Circle and Popov Criterion for Output Feedback Stabilization of Uncertain Systems

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Abstract. In this paper, we address the problem of output feedback stabilization for a class of uncertain dynamical systems. An asymptotically stabilizing controller is proposed under the assumption that the nominal system is absolutely stable.

Keywords: uncertain system, stabilization, output feedback.

1 Introduction

The problem of stabilization for dynamical systems with uncertainties has been studied by several authors; see, e.g., [1–10]. The design of a stabilizing controller is generally based on the so called mini-max approach: a control law is in fact designed as if there were no uncertainties, and a Lyapunov function is also given. Then, this known Lyapunov function is employed as a Lyapunov function candidate for the uncertain dynamical system and a control law is then chosen such that the Lyapunov function decreases along the trajectories of the uncertain dynamical system.

In this paper, we consider nonlinear uncertain systems of the following form.

$$\begin{cases} \dot{x} = Ax + Bu + f(t, x, u), \\ y = Cx, \end{cases} \tag{1}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $u,\ y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$. The pair of known matrices (A, B), defining the nominal system is assumed to be controllable with A Hurwitz. The pair (A, C) is assumed to be observable. The unknown function $f\colon \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ models plant uncertainties in the system. In the absence of uncertainties, the Lurie problem, described by [11–16] and [17], consists in finding conditions on A, B and C such that the equilibrium point $x = 0$ of the closed loop system with $u = -\psi(t, y)$, where ψ satisfies a sector condition, is globally asymptotically stable. This problem is also referred to as the absolute stability problem since it gives sufficient conditions to prove global asymptotic stability of the closed loop system for a whole class of feedback nonlinearities ψ. It was solved in [12] using two Lyapunov functions candidates: a quadratic function and a Lurie type Lyapunov function.

Our goal is to design an output feedback controller under the assumption that the nominal system is absolutely stable and the uncertainties are bounded in Euclidean norm by known functions, and such that the zero state of the system (1) is globally asymptotically stable. In most of the literature, no consideration is given to Lyapunov functions which depend on the uncertainties bounds. Here, as for the nominal system, we consider the problem of stabilizing the uncertain system (1) using two Lyapunov functions. The first one is the quadratic Lyapunov function of the nominal system, and the second one is a Lurie type Lyapunov function that depends on the uncertainties bound. This work extends in a simple way the classical absolute stability circle and Popov criterion to uncertain nonlinear systems.

2 Output feedback control

We first introduce the following definitions.

Definition 1. *A nonlinearity $\psi\colon \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is said to belong to a sector $[0, K]$ if*

$$\psi(t, y)^T \big[\psi(t, y) - Ky\big] \leq 0, \quad \forall t \geq 0,\ \forall y \in \mathbb{R}^p$$

for some symmetric positive definite matrix K.

Definition 2. *A $(p \times p)$ matrix $Z(s)$ of functions of complex variable s is called positive real if*

- $Z(s)$ *has elements that are analytic for* $\mathrm{Re}\,[s] > 0$*,*
- $Z^*(s) = Z(s^*)$ *for* $\mathrm{Re}\,[s] > 0$*, and*
- $Z^T(s^*) + Z(s)$ *is positive semi definite for* $\mathrm{Re}\,[s] > 0$,

where the asterisk $*$ *denotes complex conjugation.*

The matrix $Z(s)$ *is called strictly positive real if* $Z(s-\varepsilon)$ *is positive real for some* $\varepsilon > 0$*.*

The contents of this section depends on the following result known as the Kalman-Yakubovich-Popov lemma [12].

Lemma 1. *Let* $Z(s) = C(sI - A)^{-1}B + D$ *be a* $(p \times p)$ *transfer function matrix, where* A *is Hurwitz,* (A, B) *is controllable, and* (A, C) *is observable. Then* $Z(\cdot)$ *is strictly positive real if and only if there exist a symmetric positive definite matrix* P*, matrices* W *and* L*, and a positive constant* ε *such that*

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P, \\ PB &= C^T - L^T W, \\ W^T W &= D + D^T. \end{aligned}$$

As stated earlier, the problem is to design an output feedback controller which forces the state to converge to zero. To accomplish this goal, we propose the following controller

$$u(t, y) = -\phi(t, y) = -\psi(t, y) - v(t, y), \tag{2}$$

where $\psi(t, y)$ is a k-Lipschitz function (i.e. $\|\psi(t, y) - \psi(t, z)\| \leq k\|y - z\|$, $\forall t \geq 0,\ \forall y,\ \forall z$) which belongs to a sector $[0, K]$, where K is a symmetric positive definite matrix, and $v(t, y)$ is an auxiliary control which will be given later. We shall investigate asymptotic stability of the origin using two Lyapunov functions candidates. The first one is a simple quadratic function

$$V(x) = x^T P x, \quad P = P^T > 0$$

and the second one is a function of the form

$$V(x) = x^T P x + \eta \int_0^y \phi(\sigma)^T K d\sigma, \quad P = P^T > 0,$$

$\eta \geq 0$, which is known as a Lurie type Lyapunov function. In the latter case we assume that the nonlinearity ϕ is time invariant and satisfies some conditions to ensure that the integral is well defined and nonnegative.

2.1 Circle criterion design

If we dictate the condition

(A_1) The $(p \times p)$ matrix $Z_1(s)$ defined by

$$Z_1(s) = I + KC(sI - A)^{-1}B$$

is strictly positive real.

Then $u = -\psi(t, y)$ stabilizes exponentially and globally the nominal system. This problem is referred to as the circle criterion for absolute stability. In fact, using Lemma 1, (see [12]) there exist a symmetric positive definite matrix $P(n \times n)$, a matrix $L(p \times n)$ and $\varepsilon > 0$, such that

$$PA + A^T P = -L^T L - \varepsilon P, \tag{3}$$
$$PB = C^T K - \sqrt{2} L^T. \tag{4}$$

To achieve stabilization of the uncertain system (1) subject to the controller (2), we suppose that assumption (A_1) and the assumptions below are fulfilled.

(A_2) There exists a mapping $h \colon \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$, satisfying

$$f(t, x, u) = P^{-1} C^T h(t, x, u),$$

where P is the positive definite matrix given by (3).

(A_3) The uncertain $h(t, x, u)$ is bounded by a known function, i.e. there exists a nonnegative continuous function $\rho(\cdot, \cdot)$, such that

$$\big\| h(t, x, u) \big\| \leq \rho(t, y).$$

(A_4) There exists a nonnegative function $\rho_0(\cdot, \cdot)$, such that

$$\rho(t, y) \leq \rho_0(t, y) \|y\|$$

with

$$\rho_0(t,y) < \frac{\left(2k - \lambda_{min}(K)\right)^2}{4},$$

where $\lambda_{min}(K)$ denotes the minimum eigenvalue of the matrix K and k is the Lipschitz constant.

The proposed auxiliary controller is given by

$$v(t,y) = \alpha(t,y)K^{-1}y, \tag{5}$$

where $\alpha(t,y)$ is a positive function which will be chosen later. Therefore, we have the following result.

Theorem 1. *Consider the uncertain system described by* (1)*, satisfying assumptions* (A_1)–(A_4)*. Suppose that* $k < \frac{\lambda_{min}(K)}{2}$*. Then, there exists a positive function* $\alpha(t,y)$ *such that the closed loop system* (1)–(2) *with auxiliary control* (5) *is globally exponentially stable.*

Proof. Consider the Lyapunov function

$$V(x) = x^T P x.$$

The time derivative of V along the trajectories of (1) is

$$\dot{V} = 2x^T PAx + 2x^T PBu + 2x^T Pf(t,x,u).$$

Since (A_1) is satisfied, then we can use equations (3) and (4) to obtain

$$2x^T PAx = -\|Lx\|^2 - \varepsilon x^T Px$$

and

$$2x^T PBu = 2y^T Ku - 2\sqrt{2}(Lx)^T u.$$

Hence

$$\begin{aligned}\dot{V} &= -\|Lx\|^2 - \varepsilon x^T Px + 2y^T Ku - 2\sqrt{2}(Lx)^T u + 2x^T Pf(t,x,u)\\ &= -\|Lx + \sqrt{2}u\|^2 - \varepsilon x^T Px + 2y^T Ku2\|u\|^2 + 2x^T Pf(t,x,u)\end{aligned}$$

which implies that

$$\dot{V} \leq -\varepsilon x^T P x + 2y^T K u + 2\|u\|^2 + 2x^T P f(t,x,u).$$

Now using the controller (2) and the auxiliary controller (5), we get

$$\begin{aligned}
2y^T K u + 2\|u\|^2 &= 2\psi^T(\psi - Ky) - 2y^T K v + 2\|v\|^2 + 4\psi^T v \\
&\leq -2y^T K v + 2\|v\|^2 + 4\psi^T v \\
&= -2\alpha\|y\|^2 + 2\alpha^2\|K^{-1}y\|^2 + 4\alpha\psi^T K^{-1} y \\
&\leq -2\alpha\|y\|^2 + 2\alpha^2\|K^{-1}\|^2\|y\|^2 + 4k\alpha\|K^{-1}\|\|y\|^2 \\
&= \Big(-2\alpha + 2\frac{\alpha^2}{\lambda_{min}^2(K)} + \frac{4k\alpha}{\lambda_{min}(K)}\Big)\|y\|^2.
\end{aligned}$$

Moreover, from assumptions (A_2), (A_3) and (A_4) it follows that

$$\begin{aligned}
2x^T P f(t,x,u) &= 2y^T h(t,x,u) \leq 2\|y\|\|h(t,x,u)\| \\
&\leq 2\|y\|\rho(t,y) \leq 2\rho_0(t,y)\|y\|^2.
\end{aligned}$$

The above two inequalities in conjunction with the estimation of $\dot{V}$ yield,

$$\dot{V} \leq -\varepsilon x^T P x + 2\bigg(\frac{1}{\lambda_{min}^2(K)}\alpha^2 + \Big(\frac{2k}{\lambda_{min}(K)} - 1\Big)\alpha + \rho_0\bigg)\|y\|^2.$$

If we can choose the function $\alpha(t,y)$ in such away

$$\frac{1}{\lambda_{min}^2(K)}\alpha^2 + \Big(\frac{2k}{\lambda_{min}(K)} - 1\Big)\alpha + \rho_0 = 0, \tag{6}$$

that is the equation (6) on α admits a solution, then, we obtain

$$\dot{V} \leq -\varepsilon x^T P x$$

which achieves global exponential stability of (1).

Let us consider the quadratic equation (6). The discriminate Δ is given by

$$\Delta = \frac{\big(2k - \lambda_{min}(K)\big)^2}{\lambda_{min}^2(K)} - \frac{4\rho_0(t,y)}{\lambda_{min}^2(K)}$$

which is positive by assumption (A_4). Therefore, there are two distinct real solutions to equation (6).

$$\alpha_1 = \Big(\frac{\lambda_{min}(K) - 2k}{\lambda_{min}(K)} - \sqrt{\Delta}\Big)\frac{\lambda^2_{min}(K)}{2},$$
$$\alpha_2 = \Big(\frac{\lambda_{min}(K) - 2k}{\lambda_{min}(K)} + \sqrt{\Delta}\Big)\frac{\lambda^2_{min}(K)}{2}.$$

Since $k < \dfrac{\lambda_{min}(K)}{2}$, we get $\alpha_2 > 0$ and so is α_1. In conclusion, we can choose $\alpha = \alpha_1$ or $\alpha = \alpha_2$ to guarantee global exponential stability of the closed loop system (1)–(2). □

Remark 1. *In* [3], *output feedback stabilization of uncertain systems of the form* (1) *has been investigated. The established result is different from the one given here. In fact, in Theorem* 1, *we are concerned not with a particular stabilizing controller but with an entire family of controllers, since* $\psi(\cdot)$ *can be any nonlinearity in the sector* $[0, K]$.

2.2 Popov criterion design

Now, consider again the system (1) subject to the controller (2) and suppose that f and ϕ are time invariant. Suppose that ψ is decentralized in the sense that each ψ_i depends only on y_i, and belongs to the sector $[0, K]$ with $K = diag(\lambda_1, \ldots, \lambda_n)$. As in the former case, we start by giving conditions guaranteeing global asymptotic stability of the nominal system subject to the controller $u = -\psi(y)$, which is referred to as the Popov criterion for absolute stability.

(A'_1) There exists $\eta > 0$, with $-\frac{1}{\eta}$ not an eigenvalue of A, such that

$$Z_2(s) = I + (1 + \eta s)KC(sI - A)^{-1}B$$

is strictly positive real.

If assumption (A'_1) is satisfied, then, by Lemma 1 (see [12]), there exist a symmetric positive definite matrix P, matrices L and W and $\varepsilon > 0$ such that

$$PA + A^T P = -L^T L - \varepsilon P, \tag{7}$$
$$PB = C^T K + \eta A^T C^T K - L^T W, \tag{8}$$
$$2I + \eta KCB + \eta B^T C^T K = W^T W. \tag{9}$$

Before stating and proving our second result, let us modify the assumptions introduced above.

(A'_2) There exists a mapping $h\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ satisfying

$$f(x,u) = P^{-1}C^T h(x,u),$$

where P is the positive definite matrix given by (7).

(A'_3) There exists a nonnegative continuous function $\rho(\cdot)$ such that

$$\big\|h(x,u)\big\| \leq \rho(y).$$

(A'_4) There exists a positive constant ρ_0 such that

$$\rho(y) \leq \rho_0\|y\| \quad \text{with} \quad \rho_0 < \frac{\big(2k - \lambda_{min}(K)\big)^2}{4},$$

where $\lambda_{min}(K)$ denotes the minimum eigenvalue of the matrix K and k is the Lipschitz constant.

We are now ready to state the following theorem.

Theorem 2. *Consider system* (1) *subject to the controller*

$$u(y) = -\phi(y) = -\psi(y) - v(y),$$

where $\psi(\cdot)$ is a k-Lipschitz function which belongs to the sector $[0,K]$. Suppose that there exists η small enough satisfying (A'_1). If assumptions (A'_2)–(A'_4) *are fulfilled and the Lipschitz constant $k < \frac{\lambda_{min}(K)}{2}$, then there exists an auxiliary controller $v(\cdot)$ such that the closed loop system is globally asymptotically stable.*

Proof. The proof consists of demonstrating that the function

$$V(x) = x^T P x + 2\eta \int_0^y \phi(\sigma)^T K d\sigma = x^T P x + 2\eta \int_0^y \sum_{i=1}^p \phi_i(\sigma_i)^T \lambda_i d\sigma_i,$$

where $\eta \geq 0$ is to be chosen, is a Lyapunov function for the closed loop system. We will choose a decentralized auxiliary controller $v(\cdot)$. Thus, $\phi(\cdot)$ is decentralized and the integral term is well defined and positive. Therefore, the function V is positive definite. Its derivative along the trajectories of the system is given by

$$\begin{aligned}
\dot{V} &= 2x^T P\dot{x} + 2\eta\phi^T(y)K\dot{y} \\
&= 2x^T P\big(Ax + Bu + f(x,u)\big) + 2\eta\phi^T(y)KC\big(Ax + Bu + f(x,u)\big) \\
&= 2x^T PAx - 2x^T PB\phi(y) + 2x^T Pf(x,u) + 2\eta\phi^T(y)KCAx \\
&\quad - 2\eta\phi^T(y)KCB\phi(y) + 2\eta\phi^T(y)KCf(x,u).
\end{aligned}$$

Using equations (7)–(9), it is easy to see that

$$\begin{aligned}
\dot{V} =& - \|Lx - W\phi(y)\|^2 - \varepsilon x^T Px + 2\|\phi\|^2 \\
& - 2y^T K\phi(y) + 2x^T Pf(x,u) + 2\eta\phi^T(y)KCf(x,u) \\
\leq& - \varepsilon x^T Px + 2\|\phi\|^2 - 2y^T K\phi(y) + 2x^T Pf(x,u) + 2\eta\phi^T(y)KCf(x,u).
\end{aligned}$$

Since $\phi(y) = \psi(y) + v(y)$ and ψ is a k Lipschitz function which belongs to the sector $[0, K]$, it follows that

$$\begin{aligned}
2\|\phi\|^2 - 2y^T K\phi(y) &= 2\psi^T\psi + 4\psi^T v + 2v^T v - 2y^T K\psi - 2y^T Kv \\
&\leq 4\psi^T v + 2\|v\|^2 - 2y^T Kv \\
&\leq 4\|\psi\|\|v\| + 2\|v\|^2 - 2y^T Kv \\
&\leq 4k\|y\|\|v\| + 2\|v\|^2 - 2y^T Kv.
\end{aligned}$$

Choose a decentralized $v(\cdot)$ as follows,

$$v(y) = \alpha K^{-1} y, \text{ with } \alpha > 0.$$

Then

$$\begin{aligned}
2\|\phi\|^2 - 2y^T K\phi(y) &\leq 4k\alpha\|K^{-1}\|\|y\|^2 + 2\alpha^2\|K^{-1}\|^2\|y\|^2 - 2\alpha\|y\|^2 \\
&= \Big(\frac{4\alpha k}{\lambda_{min}(K)} + \frac{2\alpha^2}{\lambda^2_{min}(K)} - 2\alpha\Big)\|y\|^2.
\end{aligned} \tag{10}$$

Moreover, under assumptions (A'_2)–(A'_4), we have

$$\begin{aligned} &2x^T Pf(x,u) + 2\eta\phi^T KCf(x,u) \\ &\qquad = 2y^T h(x,u) + 2\eta(\psi + v)^T KCP^{-1}C^T h(x,u) \\ &\qquad \le 2\|y\|\rho(y) + 2\eta\|\psi + v\|\|KCP^{-1}C^T\|\rho(y) \\ &\qquad \le 2\|y\|^2\rho_0 + 2\eta\|KCP^{-1}C^T\|\rho_0(k + \frac{\alpha}{\lambda_{min}(K)})\|y\|^2 \\ &\qquad = 2\Big(\rho_0 + \eta m\rho_0 k + \frac{\eta m \rho_0}{\lambda_{min}(K)}\alpha\Big)\|y\|^2, \end{aligned} \tag{11}$$

where $m = \|KCP^{-1}C^T\|$.

From (10) and (11) we obtain the following upper bound on $\dot{V}$,

$$\dot{V} \le -\varepsilon x^T Px + 2\bigg(\frac{\alpha^2}{\lambda_{min}^2(K)} - \Big(1 - \frac{2k + \eta m\rho_0}{\lambda_{min}(K)}\Big)\alpha + \rho_0 + \eta m \rho_0 k\bigg)\|y\|^2.$$

Following the proof of Theorem 1, we want to show that there exists $\alpha > 0$ such that

$$\frac{\alpha^2}{\lambda_{min}^2(K)} - \Big(1 - \frac{2k + \eta m\rho_0}{\lambda_{min}(K)}\Big)\alpha + \rho_0 + \eta m \rho_0 k = 0. \tag{12}$$

First suppose that η is small enough to satisfy

$$1 - \frac{2k + \eta m\rho_0}{\lambda_{min}(K)} > 0.$$

That is

$$0 < \eta < \frac{\lambda_{min}(K) - 2k}{m\rho_0} := \eta_0.$$

It is possible, since $\lambda_{min}(K) > 2k$. If Δ is the discriminate of (12) then

$$\Delta = \frac{\big(\lambda_{min}(K) - 2k\big)^2 - 4\rho_0 + (\eta m\rho_0)^2 - 2\eta m\rho_0\lambda_{min}(K)}{\lambda_{min}^2(K)}.$$

Now consider the quadratic equation on η,

$$\big(\lambda_{min}(K) - 2k\big)^2 - 4\rho_0 + (\eta m\rho_0)^2 - 2\eta m\rho_0\lambda_{min}(K) = 0. \tag{13}$$

Its discriminate δ is given by

$$\delta = 4m^2\rho_0^2\big(\rho_0 + k(\lambda_{min}(K) - k)\big).$$

Hence, $\delta > 0$, since $\lambda_{min}(K) > 2k$. Consequently, there exist $\eta_1 < \eta_2$ solutions to equation (13), with

$$\eta_1 = \frac{\lambda_{min}(K) - 2(\rho_0 + k(\lambda_{min}(K) - k))^{1/2}}{m\rho_0}$$

which is positive. If η is small enough to satisfy $\eta < \min(\eta_0, \eta_1)$, then $\Delta > 0$ which achieves this proof. □

Remark 2. *It is important to note that the Lyapunov function used to prove Theorem* 2 *is different from the one used to prove absolute stability of the nominal system which has been given by* $V(x) = x^T Px + 2\eta \int_0^y \psi(\sigma)^T K d\sigma$, *(see* [12]*).*

3 Conclusion

We have investigated the problem of state trajectory control via output feedback for a class of nonlinear uncertain dynamical systems. We proved that the system can be globally exponentially stabilized or globally asymptotically stabilized, provided that the controlled system without uncertainties is absolutely stable with respect to the zero state and that the uncertainties are bounded in Euclidean norm by known functions of the system output. An auxiliary controller is used to obtain the stability of the system in presence of uncertainties.

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