Lie Group Analysis of Natural Convection Heat and Mass Transfer in an Inclined Surface

S. Sivasankaran, M. Bhuvaneswari, P. Kandaswamy, E. K. Ramasami

UGC-DRS Center for Fluid Dynamics
Department of Mathematics, Bharathiar University
Coimbatore-641046, India
siva_sd2000@yahoo.com

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Abstract. Natural convection heat transfer fluid flow past an inclined semi-infinite surface in the presence of solute concentration is investigated by Lie group analysis. The governing partial differential equations are reduced to a system of ordinary differential equations by the translation and scaling symmetries. An exact solution is obtained for translation symmetry and numerical solutions for scaling symmetry. It is found that the velocity increases and temperature and concentration of the fluid decrease with an increase in the thermal and solutal Grashof numbers. The velocity and concentration of the fluid decrease and temperature increases with increase in the Schmidt number.

Keywords: Lie groups, natural convection, inclined surface, heat and mass transfer.

1 Introduction

The study of natural convection flow for an incompressible viscous fluid past a heated surface has attracted the interest of many researchers in view of its important applications to many engineering problems such as cooling of nuclear reactors, the boundary layer control in aerodynamics, crystal growth, food processing and cooling towers. In this paper, symmetry methods are applied to a natural convection boundary layer problem. The main advantage of such methods is that they can successfully be applied to non-linear differential equations. The symmetries of a differential equations are those continuous groups of transformations under which the differential equations remain invariant, that is, a symmetry group maps any solution to another solution. The symmetry solutions are quite popular because they result in the reduction of the number of independent variables of the problem.

Chen [1] performed an analysis to study the natural convection flow over a permeable inclined surface with variable wall temperature and concentration. The results show that the velocity is decreased in the presence of a magnetic field. Increasing the angle of inclination decreases the effect of buoyancy force. Heat transfer rate is increased when the Prandtl number is increased. Ibrahim *et al.* [2] investigated the similarity reductions for problems of radiative and magnetic field effects on free convection and mass-transfer flow past a semi-infinite flat plate. They obtained new similarity reductions and found an analytical solution for the uniform magnetic field by using Lie group method. They also presented the numerical results for the non-uniform magnetic field.

Kalpadides and Balassas [3] studied the free convective boundary layer problem of an electrically conducting fluid over an elastic surface by group theoretic method. Their results agreed with the existing result for the group of scaling symmetry. They found that the numerical solution also does so. The Navier-Stokes and boundary layer equations for incompressible flows were derived using a convenient coordinate system by Pakdemirli [4]. The results showed that the boundary layer equations accept similarity solutions for the constant pressure gradient case. The importance of similarity transformations and their applications to partial differential equations was studied by Pakdemirli and Yurusoy [5]. They investigated the special group transformations for producing similarity solutions. They also discussed spiral group of transformations.

Using Lie group analysis, three dimensional, unsteady, laminar boundary layer equations of non-Newtonian fluids are studied by Yurusoy and Pakdemirli [6, 7]. They assumed that the shear stresses are arbitrary functions of the velocity gradients. Using Lie group analysis, they obtained two different reductions to ordinary differential equations. They also studied the effects of a moving surface with vertical suction or injection through the porous surface. They further studied exact solution of boundary layer equations of a special non-Newtonian fluid over a stretching sheet by the method of Lie group analysis. They found that the boundary layer thickness increases when the non-Newtonian behaviour increases.

They also compared the results with that for a Newtonian fluid. Yurusoy *et al.* [8] investigated the Lie group analysis of creeping flow of a second grade fluid. They constructed an exponential type of exact solution using the translation symmetry and a series type of approximate solution using the scaling symmetry. They also discussed some boundary value problems. So far no attempt has been made to study the heat and mass transfer in an inclined surface using Lie groups and hence we study the problem of natural convection heat and mass transfer flow past an inclined plate for various parameters using Lie group analysis.

2 Mathematical analysis

Consider the heat and mass transfer by natural convection in laminar boundary layer flow of an incompressible viscous fluid along a semi-infinite inclined plate with an acute angle α from the vertical. The surface is maintained at a constant temperature T_w which is higher than the constant temperature T_∞ of the surrounding fluid and the concentration C_w is greater than the constant concentration C_∞. The fluid properties are assumed to be constant. The governing equations of the mass, momentum, energy and concentration for the steady flow can be written as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty)\cos\alpha + g\beta^*(C - C_\infty)\cos\alpha, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \tag{4}$$

with the boundary conditions

$$\begin{aligned} &u = v = 0, \quad T = T_w, \quad C = C_w \quad \text{at} \quad y = 0, \\ &u = 0, \qquad T = T_\infty, \quad C = C_\infty \quad \text{as} \quad y \to \infty, \end{aligned} \tag{5}$$

where u and v are velocity components; x and y are space coordinates; T is the temperature; C is the concentration; ν is the kinematic viscosity of the fluid; g is the acceleration due to gravity; β is the coefficient of thermal expansion; β^* is the coefficient of expansion with concentration; k is the thermal conductivity of fluid; ρ is the density of the fluid; c_p is the specific heat of the fluid; D is the diffusion coefficient and α is the angle of inclination.

The nondimensional variables are

$$\begin{aligned} &\overline{x} = \frac{xU_\infty}{\nu}, \quad \overline{y} = \frac{yU_\infty}{\nu}, \quad \overline{u} = \frac{u}{U_\infty}, \quad \overline{v} = \frac{v}{U_\infty}, \\ &\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi = \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned} \tag{6}$$

Substituting (6) into equations (1)–(4) and dropping the bars, we obtain,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + Gr\theta\cos\alpha + Gc\phi\cos\alpha, \tag{8}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}, \tag{9}$$

$$u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{1}{Sc}\frac{\partial^2 \phi}{\partial y^2} \tag{10}$$

with the boundary conditions

$$\begin{aligned} &u = v = 0, \quad \theta = 1, \quad \phi = 1 \quad \text{at} \quad y = 0, \\ &u = 0, \qquad \theta = 0, \quad \phi = 0 \quad \text{as} \quad y \to \infty, \end{aligned} \tag{11}$$

where $Gr = \frac{g\beta(T_w - T_\infty)\nu}{U_\infty^3}$ is the thermal Grashof number, $Gc = \frac{g\beta^*(C_w - C_\infty)\nu}{U_\infty^3}$ is the solutal Grashof number, $Pr = \frac{\rho c_p \nu}{k}$ is the Prandtl number and $Sc = \frac{\nu}{D}$ is the Schmidt number.

3 Symmetry groups of equations

The symmetry groups of equations (7)–(10) are calculated using classical Lie group approach [9]. The one-parameter infinitesimal Lie group of transformations leaving (7)–(10) invariant is defined as

$$\begin{aligned}
x^* &= x + \epsilon\xi_1(x, y, u, v, \theta, \phi),\\
y^* &= y + \epsilon\xi_2(x, y, u, v, \theta, \phi),\\
u^* &= u + \epsilon\eta_1(x, y, u, v, \theta, \phi),\\
v^* &= v + \epsilon\eta_2(x, y, u, v, \theta, \phi),\\
\theta^* &= \theta + \epsilon\eta_3(x, y, u, v, \theta, \phi),\\
\phi^* &= \phi + \epsilon\eta_4(x, y, u, v, \theta, \phi).
\end{aligned} \tag{12}$$

By carrying out a straightforward and tedious algebra, we finally obtain the form of the infinitesimals as

$$\begin{aligned}
\xi_1 &= 2c_1x - c_2x - c_3,\\
\xi_2 &= \frac{1}{2}c_1y - \frac{1}{2}c_2y - \alpha(x),\\
\eta_1 &= c_1u,\\
\eta_2 &= -u\alpha'(x) - \frac{1}{2}c_1v + \frac{1}{2}c_2v,\\
\eta_3 &= c_2\theta - \frac{Gc}{Gr}c_4,\\
\eta_4 &= c_2\phi + c_4.
\end{aligned} \tag{13}$$

Imposing the restrictions from boundaries and from the boundary conditions on the infinitesimals, we obtain the following form for equations (13)

$$\begin{aligned}
\xi_1 &= 2c_1x - c_2x - c_3,\\
\xi_2 &= \frac{1}{2}c_1y - \frac{1}{2}c_2y,\\
\eta_1 &= c_1u,\\
\eta_2 &= -\frac{1}{2}c_1v + \frac{1}{2}c_2v,\\
\eta_3 &= c_2\theta - \frac{Gc}{Gr}c_4,\\
\eta_4 &= c_2\phi + c_4,
\end{aligned} \tag{14}$$

where the parameters c_1 and c_2 represent the scaling transformations and parameter c_3 represents translation in the x coordinate. In the following sections, solutions corresponding to the above symmetries are derived.

4 Translational symmetry

In this section, translation in x coordinate is considered and hence we take $c_1 = c_2 = 0$. The characteristic equations for finding the similarity transformations would then be

$$\frac{dx}{-c_3} = \frac{dy}{0} = \frac{du}{0} = \frac{dv}{0} = \frac{d\theta}{0} = \frac{d\phi}{0}. \tag{15}$$

The similarity variables and resulting functions are

$$\eta = y, \quad u = F_1(\eta), \quad v = F_2(\eta), \quad \theta = F_3(\eta), \quad \phi = F_4(\eta). \tag{16}$$

One now substitutes the similarity variable and the functions into the original equations of motion and obtains

$$\begin{aligned} F_1{}'' &= F_2 F_1' - Gr F_3 \cos\alpha - Gc F_4 \cos\alpha, \\ F_2{}' &= 0, \\ F_3{}'' &= Pr F_2 F_3', \\ F_4{}'' &= Sc F_2 F_4'. \end{aligned} \tag{17}$$

Integrating the above system and using the boundary conditions, we obtain the solutions of the equations as follows:

$$\begin{aligned} u &= \frac{Gr\cos\alpha}{c^2(Pr-1)Pr}(e^{-cy} - e^{-cPry}) + \frac{Gc\ \cos\alpha}{c^2(Sc-1)Sc}(e^{-cy} - e^{-cScy}), \\ v &= -c, \\ \theta &= e^{-cPry}, \\ \phi &= e^{-cScy}, \end{aligned} \tag{18}$$

where c is an arbitrary constant.

5 Scaling symmetry

In this section, parameter c_1 is taken to be arbitrary and all other parameters are zero in (14). The characteristic equations are

$$\frac{dx}{2x} = \frac{dy}{(1/2)y} = \frac{du}{u} = \frac{dv}{(-1/2)v} = \frac{d\theta}{0} = \frac{d\phi}{0} \tag{19}$$

from which the similarity variable, the velocities, the temperature and the concentration turn out to be of the form

$$\begin{aligned} &\eta = x^{-1/4}y, \quad u = x^{1/2}F_1(\eta), \quad v = x^{-1/4}F_2(\eta), \\ &\theta = F_3(\eta), \quad \phi = F_4(\eta). \end{aligned} \tag{20}$$

Substituting (20) into equations (7)–(10), we finally obtain the system of nonlinear ordinary differential equations

$$\begin{aligned} F_1{}'' &= \frac{1}{2}F_1{}^2 - \frac{1}{4}\eta F_1 F_1' + F_2 F_1' - Gr F_3 \cos\alpha - Gc F_4 \cos\alpha, \\ F_2{}' &= \frac{1}{4}\eta F_1{}' - \frac{1}{2}F_1, \\ F_3{}'' &= Pr(F_2 F_3' - \frac{1}{4}\eta F_1 F_3'), \\ F_4{}'' &= Sc(F_2 F_4' - \frac{1}{4}\eta F_1 F_4'). \end{aligned} \tag{21}$$

The appropriate boundary conditions are expressed as

$$\begin{aligned} &F_1 = F_2 = 0, \quad F_3 = 1, \quad F_4 = 1 \quad \text{at} \quad \eta = 0, \\ &F_1 = 0, \qquad\qquad F_3 = 0, \quad F_4 = 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \tag{22}$$

6 Numerical methods for solutions

Since the equations are highly nonlinear, a numerical treatment would be more appropriate. The system of transformed equations (21) together with the boundary conditions (22) is numerically solved by employing a fourth order Runge-Kutta method and Shooting techniques with a systematic guessing of $F_1'(0),\ F_3'(0)$ and $F_4'(0)$. The procedure is repeated until we get the results upto the desired degree of accuracy, namely 10^{-5}. A code is written in MATHEMATICA package and solutions are presented graphically.

7 Results and discussions

Numerical solutions are carried out for various values of the Prandtl number, thermal Grashof number, solutal Grashof number and Schmidt number. Prandtl number Pr is varied from 0.1 to 13.67, thermal Grashof number Gr from 0.1 to

2.5, solutal Grashof number Gc from 0.1 to 1.0 and Schmidt number Sc from 1 to 10 with the angle of inclination α taking the values $0°$, $30°$ and $45°$. The numerical results are depicted graphically in the form of velocity, temperature and concentration profiles. Most of the investigations are carried out for $\alpha = 45°$. Some results are taken for $\alpha = 0°$ (vertical plate case) and $30°$.

Fig. 1 show the effect of Schmidt number on velocity, temperature and concentration of the water $(Pr = 13.67)$ boundary layer for $Gr = Gc = 0.1$. It is clearly seen that the velocity is decreased by increasing the Schmidt number. The thickness of the concentration boundary layer is also decreased. The variation in the thermal boundary layer is very small corresponding to a moderate change in Schmidt number.

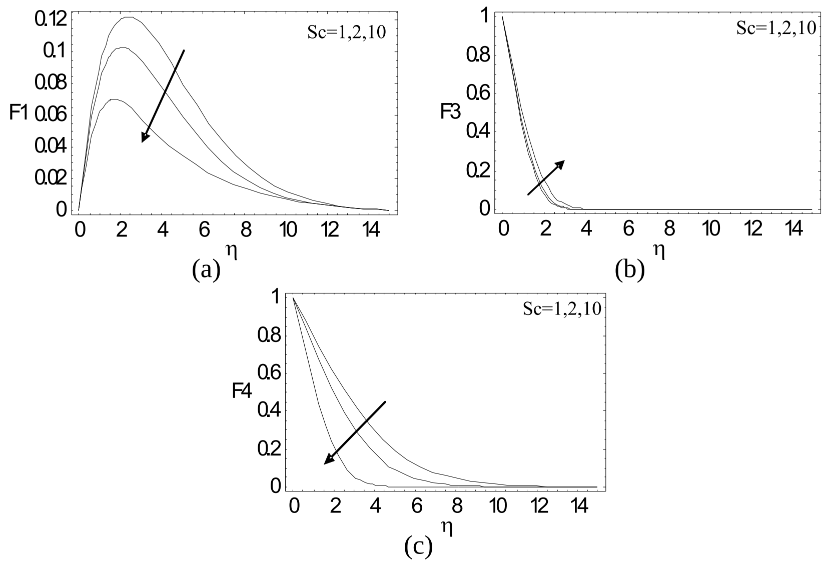

Fig. 1. The velocity (a), temperature (b) and concentration (c) profiles for $Pr = 13.67$, $Gr = 0.1$ and $Gc = 0.1$.

Fig. 2 show the velocity, temperature and concentration profiles for $Pr = 0.71$, $Gr = 0.1$ and $Sc = 1$. Increasing the solutal Grashof number increases the velocity whereas it decreases the temperature and concentration.

The effect of the thermal Grashof number on heat and mass fluid flow behaviour is depicted in Fig. 3. It is found that the velocity increases rapidly and suddenly falls near the boundary due to favourable buoyancy force. The thermal

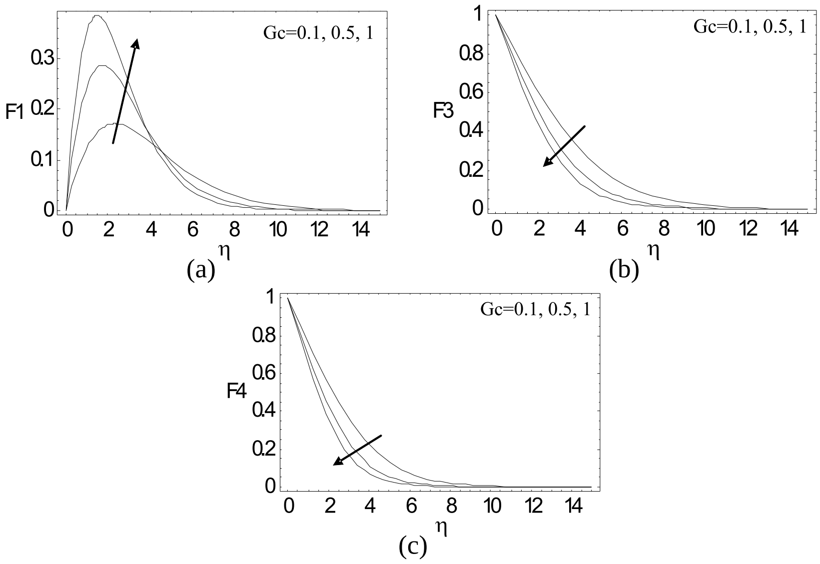

Fig. 2. The velocity (a), temperature (b) and concentration (c) profiles for $Pr = 0.71$, $Gr = 0.1$ and $Sc = 1$.

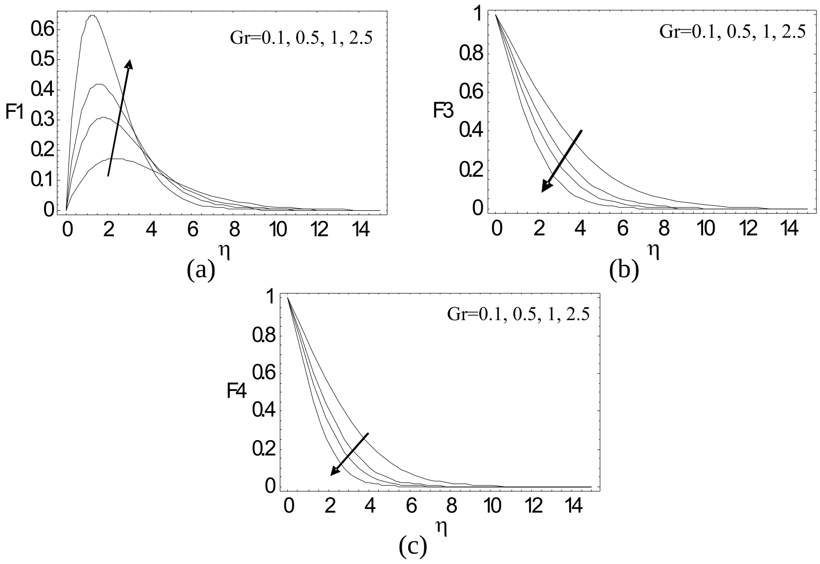

Fig. 3. The velocity (a), temperature (b) and concentration (c) profiles for $Pr = 0.71$, $Gc = 0.1$ and $Sc = 1$.

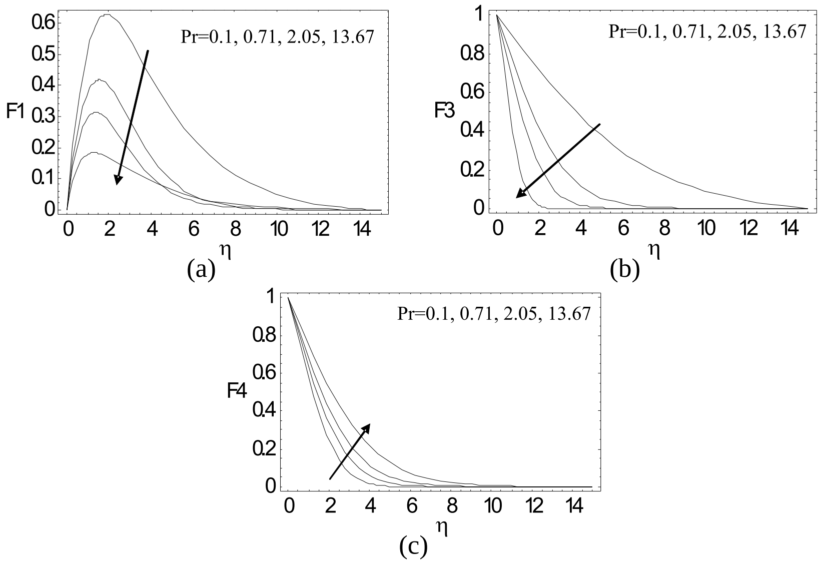

Fig. 4. The velocity (a), temperature (b) and concentration (c) profiles for $Gr = 1$, $Gc = 0.1$ and $Sc = 1$.

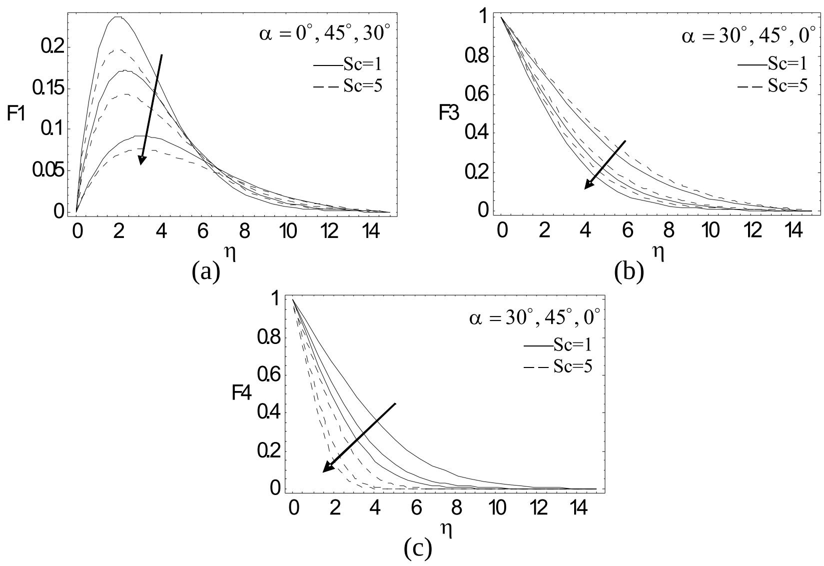

Fig. 5. The velocity (a), temperature (b) and concentration (c) profiles for $Pr = 0.71$, $Gr = 0.1$ and $Gc = 0.1$.

and solutal boundary layer thicknesses are monotonically decreased on increasing the thermal Grashof number. In the presence of uniform Schmidt number it is seen that the increase in the Prandtl number leads to a fall in the velocity and temperature of the fluid and a rise in the concentration of the fluid along the inclined surface as shown in Fig. 4.

The effect of inclination of the surface for different parameters is depicted in Fig. 5. At a fixed value of the Schmidt number the velocity is decreased for all angles. The fluid has higher velocity when the surface is vertical than when inclined because of the fact that the buoyancy effect decreases due to gravity components $(g \cos \alpha)$ as the plate is inclined. For a fixed value of Schimidt number, the fluid has higher temperature when $\alpha = 30^\circ$. Increasing the Schmidt number increases the temperature and decreases the concentration of the fluid along the surface. The inclination angle $\alpha = 30^\circ$ gives the enhanced heat and mass distribution of the convective fluid.

8 Conclusions

By using Lie group analysis, we first find the symmetries of the partial differential equations and then reduce the equations to ordinary differential equations by using scaling and translational symmetries. Exact solutions for translation symmetry and numerical solution for scaling symmetry are obtained. From the numerical results, it is found that the velocity increases and temperature and concentration of the fluid decrease with an increase in the thermal and solutal Grashof numbers. The temerature and velocity of the fluid decrease at a very fast rate in the case of water in comparison with air. Increasing the Prandtl number decreases the temperature and velocity of the fluid and increases the concentration. The velocity and concentration of the fluid decrease and temperature increases with increase in the Schmidt number.

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