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Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique

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Abstract. A class of semilinear fractional difference equations is introduced in this paper. The fixed point theorem is adopted to find stability conditions for fractional difference equations. The complete solution space is constructed and the contraction mapping is established by use of new equivalent sum equations in form of a discrete Mittag-Leffler function of two parameters. As one of the application, finite-time stability is discussed and compared. Attractivity of fractional difference equations is proved, and Mittag-Leffler stability conditions are provided. Finally, the stability results are applied to fractional discrete-time neural networks with and without delay, which show the fixed point technique's efficiency and convenience.

Keywords: fractional difference equations, fractional discrete-time neural networks, Mittag-Leffler stability.

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1 Introduction

Recurrent neural networks of discrete time (RNN) are described by the following difference equation:

$$x(k+1) = -Ax(k) + Bf\big(x(k)\big) + I, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, the diagonal matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$ with $0 < a_i$ is the state feedback coefficients, $B = (b_{ij})_{n\times n}$ is the network's interconnection matrix, $f(x)$ is the activation function, and I denotes the constant external input.

Equation (1) can be considered as a discrete analog of the recurrent neural network

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + I_i, \quad i = 1, 2, \dots, n.$$

In order to use the past information, two classes of memory models are suggested.

One is the neural network with delay. The time delay τ is introduced into Eq. (1) as

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t-\tau)\big) + I_i, \quad i = 1, 2, \dots, n,$$

or a neural active function $g(t)$ with delay is suggested:

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(x_j(t-\tau)\big) + I_i,$$

which is a recurrent neural network. Both of them belong to short-time memory models.

The fractional derivative also holds the memory effects, and the fractional order α depicts the history dependence on the past states. The fractional neural network (FNN) was proposed:

$${}_{t_0}^{C}D_t^{\alpha} x_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j(t)\big) + I_i, \quad i = 1, 2, \dots, n, \tag{2}$$

where ${}_{t_0}^{C}D_t^{\alpha}$ denotes the Caputo derivative [32]. One can see that the fractional neural network has a long memory from the starting point $t = t_0$. This feature can fully use all of the past information for better control or predictions. The FNN has been extensively investigated in [8, 19, 22, 23, 28, 31, 33–37, 41, 42]. Naturally, one question may arise whether there is a discrete analog of the fractional neural network (2). This is not only of theoretical value, but also of great interests in developing discrete-time neural network with long-memory effects. This is the main purpose of this study. Generally, two methods can be applied to derive fractional discrete-time systems. One is numerical discretization of (2), for example, by finite difference methods [21, 27]. Unfortunately, it is well known that the numerical discretization will readily result in cumulate errors even in short-term domains. It becomes difficult in the real-world applications for long-term issues in the practical view. This paper aims to address this problem and propose a new kind of discrete-time neural network by use of discrete fractional calculus on an isolated time scale [1–6, 9, 24].

Many efforts have been made to the stability theory of fractional differential equations. Lyapunov direct method was proposed by Li et al. [30]. The method was given based on the comparison theorem and Mittag-Leffler function's asymptotic. The main idea is to construct Lyapunov function whose fractional derivative is negative definite, and this implies the fractional systems' asymptotic stability. The method has gained much success in stability analysis, and we consider the method for fractional difference equations in [7, 38–40]. However, for fractional delay equations, it is still an open problem to construct Lyapunov function due to the complicated structures of fractional operators, or the existing technique is suitable for some special fractional delay differential equations.

Frequency analysis is another often used method. Cermak et al. investigated stability of linear fractional delay equations as well as difference equations [15, 16]. In fact, asymptotic stability can be analyzed by the corresponding Jacobian linearization equation. The characteristic root equation is established by Laplace or Z-transform. Negative real part areas of characteristic roots are derived on critical surfaces. This method is useful for constant coefficient differential equations. Some applications of stability results have been considered in chaos synchronization and image encryption.

Burton et al. introduced fixed point technique for stability analysis of differential equations [11, 12] and fractional ones [10, 13, 14]. By use of the technique, Chen et al. investigated asymptotic stability results of fractional difference equations in [17, 18]. Particularly, the method is very successful for semilinear differential equations where a resolvent equation is used. The solution space is constructed according to the stability conditions. Various fixed point theorems are adopted for different existence results. Mittag-Leffler stability is an important concept, which often used in fractional neural network with delay. However, there is less effort contributed to this topic within fixed point theorems.

This paper is structured as follows: In Section 2, some preliminaries of fractional differences and sum are revisited. Section 3 introduces an equivalent fractional sum equation, which constructs a mapping for stability analysis. Finite-time stability is compared by use of two fractional sum equations. The new sum equation containing the Mittag-Leffler kernel function shows the efficiency. Furthermore, in Section 4, the applications to discrete-time neural network are considered, and some numerical examples are illustrated. Finally, we give some perspective of this study and arrive at the conclusion.

2 Preliminaries

We use the following definitions in this paper.

Definition 1. (See [6, 9, 24].) Let $\mathbb{N}_a := \{a,\, a+1,\, a+2,\, \dots\}$. $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu$ be given. The νth-order sum is given by

$$\Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \bigl(t - \sigma(s)\bigr)^{(\nu-1)} u(s), \quad t \in \mathbb{N}_{a+\nu},$$

where $\sigma(s) = s + 1$, $a \in \mathbb{R}$, and $t^{(\nu)}$ is the falling factorial functional defined by

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}.$$

Definition 2. (See [6, 9, 24].) Let $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu$ be given. The νth-order Riemann–Liouville difference is given by

$$\Delta_a^\nu u(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} \bigl(t - \sigma(s)\bigr)^{(-\nu-1)} u(s), \quad t \in \mathbb{N}_{a+m-\nu},$$

$\sigma(s) = s + 1$, $a \in \mathbb{R}$, $m = [\nu] + 1$.

Definition 3. (See [1, 5, 24].) For $u(t)$ defined on $\mathbb{N}_a$ and $0 < \nu$, $\nu \notin \mathbb{N}$, the Caputo difference is defined by

$${}^C\Delta_a^\nu u(t) := \Delta_a^{-(m-\nu)} \Delta^m u(t), \quad t \in \mathbb{N}_{a+m-\nu},$$

$m = [\nu] + 1$, where $\Delta u(t) = u(t+1) - u(t)$. For $\nu = m$, ${}^C\Delta_a^\nu u(t) := \Delta^m u(t)$.

For more details of discrete fractional calculus, readers are suggested to read the monograph [24]. We consider discrete fractional calculus and introduce generalized neural network with memory

$${}^C\Delta_a^\nu x(t) = -Ax(t+\nu) + Bf\bigl(x(t+\nu)\bigr), \quad t \in \mathbb{N}_{a+1-\nu}, \tag{3}$$

where $A = \operatorname{diag}[a_1, \dots, a_n]$, and f is a continuous function with respect to x. It is equivalent to

$$x(a+k) = x(a) + \frac{1}{\Gamma(\nu)} \sum_{j=0}^{k} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} \bigl(-Ax(a+j) + Bf\bigl(x(a+j)\bigr)\bigr),$$

where $k \in \mathbb{N}_1$. We revisit some basics in stability theory in the discrete fractional calculus. Assume that Eq. (3) has a zero solution. Let $|\cdot|$ be the norm l_1 of $\mathbb{R}^n$. Considering the matrix $C = (c_{ij})_{n\times n}$, the matrix norm is used as $|C| = \max_{1\leqslant j\leqslant n} \sum_{i=1}^n |c_{ij}|$, accordingly. Denote by ℓ_a^∞ the set of all real sequences $x = \{x(t)\}_{t=a}^\infty$ from the starting point $t = a$. The space is endowed with the supremum norm $\|x\| = \sup_{t\in\mathbb{N}_a} |x(t)|$. ℓ_a^∞ is a Banach space (for more details, see [20]).

Definition 4. The zero solution of Eq. (3) is said to be

(i) stable if for any $\varepsilon > 0$ and $t_0 = a \in \mathbb{R}$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\bigl|x(t, x_0, t_0)\bigr| < \varepsilon, \quad t \geqslant t_0,$$

for $|x_0| \leqslant \delta(t_0, \varepsilon)$;

(ii) attractive if there exists $\eta(t_0) > 0$ such that $|x_0| \leqslant \eta$ implies

$$\lim_{t\to\infty} x(t, x_0, t_0) = 0;$$

(iii) asymptotically stable if it is stable and attractive.

Lemma 1. (*See* [7].) *The equation*

$$\begin{aligned} &{}^{C}\Delta_a^{\nu} u(t) = \lambda u(t+\nu), \quad t \in \mathbb{N}_{a+1-\nu}, \\ &u(a) = 1, \end{aligned} \tag{4}$$

has a unique solution

$$u(t) = e_\nu(\lambda,\, t-a),$$

where $e_\nu(\lambda,\, t-a) := \sum_{k=0}^{\infty} \lambda^k (t-a-1+k\nu)^{(k\nu)}/\Gamma(k\nu+1)$.

Remark 1. We assume $|\lambda| < 1$ for the convergence of the discrete Mittag-Leffler function $e_\nu(\lambda,\, t-a)$. For $t, t_1, t_2 \in \mathbb{N}_a$, some other properties hold:

(i) the Mittag-Leffler function is positive [7]:

$$e_\nu(\lambda,\, t-a) > 0, \quad t \geqslant a;$$

(ii) the Mittag-Leffler function is asymptotically stable [7]:

$$e_\nu(\lambda,\, t-a) \to 0, \quad t \to +\infty,$$

where $-1 < \lambda < 0$;

(iii) the discrete Mittag-Leffler function of two parameter is defined by

$$e_{\nu,\nu}(\lambda,\, t-a) = \sum_{k=0}^{\infty} \lambda^k \frac{(t-a+k\nu)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)}.$$

Remark 2. Besides, we note the following results hold:

$$\sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(\lambda,\, t-\sigma(s)\bigr) = \frac{e_\nu(\lambda,\, t-a) - 1}{\lambda},$$

and, when $-1 < \lambda < 0$,

$$\begin{aligned} &e_{\nu,\nu}(\lambda,\, t-a+\nu-1) = \frac{1}{\lambda}\Delta e_\nu(\lambda,\, t-a) > 0 \quad \text{if } t \in \mathbb{N}_{a+1}, \\ &e_{\nu,\nu}(\lambda,\, t-a+\nu-1) \to 0 \quad \text{if } t \to +\infty. \end{aligned}$$

These lemmas and remarks are useful to analyze stability conditions by the fixed point technique in the rest.

Definition 5 [Mittag-Leffler stability]. (3) is said to be Mittag-Leffler stable if $x(t)$ satisfies

$$\bigl|x(t)\bigr| \leqslant m\bigl(x(t_0)\bigr) e_\nu\bigl(\lambda^*, (t-a)\bigr), \quad t \in \mathbb{N}_{a+1},$$

where $-1 < \lambda^* < 0$, $m(x(t_0))$ is locally Lipschitz on $D \in \mathbb{R}$, $m(x) \geqslant 0$ and $m(\mathbf{0}) = 0$.

3 A new equivalent sum equation

3.1 Picard's method

We consider linear fractional difference equations with nonhomogeneous term. Generally, we can use Laplace transform or Z-transform to get fractional sum equations. In this section, we adopt Picard's method to derive a fractional sum equation.

Lemma 2. *Consider the nonhomogeneous equation*

$$\begin{aligned} &{}^{C}\Delta_a^{\nu} y(t) = \lambda y(t+\nu) + h(t+\nu), \quad t \in \mathbb{N}_{a+1-\nu}, \\ &y(a) = y_a, \end{aligned} \tag{5}$$

where λ is a constant, and $|\lambda| < 1$. The equation has an exact solution

$$y(t) = y(a) e_{\nu}(\lambda, t-a) + \sum_{s=a+1-\nu}^{t-\nu} \bigl[e_{\nu,\nu}\bigl(\lambda, t-\sigma(s)\bigr) h(s+\nu)\bigr], \quad t \in \mathbb{N}_{a+1}.$$

Proof. When using the fractional sum equation of (5), we have $t \in \mathbb{N}_a$. By Picard's method, we get the successive iteration as

$$y_{m+1}(t) = y_0 + \lambda \Delta_{a+1-\nu}^{-\nu} y_m(t+\nu), \quad m = 0, 1, 2, \ldots,$$

where $y_0 = y(a) + \Delta_{a+1-\nu}^{-\nu} h(t+\nu)$.

For $m = 0$, we get

$$\begin{aligned} y_1(t) &= y_0 + \lambda \Delta_{a+1-\nu}^{-\nu} y_0(t+\nu) \\ &= y(a) + \lambda \Delta_{a+1-\nu}^{-\nu} y(a) + \Delta_{a+1-\nu}^{-\nu} h(t+\nu) + \lambda \Delta_{a+1-\nu}^{-\nu} \Delta_{a+1-\nu}^{-\nu} h(t+\nu) \\ &= y(a) + y(a) \frac{\lambda (t-a-1+\nu)^{(\nu)}}{\Gamma(\nu+1)} + \Delta_{a+1-\nu}^{-\nu} h(t+\nu) \\ &\quad + \lambda \Delta_{a+1-\nu}^{-\nu} \Delta_{a+1-\nu}^{-\nu} h(t+\nu). \end{aligned}$$

For $m = 1$, we get

$$\begin{aligned} y_2(t) &= y_0 + \lambda \Delta_{a+1-\nu}^{-\nu} y_1(t+\nu) \\ &= y(a) \biggl(1 + \frac{\lambda (t-a-1+\nu)^{(\nu)}}{\Gamma(\nu+1)} + \frac{\lambda^2 (t-a-1+2\nu)^{(2\nu)}}{\Gamma(2\nu+1)}\biggr) \\ &\quad + \sum_{k=1}^{3} \lambda^{k-1} \underbrace{\Delta_{a+1-\nu}^{-\nu} \cdots \Delta_{a+1-\nu}^{-\nu}}_{k \text{ factors}} h(t+\nu). \end{aligned}$$

More generally, we have

$$y_m(t) = y(a)\left(1 + \sum_{k=1}^{m} \frac{\lambda^k (t-a-1+k\nu)^{(k\nu)}}{\Gamma(k\nu+1)}\right) + \sum_{k=1}^{m+1} \lambda^{k-1} \underbrace{\Delta_{a+1-\nu}^{-\nu} \cdots \Delta_{a+1-\nu}^{-\nu}}_{k \text{ factors}} h(t+\nu). \tag{6}$$

In order to simplify the second term in the r.h.s. of (6), we only need to discuss the $k = 2$ since the general case of arbitrary k can be done by induction. After interchange of the order of summation, we derive that

$$\begin{aligned}
&\Delta_{a+1-\nu}^{-\nu} \Delta_{a+1-\nu}^{-\nu} h(t+\nu) \\
&\quad = \frac{1}{\Gamma(\nu)} \sum_{r=a+1-\nu}^{t-\nu} \big(t-\sigma(r)\big)^{(\nu-1)} \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{r} \big(r+\nu-\sigma(s)\big)^{(\nu-1)} h(s+\nu) \\
&\quad = \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} \frac{1}{\Gamma(\nu)} \sum_{r=s}^{t-\nu} \big(t-\sigma(r)\big)^{(\nu-1)} \big(r+\nu-\sigma(s)\big)^{(\nu-1)} h(s+\nu) \\
&\quad = \sum_{s=a+1-\nu}^{t-\nu} \Delta_{a^*+\nu-1}^{-\nu} \frac{(t-a^*)^{(\nu-1)}}{\Gamma(\nu)} h(s+\nu), \quad a^* = s+1-\nu.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\Delta_{a+1-\nu}^{-\nu} \Delta_{a+1-\nu}^{-\nu} h(t+\nu) \\
&\quad = \sum_{s=a+1-\nu}^{t-\nu} \frac{(t-a^*)^{(2\nu-1)}}{\Gamma(2\nu)} h(s+\nu) = \sum_{s=a+1-\nu}^{t-\nu} \frac{(t-\sigma(s)+\nu)^{(2\nu-1)}}{\Gamma(2\nu)} h(s+\nu).
\end{aligned}$$

Repeat the above procedure, and we arrive at

$$\underbrace{\Delta_{a+1-\nu}^{-\nu} \cdots \Delta_{a+1-\nu}^{-\nu}}_{k \text{ factors}} h(t+\nu) = \sum_{s=a+1-\nu}^{t-\nu} \frac{(t-\sigma(s)+(k-1)\nu)^{(k\nu-1)}}{\Gamma(k\nu)} h(s+\nu).$$

Let $m \to \infty$ in (6). Then, for $t \in \mathbb{N}_{a+1}$,

$$y(t) = y(a)\left(1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t-a-1+k\nu)^{(k\nu)}}{\Gamma(k\nu+1)}\right) + \sum_{k=1}^{\infty} \lambda^{k-1} \sum_{s=a+1-\nu}^{t-\nu} \frac{(t-\sigma(s)+(k-1)\nu)^{(k\nu-1)}}{\Gamma(k\nu)} h(s+\nu)$$

$$
\begin{aligned}
&= y(a)e_\nu(\lambda,\, t-a) + \sum_{s=a+1-\nu}^{t-\nu} h(s+\nu) \sum_{k=0}^{\infty} \lambda^k \frac{(t-\sigma(s)+k\nu)^{(k\nu+\nu-1)}}{\Gamma(k\nu+\nu)} \\
&= y(a)e_\nu(\lambda,\, t-a) + \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(\lambda,\, t-\sigma(s)\big)h(s+\nu). \qquad \square
\end{aligned}
$$

Theorem 1. *Equation* (3) *is equivalent to the fractional sum equation*

$$
\begin{aligned}
x(t) &= e_\nu(-A,\, t-a)x(a) \\
&\quad + \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)Bf\big(t+\nu,\, x(t+\nu)\big), \quad t \in \mathbb{N}_{a+1}, \qquad (7)
\end{aligned}
$$

where the matrix Mittag-Leffler function is defined as

$$
e_{\nu,\nu}(-A,\, t-a) := \begin{pmatrix} e_{\nu,\nu}(-a_1,\, t-a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_{\nu,\nu}(-a_n,\, t-a) \end{pmatrix}.
$$

3.2 Application to finite-time stability

Let us revisit the definition of finite-time stability. Assume that T is a positive integer and $J = \{a,\, a+1,\, a+2,\, \dots,\, T\}$.

Definition 6. (See [26].) System (1) is finite-time stable w.r.t. $\{\delta, \varepsilon, J\}$, $\delta < \varepsilon$, if and only if $|\phi| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \in J$.

Concerning the finite-time stability of the following fractional difference equation

$$
{}^C\!\Delta_a^\nu y(t) = \lambda y(t+\nu) + \omega g\big(y(t+\nu)\big), \quad t \in \{a+1-\nu, \dots, T+1-\nu\}, \qquad (8)
$$

where $-1 < \lambda < 0$, fractional sum equations should be used. We can have two kinds:

$$
y(t) = y(a) + \Delta_{a+1-\nu}^{-\nu}\big(\lambda y(t+\nu) + \omega g\big(y(t+\nu)\big)\big),
$$

$|g(y(t+\nu))| \leqslant \widetilde{M}$, $0 < \widetilde{M}$, which is directly derived by taking the fractional sum to both sides of (8), and

$$
y(t) = y(a)e_\nu(\lambda,\, t-a) + \omega \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}(\lambda,\, t-s)g\big(y(s+\nu)\big)
$$

derived from (7). Hence, for $t \in \mathbb{N}_{a+1}$, we can have two inequalities

$$
\begin{aligned}
\big|y(t)\big| &\leqslant \big|y(a)\big| + \big|\Delta_{a+1-\nu}^{-\nu}\big(\lambda y(t+\nu) + \omega g\big(y(t+\nu)\big)\big)\big| \\
&\leqslant \big|y(a)\big| + |\lambda|\Delta_{a+1-\nu}^{-\nu}\big|y(t+\nu)\big| + |\omega|\Delta_{a+1-\nu}^{-\nu}\big|g\big(x(t+\nu)\big)\big|
\end{aligned}
$$

from which we can obtain

$$\max_{t\in\{a+1,\dots,T\}} \big|y(t)\big| \leqslant \frac{|y(a)| + \frac{|\omega|\widetilde{M}(t-a-1+\nu)^{(\nu)}}{\Gamma(\nu+1)}}{1 - \frac{|\lambda|\widetilde{M}(t-a-1+\nu)^{(\nu)}}{\Gamma(\nu+1)}}.$$

Consider the condition $-1 < \lambda < 0$ and $0 < e_\nu(\lambda, t-a) < 1$ so that, for $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned}\big|y(t)\big| &\leqslant \big|y(a)e_\nu(\lambda, t-a)\big| + \left|\sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(\lambda, t-\sigma(s)\big)\omega g\big(x(t+\nu)\big)\right| \\ &\leqslant \big|y(a)\big|e_\nu(\lambda, t-a) + \left|\omega\left|\sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(\lambda, t-\sigma(s)\big)\right| g\big(x(t+\nu)\big)\right|\end{aligned}$$

and

$$\big|y(t)\big| \leqslant \big|y(a)\big|e_\nu(\lambda, t-a) + |\omega|\widetilde{M}\frac{e_\nu(\lambda, t-a) - 1}{\lambda}.$$

It is clear that the estimation of $|y(t)|$ derived from (7) has no restriction of $|\lambda|$ and becomes more convenient and accurate.

4 Mittag-Leffler stability

Let us give some hypotheses for Mittag-Leffler stability analysis.

(H1) There exist constants $L > 0$ and $M > 0$ such that

$$f(\mathbf{0}) = 0, \qquad \big|f(x) - f(y)\big| \leqslant L|x-y|$$

for all $x, y \in C_M := \{u \in \ell_a^\infty\colon \|u\| \leqslant M,\ t \in \mathbb{N}_a\}$.

(H2) There exists $0 < r < 1$ such that

$$L \sum_{s=a+1-\nu}^{t-\nu} \big|e_{\nu,\nu}\big(-A, t-\sigma(s)\big)B\big| < r.$$

(H3) Let $\lambda = \max(-a_1, -a_2, \dots, -a_n)$ and $0 < a_i < 1$. λ, L and the coefficient matrix B satisfy

$$-1 < \lambda + L|B| < 0.$$

Lemma 3. *The following inequality holds*:

$$u(t) \leqslant u(a)e_\nu(\omega, t-a) + l \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(\omega, t-\sigma(s)\big)u(s+\nu), \quad t \in \mathbb{N}_{a+1},$$

$|\omega| < 1$, $l > 0$, $|\omega + l| < 1$, *then*

$$u(t) \leqslant u(a)e_\nu(\omega + l, t-a).$$

Proof. We use a nonnegative discrete function $m(t)$ to construct the following equality:

$$u(t) = u(a)e_\nu(\omega,\, t-a) + l \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(\omega,\, t-\sigma(s)\bigr)u(s+\nu) - m(t).$$

We take the discrete Laplace transform $\mathcal{L}_{a+1}$ [25, 29] to both sides. Then we arrive at

$$\mathcal{L}_{a+1}[u] = \frac{u(a)s^{\nu-1}}{s^\nu - (\omega+l)(s+1)^\nu} - \mathcal{L}_{a+1}[m] - \frac{l(s+1)^\nu}{s^\nu - (\omega+l)(s+1)^\nu}\mathcal{L}_{a+1}[m].$$

It follows that

$$u(t) = u(a)e_\nu(\omega+l,\, t-a) - m(t) - l \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(\omega+l,\, t-\sigma(s)\bigr)m(s+\nu)$$

from which we obtain

$$u(t) \leqslant u(a)e_\nu(\omega+l,\, t-a)$$

and complete the proof. □

Theorem 2. *If* (H1), (H2) *and* (H3) *hold, system* (3) *is asymptotically stable.*

Proof. Define the space

$$S = \Bigl\{x\colon\, x \in C_M,\, \lim_{t\to+\infty} x(t) = 0\Bigr\}$$

with the distance $\rho(x,y) = \|x-y\|$, and (S,ρ) is a complete space. We assume $|x(a)| < \delta = (1-r)M$. In the space S, define the contraction mapping $T\colon S \to \ell_a^\infty$, then

$$\begin{aligned}(Tx)(t) &= e_\nu(-A,\, t-a)x(a)\\ &\quad + \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)Bf\bigl(x(s+\nu)\bigr), \quad t \in \mathbb{N}_{a+1}.\end{aligned}$$

It is clear that T is continuous on S since f is continuous with respect to x.

Besides, if $\|x\| \leqslant M$ and $t \in \mathbb{N}_{a+1}$, then $\|(Tx)\| \leqslant M$. In fact,

$$\begin{aligned}\bigl|(Tx)(t)\bigr| &\leqslant \bigl|e_\nu(-A,\, t-a)x(a)\bigr| + \sum_{s=a+1-\nu}^{t-\nu} \bigl|e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)B\bigr|\bigl|f\bigl(x(s+\nu)\bigr)\bigr|\\ &\leqslant \bigl|e_\nu(-A,\, t-a)x(a)\bigr| + L\sum_{s=a+1-\nu}^{t-\nu} \bigl|e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)B\bigr|\bigl|x(s+\nu)\bigr|\\ &\leqslant \bigl|e_\nu(-A,\, t-a)x(a)\bigr| + ML\sum_{s=a+1-\nu}^{t-\nu} \bigl|e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)B\bigr|\\ &\leqslant \delta + rM \leqslant M.\end{aligned}$$

Furthermore, by use of the properties of discrete Mittag-Leffler functions in Remark 1, we can claim that

$$\lim_{t\to+\infty}(Tx)(t) = 0.$$

In fact, we have

$$\begin{aligned}\big|(Tx)(t)\big| &\leqslant \big|e_\nu(-A,\, t-a)x(a)\big| + \sum_{s=a+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)B\big|\big|f\big(x(s+\nu)\big)\big| \\ &\leqslant \big|e_\nu(-A,\, t-a)x(a)\big| + L|B|\sum_{s=a+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)\big|\big|x(s+\nu)\big|,\end{aligned}$$

where the first term tends to zero for $t \to +\infty$. Since $e_{\nu,\nu}(\lambda,\, t-a)$ and $x(t) \to 0$ for $t \to +\infty$, we can select N enough large such that the second term is decoupled into

$$\begin{aligned}&\sum_{s=a+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)\big|\big|x(s+\nu)\big| \\ &\quad= \sum_{s=a+1-\nu}^{a+N-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)\big|\big|x(s+\nu)\big| \\ &\qquad+ \sum_{s=a+N+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)\big|\big|x(s+\nu)\big| \\ &\quad\to 0.\end{aligned}$$

Finally, for arbitrary $x,\ y \in S$, we have

$$\begin{aligned}&\big|(Tx)(t) - (Ty)(t)\big| \\ &\quad= \left|\sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)B\big(f\big(x(s+\nu)\big) - f\big(y(s+\nu)\big)\big)\right| \\ &\quad\leqslant L\sum_{s=a+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)B\big|\big|x(s+\nu)-y(s+\nu)\big| \\ &\quad\leqslant L\sum_{s=a+1-\nu}^{t-\nu}\big|e_{\nu,\nu}\big(-A,\, t-\sigma(s)\big)B\big|\|x-y\| = r\|x-y\|.\end{aligned}$$

T is a contraction mapping if $0 < r < 1$. By Banach fixed point theorem, the mapping T has a unique fixed point $x(t)$ in S, which is also the unique solution of (3) and $\lim_{t\to+\infty} x(t) = 0$. That means the zero solution is attractive, and it is easy to prove that the zero solution is also stable, which completes the proof. □

Theorem 3. *If* (H1), (H2) *and* (H3) *hold, system* (3) *is Mittag-Leffler stable.*

Proof. From the sum equation of fractional order (7), we get

$$\begin{aligned}\left|x(t)\right| &\leqslant \left|e_\nu(-A,\, t-a)x(a)\right| + \left|\sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)Bf\bigl(x(s+\nu)\bigr)\right| \\ &\leqslant \left|x(a)\right|\left|e_\nu(-A,\, t-a)\right| + L|B| \sum_{s=a+1-\nu}^{t-\nu} \left|e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)\right|\left|x(s+\nu)\right|.\end{aligned}$$

By use of Lemma 3, it leads to

$$\left|x(t)\right| \leqslant |x_0| e_\nu\bigl(\lambda + L|B|,\, t-a\bigr).$$

The system is Mittag-Leffler stable. □

Theorem 4. *The following fractional difference equation with delay*

$$^C\Delta_a^\nu x(t) = -Ax(t+\nu) + Bf\bigl(x(t+\nu-1)\bigr), \quad t \in \mathbb{N}_{a+1-\nu}$$

is asymptotically stable if (H1), (H2) *and* (H3) *hold.*

We proposed Lyapunov direction method for the fractional difference equation (3) in [7,40]. But it fails here since there exists a delay term. In order to address this problem, we can construct Lyapunov functions as that in fractional differential equations [36], i.e., the comparison principle, discrete Mittag-Leffler function's positivity and monotonicity. On the other hand, we can adopt the fixed point technique and it becomes much easier. In fact, the space can be defined as

$$S = \Bigl\{x\colon \|x_t\| \leqslant M,\ \lim_{t\to+\infty} x(t) = 0,\ x \in \ell_a^\infty,\ t \in \mathbb{N}_{a+1}\Bigr\},$$

where $x_t = x(t-1)$. We can obtain the same conditions for asymptotic stability. We construct the contraction map as

$$\begin{aligned}(Tx)(t) &= e_\nu(-A,\, t-a)x(a) \\ &\quad + \sum_{s=a+1-\nu}^{t-\nu} e_{\nu,\nu}\bigl(-A,\, t-\sigma(s)\bigr)Bf\bigl(x(t+\nu-1)\bigr), \quad t \in \mathbb{N}_{a+1}.\end{aligned}$$

We can prove that $\lim_{t\to+\infty} Tx(t) = 0$ and the rest is similar as that in Theorem 2.

As a special case, we can get the following theorem, which can be considered as a discrete analogy of the fractional delay differential equation for continuous-time neural network

$$\begin{aligned}&{}_{t_0}^{C}D_t^\alpha z(t) = -\lambda_1 z(t) + \lambda_2 z(t-\tau), \quad t \in (t_0, +\infty), \\ &z(t) = \phi(t), \quad t \in [t_0-\tau,\, t_0],\end{aligned}$$

where $0 < \alpha \leqslant 1$, and ${}_{t_0}^{C}D_t^\alpha$ is the Caputo derivative of the differentiable function $z(t)$ and τ is the time delay.

Theorem 5. *Let $z(t)$: $\mathbb{N}_a \to \mathbb{R}$. The following fractional difference equation with delay*

$$\begin{aligned}&{}^C\Delta_a^\nu z(t) = -\lambda_1 z(t+\nu) + \lambda_2 z(t+\nu-1), \quad t \in \mathbb{N}_{a+1-\nu},\\ &z(a) = z_a,\end{aligned}$$

is asymptotically stable if $-1 < \lambda_2 - \lambda_1 < 0$, $0 < \lambda_2$ *and* $0 < \lambda_1 < 1$.

5 Numerical examples

We apply the stability theory to fractional discrete-time neural networks through the following examples.

Example 1. Consider the following neural network of fractional discrete time

$$\begin{aligned}{}^C\Delta_a^\nu x_1(t) &= -a_1 x_1(t+\nu) + b_{11}\tanh\bigl(x_1(t+\nu)\bigr) + b_{12}\tanh\bigl(x_2(t+\nu)\bigr),\\ {}^C\Delta_a^\nu x_2(t) &= -a_2 x_2(t+\nu) + b_{21}\tanh\bigl(x_1(t+\nu)\bigr) + b_{22}\tanh\bigl(x_2(t+\nu)\bigr),\end{aligned} \tag{9}$$

where $t \in \mathbb{N}_{a+1-\nu}$, $0 < \nu \leqslant 1$, $x_1(a) = 0.2$ and $x_2(a) = -0.3$.

We list the numerical formulae for the solutions for $k \geqslant 1$

$$\begin{aligned}x_1(k) = x_1(0) + \frac{1}{\Gamma(\nu)}\sum_{j=1}^{k}\frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)}\bigl[&-a_1x_1(j) + b_{11}\tanh\bigl(x_1(j)\bigr)\\ &+ b_{12}\tanh\bigl(x_2(j)\bigr)\bigr],\\ x_2(k) = x_2(0) + \frac{1}{\Gamma(\nu)}\sum_{j=1}^{k}\frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)}\bigl[&-a_2x_2(j) + b_{21}\tanh\bigl(x_1(j)\bigr)\\ &+ b_{22}\tanh\bigl(x_2(j)\bigr)\bigr].\end{aligned}$$

We use the following parameters: $a = 0$, $\nu = 0.9$, $L = 1$, $k = 80$ and $a_1 = 0.8$, $a_2 = 0.85$, $b_{11} = 0.1$, $b_{12} = 0.11$, $b_{21} = 0.12$, $b_{22} = 0.13$. We can check these parameters satisfy Theorem 2. Figure 1 illustrates the behavior of $x_1(t)$ and $x_2(t)$, respectively. Each tends to zero for $t \to +\infty$. Figure 2 shows that system (9) is Mittag-Leffler stable.

Example 2. Consider the delay case

$$\begin{aligned}{}^C\Delta_a^\nu x_1(t) &= -a_1x_1(t+\nu) + b_{11}\tanh\bigl(x_1(t+\nu-1)\bigr)\\ &\quad + b_{12}\tanh\bigl(x_2(t+\nu-1)\bigr),\\ {}^C\Delta_a^\nu x_2(t) &= -a_2x_2(t+\nu) + b_{21}\tanh\bigl(x_1(t+\nu-1)\bigr)\\ &\quad + b_{22}\tanh\bigl(x_2(t+\nu-1)\bigr),\end{aligned} \tag{10}$$

where $t \in \mathbb{N}_{a+1-\nu}$, $0 < \nu \leqslant 1$ and the numerical formula is derived as

$$\begin{aligned}x_1(k) = x_1(0) + \frac{1}{\Gamma(\nu)}\sum_{j=1}^{k}\frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)}\bigl[&-a_1x_1(j) + b_{11}\tanh\bigl(x_1(j-1)\bigr)\\ &+ b_{12}\tanh\bigl(x_2(j-1)\bigr)\bigr],\end{aligned}$$

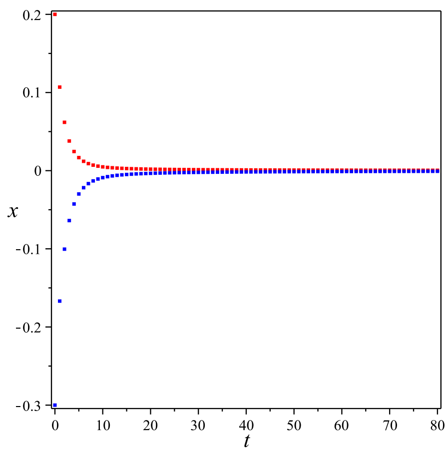

Figure 1. Numerical solutions of (9): $x_1(t)$ (the red) and $x_2(t)$ (the blue).

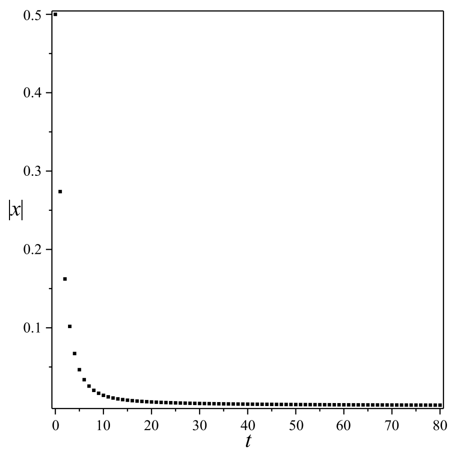

Figure 2. (9) is Mittag-Leffler stable.

$$x_2(k) = x_2(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{k} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} \bigl[-a_2 x_2(j) + b_{21} \tanh\bigl(x_1(j-1)\bigr) + b_{22} \tanh\bigl(x_2(j-1)\bigr)\bigr].$$

We obtain a recurrence relationship

$$\begin{aligned} x_1(k) &= \frac{1}{1+a_1} x_1(0) - \frac{a_1}{1+a_1} \frac{1}{\Gamma(\nu)} \sum_{j=1}^{k-1} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} x_1(j) \\ &\quad + \frac{1}{1+a_1} \frac{1}{\Gamma(\nu)} \sum_{j=1}^{k} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} \bigl[b_{11} \tanh\bigl(x_1(j-1)\bigr) + b_{12} \tanh\bigl(x_2(j-1)\bigr)\bigr], \\ x_2(k) &= \frac{1}{1+a_2} x_2(0) - \frac{a_2}{1+a_2} \frac{1}{\Gamma(\nu)} \sum_{j=1}^{k-1} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} x_2(j) \\ &\quad + \frac{1}{1+a_2} \frac{1}{\Gamma(\nu)} \sum_{j=1}^{k} \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} \bigl[b_{21} \tanh\bigl(x_1(j-1)\bigr) + b_{22} \tanh\bigl(x_2(j-1)\bigr)\bigr], \end{aligned} \tag{11}$$

and the numerical results are illustrated in Figs. 3 and 4 from which we can see system (10) is asymptotically stable.

We need to point out, Eq. (9) is an implicit system whose both sides contain $x(k)$. Equation (10) can be reduced to an explicit one (11) with which we can obtain the exact numerical value of $x(t)$ involving no numerical errors. This is very important for long-term calculation and can depict the delay dynamics more accurately.

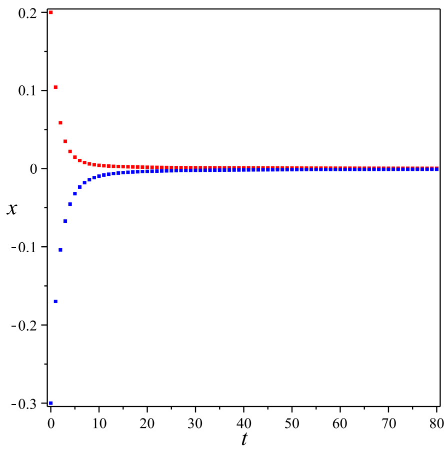

Figure 3. Numerical solutions of (10): $x_1(t)$ (the red) and $x_2(t)$ (the blue).

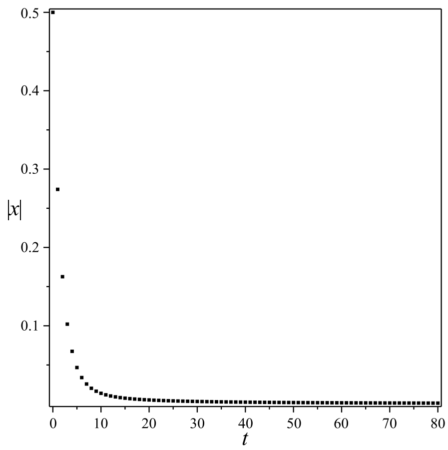

Figure 4. (10) is asymptotically stable.

6 Conclusions

Fractional difference equations can be considered a class of generalized difference equations. On the other hand, we consider the extensive applications of recurrent neural network. The fractional discrete-time systems are paid much attention less results than the continuous-time case. This paper introduces a kind of fractional discrete-time neural network and investigates stability, which is defined in a new sense within a discrete Mittag-Leffler function. The conditions for the stability conditions are given and numerical example is provided to support the analysis.

We give our view on other possible applications:

1. Since we adopt the discrete fractional calculus, which is defined in form of a finite sum, memory effects are exact such the application to big data and long-term models becomes possible. Particularly for the following fractional discrete-time neural networks:

$$ {}^{C}\Delta_a^{\nu} x(t) = -Ax(t+\nu-1) + Bf\big(x(t+\nu-1)\big), \quad t \in \mathbb{N}_{a+1-\nu}. \tag{12}$$

   In fact, for the linear one

$$ {}^{C}\Delta_a^{\nu} x(t) = \lambda x(t+\nu-1), \quad 0 < \nu \leqslant 1,\ t \in \mathbb{N}_{a+1-\nu}, $$
$$ x(a) = x_a > 0, $$

   the exact solution $x(t) > 0$ if $\lambda > -\nu$, and a different stability condition from that of (4) holds. We will consider these in the nearest future and show the convenience by fixed point theorems.

2. The fixed point technique is adopted in this study. This technique well illustrates its efficiency. In fact, we also can adopt the Lyapunov method and construct Lyapunov functions. However, it becomes much complicated since we need to have

the negative estimation of the fractional difference of the V functions. Particularly, it is a challenging work to consider the fractional delay equation (12).

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References

1. T. Abdeljawad, On Riemann and Caputo fractional differences, *Comput. Math. Appl.*, **62**:1602–1611, 2011.
2. T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, *Chaos Solitons Fractals*, **116**:146–156, 2018.
3. T. Abdeljawad, F. Jarad, J. Alzabut, Fractional propertional differences with memory, *Eur. Phys. J., Spec. Top.*, **226**:3333–3354, 2017.
4. T. Abdeljawad, F. Jarad, D. Baleanu, A semigroup-like property for discrete Mittag-Leffler functions, *Adv. Difference Equ.*, **2012**:72, 2012.
5. G.A. Anastassiou, About discrete fractional calculus with inequalities, in *Intelligent Mathematics: Computational Analysis*, Intell. Syst. Ref. Libr., Vol. 5, Springer, Berlin, Heidelberg, 2011, pp. 575–585.
6. F.M. Atici, P.W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Am. Math. Soc.*, **137**:981–989, 2007.
7. D. Baleanu, G.C. Wu, Y.R. Bai, F. Chen, Stability analysis of Caputo-like discrete fractional systems, *Commun. Nonlinear Sci. Numer. Simul.*, **48**:520–530, 2017.
8. H. Bao, J. Park, J. Cao, Adaptive synchronization of fractional-order memristor-based neural networks with time delay, *Nonlinear Dyn.*, **82**(3):1–12, 2015.
9. N.R.O. Bastos, R.A.C. Ferreira, D.F.M. Torres, Discrete-time fractional variational problems, *Sign. Process.*, **91**:513–524, 2011.
10. L.C. Becker, Resolvents for weakly singular kernels and fractional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **75**:4839–4861, 2012.
11. T. Burton, T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, *Dyn. Syst. Appl.*, **11**:499–519, 2002.
12. T. Burton, T. Furumochi, Krasnoselskii's fixed point theorem and stability, *Nonlinear Anal., Theory Methods Appl.*, **49**:445–454, 2002.
13. T. Burton, B. Zhang, Fractional equations and generalizations of Schaefer's and Krasnoselskii's fixed point theorems, *Nonlinear Anal., Theory Methods Appl.*, **75**:6485–6495, 2012.
14. T. Burton, B. Zhang, Fixed points and fractional differetial equations: Examples, *Fixed Point Theory*, **14**:313–325, 2013.
15. J. Cermak, Z. Dosla, T. Kisela, Fractional differential equations with a constant delay: Stability and asymptotics of solutions, *Appl. Math. Comput.*, **298**:336–350, 2017.

16. J. Cermak, I. Gyori, L. Nechvatal, On explicit stability conditions for a linear fractional difference system, *Fract. Calc. Appl. Anal.*, **18**:651–672, 2015.
17. F. Chen, Fixed points and asymptotic stability of nonlinear fractional difference equations, *Electron. J. Qual. Theory Differ. Equ.*, **2011**:30, 2011.
18. F. Chen, Asymptotic stability of nonlinear fractional difference equations, *J. Appl. Math.*, **2012**:879657, 2012.
19. L.P. Chen, J. Cao, R.C. Wu, J. Machado, A. Lopes, H. Yang, Stability and synchronization of fractional-order memristive neural networks with multiple delays, *Neural Netw.*, **94**:76–85, 2017.
20. S.S. Cheng, W.T. Patula, An existence theorem for a nonlinear difference equation, *Nonlinear Anal., Theory Methods Appl.*, **20**:193–203, 1993.
21. M. Cui, Compact finite difference method for the fractional diffusion equation, *J. Comput. Phys.*, **228**(20):7792–7804, 2009.
22. X. Ding, X. Zhao J. Cao, F. Alsaadi, Mittag-leffler synchronization of delayed fractional-order bidirectional associative memory neural networks with discontinuous activations: State feedback control and impulsive control schemes, *Proc. R. Soc. A*, **473**:20170322, 2017.
23. Y.J. Fan, X. Huang, Z. Wang, Y.X. Li, Improved quasi-synchronization criteria for delayed fractional-order memristor-based neural networks via linear feedback control, *Neurocomputing*, **306**:68–79, 2018.
24. C. Goodrich, A.C. Peterson, *Discrete Fractional Calculus*, Springer International, 2015.
25. M.T. Holm, The Laplace transform in discrete fractional calculus, *Comput. Math. Appl.*, **62**: 1591–1601, 2011.
26. M.P. Lazarevic, A.M. Spasic, Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach, *Math. Comput. Model.*, **49**:475–481, 2009.
27. C. Li, F. Zeng, The finite difference methods for fractional ordinary differential equations, *Numer. Funct. Anal. Optim.*, **34**(2):149–179, 2013.
28. R. Li, J. Cao, A. Alsaedi, F. A. Alsaedi, Stability analysis of fractional-order delayed neural networks, *Nonlinear Anal. Model. Control*, **22**(4):505–520, 2017.
29. X.Y. Li, W. jiang, Solving fractional difference equations using the Laplace transform method, *Abstr. Appl. Anal.*, **2014**:Article 230850, 2014.
30. Y. Li, Y.Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, **59**: 1810–1821, 2010.
31. P. Liu, X. Nie, J. Liang, J. Cao, Multiple Mittag-Leffler stability of fractional-order competitive neural networks with Gaussian activation functions, *Neural Netw.*, **108**:452–465, 2018.
32. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.
33. A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, J. Cao, G. Rajchakit, Robust generalized Mittag-Leffler synchronization of fractional order neural networks with discontinuous activation and impulses, *Neural Netw.*, **103**:128–141, 2018.
34. I. Stamova, G. Stamov, S. Simeonov, A. Ivanov, Mittag-leffler stability of impulsive fractional-order bi-directional associative memory neural networks with time-varying delays, *Tran. Inst. Meas. Control*, **40**:3068–3077, 2018.

35. G. Velmurugan, R. Rakkiyappan, J. Cao, Finite-time synchronization of fractional-order memristor-based neural networks with time delays, *Neural Netw.*, **73**:36–46, 2016.

36. H. Wang, Y. Yu, G. Wen, S. Zhang, J.Z. Yu, Global stability analysis of fractional-order Hopfield neural networks with time delay, *Neurocomputing*, **154**:15–23, 2015.

37. Z. Wang, X.H. Wang, Y. Li, Stability and Hopf bifurcation of fractional-order complex-valued single neuron model with time delay, *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, **27**:1750209, 2017.

38. G.C. Wu, D. Baleanu, Stability analysis of impulsive fractional difference equations, *Fract. Calc. Appl. Anal.*, **21**:354–375, 2018.

39. G.C. Wu, D. Baleanu, L.L. Huang, Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse, *Appl. Math. Lett.*, **82**:71–78, 2018.

40. G.C. Wu, D. Baleanu, W.H. Luo, Lyapunov functions for Riemann–Liouville-like fractional difference equations, *Appl. Math. Comput.*, **314**:228–236, 2017.

41. J. Zhang, J. Wu, H. Bao, J. Cao, Synchronization analysis of fractional-order three-neuron BAM neural networks with multiple time delays, *Appl. Math. Comput.*, **339**:441–450, 2018.

42. W. Zhang, J. Cao, R. Wu, A. Alsaedi, F. Alsaadi, Projective synchronization of fractional-order delayed neural networks based on the comparison principle, *Adv. Difference Equ.*, **2018**:73, 2018.