \notin \mathcal{M}$ so $f^{-1}(\{x_1\})$ does not lie in $\mathcal{M}$ or $x_3 \in f^{-1}(\{x_2\}) \notin \mathcal{M}$ which implies that $f\colon (X, \mathcal{M}) \to (X, \mathcal{M})$ is not m-continuous.

Next result shows that fixed point property is invariant under m-homeomorphisms.

Proposition 5. *Suppose X is m-homeomorphic to Y. Then Y has the fixed point property if X has this property too.*

Proof. Suppose $h\colon X \to Y$ is an m-homeomorphism and $g\colon Y \to Y$ is an m-continuous function. Since $h^{-1}ogoh\colon X \to X$ is m-continuous, applying Theorem 1 and fixed point property of X, there exists $x_0 \in X$ in which $h^{-1}ogoh(x_0) = x_0$. Set $y_0 = h(x_0)$, then $h^{-1}og(y_0) = x_0$. Therefore, $g(y_0) = h(x_0) = y_0$ which is required. $\square$

Definition 12. *A subset A of a minimal space $(X, \mathcal{M})$ is a minimal retract of X if there is an m-continuous function $r\colon X \to A$ by $r(a) = a$ for all $a \in A$. In this case, r is called minimal retraction.*

In following we prove the fixed point property for some subset of a set with this property.

Proposition 6. *Suppose $(X, \mathcal{M})$ has the fixed point property and A is a minimal retract of X. Then A has the fixed point property too.*

Proof. Suppose $f\colon A \to A$ is an m-continuous function and $r\colon X \to A$ is a minimal retraction. Consider the following compositions,

$$X \xrightarrow{r} A \xrightarrow{f} A \xrightarrow{i} X,$$

where i is the inclusion map. From Proposition 3, i is an m-continuous function. Since X has the fixed point property, there exists $x_0 \in X$ such that $iofor(x_0) = x_0$ and so $for(x_0) = x_0$. Put $a = r(x_0) \in A$, then $f(a) = x_0$. Consequently, $r(f(a)) = r(x_0)$ which implies that $f(a) = a$. ☐

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