

## Parabolic Nonlinear Second Order Slip Reynolds Equation: Approximation and Existence

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**Abstract.** This work studies an initial boundary value problem for nonlinear degenerate parabolic equation issued from a lubrication slip model. Existence of solutions is established through a semi discrete scheme approximation combined with some a priori estimates.

**Keywords:** nonlinear parabolic boundary value problem, semi-discrete scheme, a priori estimates.

### 1 Introduction

Small scale gaseous lubrication theory is used widely in microelectromechanical systems (MEMS) such as microbearings, micropumps and microturbines. Because of microsize or even nanosize geometries flow can no longer be considered as a continuum. This failure of the Navier-Stokes description as the characteristic system scale approaches the mean free path makes this problem both challenging and interesting from a scientific point of view.

The deviation from Navier-Stokes is quantified by the Knudsen number  $K_n$  [1, 2] typically defined as the ratio of the molecular mean free path to the characteristic system scale. It is well known that flow can be classified into three categories [3]:  $K_n \leq 10^{-3}$  the flow can be considered as a continuum;  $K_n > 10$  the flow is considered to be a free molecular flow;  $10^{-3} \leq K_n \leq 10$  the flow can neither be a continuum flow nor a free molecular one. The conventional Navier-Stokes system is based on a continuum assumption and it is no longer valid if the Knudsen number is beyond a certain limit [4]. This involves the selection of an appropriate model and boundary conditions.

The well-known Reynolds equation in the continuum regime is [5, 6]:

$$\frac{\partial}{\partial x_1} \left( \frac{\rho h^3}{\mu} \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{\rho h^3}{\mu} \frac{\partial p}{\partial x_2} \right) = 6 \left( 2 \frac{\partial(\rho h)}{\partial t} + \frac{\partial(\rho U_0 h)}{\partial x_1} \right),$$

where  $h$  is the local gas bearing thickness,  $p$  the local pressure,  $\rho$  the local gas density,  $\mu$  the viscosity,  $U_0$  is the moving plate velocity and  $x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$  (with smooth boundary  $\partial\Omega$ ).

In the slip regime the above equation needs modifications. Taking the Hsia's second order model, the boundary condition are given as follows [7]:

$$\begin{aligned} U_{x_1}(x_3 = 0) &= U_0 + \frac{2 - \tau}{\tau} \lambda \frac{\partial U_{x_1}}{\partial x_3} \Big|_{x_3=0} - \frac{\lambda^2}{2} \frac{\partial^2 U_{x_1}}{\partial x_3^2} \Big|_{x_3=0} + \dots \\ U_{x_1}(x_3 = h) &= -\frac{2 - \tau}{\tau} \lambda \frac{\partial U_{x_1}}{\partial x_3} \Big|_{x_3=h} - \frac{\lambda^2}{2} \frac{\partial^2 U_{x_1}}{\partial x_3^2} \Big|_{x_3=h} + \dots \\ U_{x_2}(x_3 = 0) &= \frac{2 - \tau}{\tau} \lambda \frac{\partial U_{x_2}}{\partial x_3} \Big|_{x_3=0} - \frac{\lambda^2}{2} \frac{\partial^2 U_{x_2}}{\partial x_3^2} \Big|_{x_3=0} + \dots \\ U_{x_2}(x_3 = h) &= -\frac{2 - \tau}{\tau} \lambda \frac{\partial U_{x_2}}{\partial x_3} \Big|_{x_3=h} - \frac{\lambda^2}{2} \frac{\partial^2 U_{x_2}}{\partial x_3^2} \Big|_{x_3=h} + \dots \end{aligned}$$

$U_{x_1}$ ,  $U_{x_2}$  are the velocity distributions,  $\tau$  is the surface accommodation coefficient and  $\lambda$  is the mean free path.

For these boundary conditions, the velocity distributions are obtained by solving the momentum equation [7]:

$$\begin{aligned} U_{x_1} &= \frac{1}{2\mu} \frac{\partial p}{\partial x_1} (x_3^2 - hx_3 - h\lambda - \lambda^2) + U_0 \left(1 - \frac{\lambda + x_3}{h + 2\lambda}\right), \\ U_{x_2} &= \frac{1}{2\mu} \frac{\partial p}{\partial x_2} (x_3^2 - hx_3 - h\lambda - \lambda^2). \end{aligned}$$

The second order modified Reynolds equation can hence be obtained by incorporating the expressions of  $U_{x_1}$  and  $U_{x_2}$  into the continuity equation and then integrating from  $x_3 = 0$  to  $x_3 = h$

$$\begin{aligned} \frac{\partial(\rho h)}{\partial t} + \frac{1}{2} \frac{\partial(\rho U_0 h)}{\partial x_1} &= \frac{\partial}{\partial x_1} \left[ \frac{1}{2\mu} \frac{\partial p}{\partial x_1} \rho \left( \frac{h^3}{6} + \lambda h^2 + \lambda^2 h \right) \right] \\ &+ \frac{\partial}{\partial x_2} \left[ \frac{1}{2\mu} \frac{\partial p}{\partial x_2} \rho \left( \frac{h^3}{6} + \lambda h^2 + \lambda^2 h \right) \right]. \end{aligned}$$

Normally, the non-dimensional second order slip Reynolds equation is used which is given by [6]:

$$\frac{\partial(\rho h)}{\partial t} - \nabla \left[ \left( h^3 p + 6K_n h^2 + 6K_n^2 \frac{h}{p} \right) \nabla p \right] = \Lambda \nabla(\rho h), \quad (1)$$

with  $\Lambda$  is the bearing vector.

In earlier works [8, 9], existence and uniqueness for the stationary version of (1) was proved under some hypotheses on the data. In this paper, we continue our investigation concerning the same problem and we plan to establish existence of weak solutions to (1) through a procedure of semi-discrete scheme as in [10] and to present a priori estimates

for this scheme. First, in Section 2, we present the problem with corresponding boundary and initial data and we introduce a new formulation, Section 3 is devoted to some notations and to semi-discrete scheme. A priori estimates for this scheme are established in Section 4 and existence of weak solution is proved in the last section.

## 2 Formulation of the problem

Here, we consider the following problem

$$(\mathcal{P}) \quad \begin{cases} \frac{\partial(ph)}{\partial t} - \nabla \left[ \left( h^3 p + 6K_n h^2 + 6K_n^2 \frac{h}{p} \right) \nabla p \right] = \Lambda \nabla(ph), \\ p(x, t) = \Psi, \quad x \in \partial\Omega, \quad t \in ]0, T[, \\ p(x, 0) = p_0(x), \end{cases}$$

where  $\Psi > 0$  is the ambient pressure and  $T$  a positive number.

We assume that the functions  $h: \Omega \times ]0, T[ \rightarrow \mathbb{R}$  satisfies the following hypotheses:

$$(\mathcal{A}_1) \quad \begin{cases} h = h(x, t) \text{ is a continuous function, } t \rightarrow h(x, t) \text{ is a Lipschitz} \\ \text{function,} \\ 0 < a \leq h(x, t) \leq b, \quad x \in \Omega, \quad t \in ]0, T[, \\ |\nabla h| \leq C \text{ a.e. in } \Omega \text{ and } t \in ]0, T[, \\ h \in H^1(\Omega \times ]0, T[) \text{ and } \frac{\partial h}{\partial t} \in L^\infty(\Omega \times ]0, T[). \end{cases}$$

We introduce the new unknown function

$$u = \frac{p^2}{2} + 6K_n \frac{p}{h} + 6K_n^2 \frac{\log(p)}{h^2}, \quad (2)$$

and we consider the function  $g: ]0, +\infty[ \rightarrow \mathbb{R}$  such that

$$g(z) = \frac{z^2}{2} + 6K_n z + 6K_n^2 \log(z).$$

It is easy to see that  $g$  is an increasing and bijective function. We have from the above equality

$$p = \frac{1}{h} \kappa(x, t, u), \quad (3)$$

with

$$\kappa(x, t, u) = g^{-1}(h^2(x, t)u + 6K_n^2 \log h(x, t)). \quad (4)$$

Our initial problem ( $\mathcal{P}$ ) becomes in  $u$

$$(\mathcal{P}_u) \begin{cases} \frac{\partial(\kappa(x, t, u))}{\partial t} = \nabla(h^3 \nabla u) \\ \quad - \nabla[(\Lambda - 6K_n \nabla h) \kappa(x, t, u) - 12K_n^2 \log \kappa(x, t, u) \nabla h] \\ \quad + \nabla[12K_n \log h \nabla h] \quad \text{in } \Omega, \\ u(x, t) = \Psi_u = \frac{\Psi^2}{2} + 6K_n \frac{\Psi}{h(x, t)} + 6K_n^2 \frac{\log(\Psi)}{h^2(x, t)}, \quad x \in \partial\Omega, \quad t \in ]0, T[, \\ u(x, 0) = u_0(x) = \frac{p_0^2(x)}{2} + 6K_n \frac{p_0(x)}{h(x, 0)} + 6K_n^2 \frac{\log(p_0(x))}{h^2(x, 0)}. \end{cases}$$

**Definition 1.** By the weak solution of problem ( $\mathcal{P}_u$ ), we mean a function  $u$  such that:

$$\begin{aligned} \text{(i)} \quad & u \in \Psi_u + L^2(0, T, H_0^1(\Omega)), \quad u(x, 0) = u_0(x), \\ & \frac{\partial(\kappa(\cdot, \cdot, u))}{\partial t} \in L^2(0, T, H^{-1}(\Omega)), \\ \text{(ii)} \quad & \int_0^T \left\langle \frac{\partial(\kappa(\cdot, \cdot, u))}{\partial t}, \psi \right\rangle dt = - \int_0^T \int_{\Omega} \kappa(x, t, u) \frac{\partial \psi}{\partial t} dx dt \\ & \quad - \int_{\Omega} \kappa(x, 0, u_0) \psi(x, 0) dx \quad \forall \psi \in \mathcal{D}(\Omega \times [0, T]), \\ \text{(iii)} \quad & \int_0^T \left\langle \frac{\partial(\kappa(\cdot, \cdot, u))}{\partial t}, v \right\rangle dt + \int_0^T \int_{\Omega} h^3 \nabla u \nabla v dx dt \\ & \quad - \int_0^T \int_{\Omega} \chi(x, t, u) \nabla v dx dt = 0, \quad \forall v \in L^2(0, T, H_0^1(\Omega)), \end{aligned} \tag{5}$$

$$\chi(x, t, u) = (\Lambda - 6K_n \nabla h) \kappa(x, t, u) - 12K_n^2 \log \kappa(x, t, u) \nabla h - 12K_n \log h \nabla h.$$

**Remark 1.** Due to the definition

$$p(x, t) = \frac{1}{h(x, t)} g^{-1}(h^2(x, t)u + 6K_n^2 \log h(x, t)),$$

we will prove existence of a positive solution  $p$  for the problem ( $\mathcal{P}$ ) without any knowledge on the sign of  $u$ .

### 3 A semi-discrete time scheme

We replace in  $(\mathcal{P}_u)$  the time derivative by the backward difference quotient. Let  $N$  be a positive integer and  $\tau = \frac{T}{N}$ , denote by  $\{U^j\}_{j=0,1,\dots,N}$  the solution of the elliptic system:

$$(\mathcal{P}_u)^\tau \begin{cases} \left( \frac{\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})}{\tau} \right) - \nabla(h_j^3(x)\nabla U^j) \\ \quad + \nabla(\chi(x, j\tau, U^j)) = 0 \quad \text{in } \Omega, \\ U^j(x) = \Psi_u^j = \frac{1}{\tau} \int_{(j-1)\tau}^{j\tau} \Psi_u(x, t) dt, \quad x \in \partial\Omega, \\ U^0(x) = u_0(x). \end{cases}$$

The variational formulation associated to problem  $(\mathcal{P}_u)^\tau$  is given by:

$$(\mathcal{P}_u)^\tau_V \begin{cases} \text{Find } U^j \in \Psi_u^j + H_0^1(\Omega) \quad (j = 1, 2, \dots, N) \text{ such that} \\ \int_{\Omega} \left( \frac{\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})}{\tau} \right) v dx \\ \quad + \int_{\Omega} h_j^3(x)\nabla U^j \nabla v dx - \int_{\Omega} \chi(x, j\tau, U^j) \nabla v dx = 0, \quad \forall v \in H_0^1(\Omega). \end{cases}$$

This stationary problem is very close to the one studied in [8], so we can use the same arguments as in [8] to prove existence and uniqueness, under some hypotheses on the data (6), of solution for the problem  $(\mathcal{P}_u)^\tau_V$  at each time step.

**Theorem 1.** For  $j = 1, 2, \dots, N$  and under the hypothesis

$$\frac{a^3}{C_p b^2 \left( \frac{\|\Lambda\|_e}{K_n} + 3 \|\nabla h_j\|_{L^2(\Omega)} \right)} > 1, \quad (6)$$

(where  $C_p$  is the constant of Poincaré [11] and  $\|\Lambda\|_e$  is the Euclidean norm of  $\Lambda$ ), there exists a weak solution  $U^j \in H^1(\Omega)$  to the problem  $(\mathcal{P}_u)^\tau_V$ .

In addition, We have uniqueness among all weak solutions to problem  $(\mathcal{P}_u)^\tau_V$ . Further, suppose that  $U_i^j$  is a weak solution to  $(\mathcal{P}_u)^\tau_V$  corresponding to the boundary data  $(\Psi_u^j)_i$ ,  $i = 1, 2$ , if  $(\Psi_u^j)_1 \geq (\Psi_u^j)_2$  a.e. on  $\partial\Omega$ , then  $U_1^j \geq U_2^j$  a.e. on  $\Omega$ .

**Remark 2.** For small  $C_p$  and particular values of  $h$ , assumption (6) is satisfied. However it is not always the case and existence result in hypotheses less restrictive than (6) remains an interesting and open problem.

#### 4 A priori estimates

From the sequence  $\{U^j\}_{j=0,1,\dots,N}$ , we define the following global functions on  $Q = \Omega \times [0, T]$  such that:

$$\begin{aligned} U_\tau(x, t) &= \sum_{j=1}^N U^j(x) \mathbb{I}_{(j-1)\tau, j\tau]}(t), \\ \kappa_\tau(x, t, U_\tau) &= \sum_{j=1}^N \kappa(x, j\tau, U^j) \mathbb{I}_{(j-1)\tau, j\tau]}(t), \\ \chi_\tau(x, t, U_\tau) &= \sum_{j=1}^N \chi(x, j\tau, U^j) \mathbb{I}_{(j-1)\tau, j\tau]}(t), \\ \Psi_u^\tau(x, t) &= \sum_{j=1}^N \Psi_u^j(x) \mathbb{I}_{(j-1)\tau, j\tau]}(t), \end{aligned}$$

where  $\mathbb{I}_{(j-1)\tau, j\tau]}$  is the characteristic function of  $](j-1)\tau, j\tau]$ .

For the following,  $C$  denotes a generic constant which can take different values in different occurrences and we consider that (6) is verified for  $j = 1, 2, \dots, N$  and that  $U^0(x) = u_0(x) \geq 0$ .

**Proposition 1.** *For a big enough value of  $K_n$ , there exists a constant  $C$  which don't rely on  $\tau$  such that*

$$\|U_\tau\|_{L^2(Q)} \leq C.$$

*Proof.* By choosing  $v = (U^j - \Psi_u^j)\tau$  in  $(\mathcal{P}_u)_V^\tau$  and summing on  $j$ , we get

$$\begin{aligned} & \sum_{j=1}^N \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1}))(U^j - \Psi_u^j) dx \\ & + \tau \sum_{j=1}^N \int_{\Omega} h_j^3(x) \nabla U^j (\nabla U^j - \nabla \Psi_u^j) dx \\ & - \tau \sum_{j=1}^N \int_{\Omega} \chi(x, j\tau, U^j) (\nabla U^j - \nabla \Psi_u^j) dx = 0, \end{aligned}$$

which leads, according to  $(\mathcal{A}_1)$ , to

$$\begin{aligned} \|\nabla U_\tau\|_{L^2(Q)}^2 & \leq C \|\chi_\tau\|_{L^2(Q)} (\|\nabla U_\tau\|_{L^2(Q)} + \|\nabla \Psi_u^\tau\|_{L^2(Q)}) \\ & + C \|\nabla U_\tau\|_{L^2(Q)} \|\nabla \Psi_u^\tau\|_{L^2(Q)} \\ & - \sum_{j=1}^N \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1}))(U^j - \Psi_u^j) dx. \end{aligned} \tag{7}$$

However, from the definition of  $\chi_\tau$  and the fact that

$$0 \leq \frac{dg^{-1}}{ds}(s) = \frac{g^{-1}(s)}{(g^{-1})^2(s) + 6K_n g^{-1}(s) + 6K_n^2} \leq \frac{1}{6K_n}, \quad (8)$$

$$0 \leq \frac{d}{ds} \log(g^{-1}(s)) = \frac{1}{(g^{-1})^2(s) + 6K_n g^{-1}(s) + 6K_n^2} \leq \frac{1}{6K_n^2}, \quad (9)$$

we obtain

$$\|\chi_\tau\|_{L^2(Q)} \leq \left( \frac{C}{6K_n} + \frac{C}{6K_n^2} \right) \|U_\tau\|_{L^2(Q)} + C.$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\nabla \Psi_u^\tau\|_{L^2(Q)} &= \tau \sum_{j=1}^N \int_{\Omega} (\nabla \Psi_u^j)^2 dx \leq \frac{1}{\tau} \sum_{j=1}^N \int_{\Omega} \left( \int_{(j-1)\tau}^{j\tau} \nabla \Psi_u dt \right) \left( \int_{(j-1)\tau}^{j\tau} \nabla \Psi_u dt \right) dx \\ &\leq \sum_{j=1}^N \int_{\Omega} \int_{(j-1)\tau}^{j\tau} \nabla \Psi_u \nabla \Psi_u dt dx = \int_{\Omega} \int_0^T \nabla \Psi_u \nabla \Psi_u dt dx = \|\nabla \Psi_u\|_{L^2(Q)}. \end{aligned}$$

The last term on the right hand side of (7) can be estimated as:

$$\begin{aligned} J &= - \sum_{j=1}^N \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})) (U^j - \Psi_u^j) dx \\ &= - \sum_{j=1}^N \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})) U^j dx \\ &\quad + \sum_{j=1}^N \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})) \Psi_u^j dx \\ &= \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1}) (U^j - U^{j-1}) dx \\ &\quad + \int_{\Omega} \kappa(x, 0, U^0) U^0 dx - \int_{\Omega} \kappa(x, N\tau, U^N) U^N dx \\ &\quad + \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1}) (\Psi_u^{j-1} - \Psi_u^j) dx \\ &\quad - \int_{\Omega} \kappa(x, 0, U^0) \Psi_u^0 dx + \int_{\Omega} \kappa(x, N\tau, U^N) \Psi_u^N dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1})(U^j - U^{j-1}) dx \\
&\quad + \int_{\Omega} \kappa(x, N\tau, U^N)(\Psi_u^N - U^N) dx \\
&\quad + \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1})(\Psi_u^{j-1} - \Psi_u^j) dx \\
&\quad - \int_{\Omega} \kappa(x, 0, U^0)\Psi_u^0 dx + \int_{\Omega} \kappa(x, 0, U^0)U^0 dx,
\end{aligned}$$

which can be written  $J = J_1 + J_2$  such that

$$\begin{aligned}
J_1 &= \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1})(U^j - U^{j-1}) dx \\
&\quad + \int_{\Omega} \kappa(x, N\tau, U^N)(\Psi_u^N - U^N) dx, \\
J_2 &= \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1})(\Psi_u^{j-1} - \Psi_u^j) dx \\
&\quad - \int_{\Omega} \kappa(x, 0, U^0)\Psi_u^0 dx + \int_{\Omega} \kappa(x, 0, U^0)U^0 dx.
\end{aligned}$$

Due to the monotonicity of the function  $s \rightarrow \kappa(\cdot, \cdot, s)$ , we have that

$$\int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1})(U^j - U^{j-1}) dx \leq \int_{\Omega} \int_{U^{j-1}}^{U^j} \kappa(x, (j-1)\tau, s) ds dx,$$

therefore,

$$\begin{aligned}
J_1 &\leq \sum_{j=1}^N \int_{\Omega} \int_{U^{j-1}}^{U^j} \kappa(x, (j-1)\tau, s) ds dx - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx \\
&= \int_{\Omega} \int_0^{U^N} \kappa(x, (N-1)\tau, s) ds dx - \int_{\Omega} \int_0^{U^0} \kappa(x, 0, s) ds dx \\
&\quad + \sum_{j=1}^{N-1} \int_{\Omega} \int_0^{U^j} (\kappa(x, (j-1)\tau, s) - \kappa(x, j\tau, s)) ds dx \\
&\quad - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx,
\end{aligned}$$



since the term  $\int_{\Omega} \int_0^{U^0} \kappa(x, 0, s) ds dx$  is nonnegative, we have

$$\begin{aligned}
 J_1 &\leq \sum_{j=1}^{N-1} \int_{\Omega} \int_0^{U^j} (\kappa(x, (j-1)\tau, s) - \kappa(x, j\tau, s)) ds dx \\
 &\quad - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx + \int_{\Omega} \int_0^{U^N} \kappa(x, (N-1)\tau, s) ds dx \\
 &\leq \sum_{j=1}^{N-1} \int_{\Omega} \int_0^{U^j} (\kappa(x, (j-1)\tau, s) - \kappa(x, j\tau, s)) ds dx \\
 &\quad + \int_{\Omega} \int_0^{\Psi_u^N} \kappa(x, (N-1)\tau, s) ds dx - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx \\
 &\quad + \int_{\Omega} \int_{\Psi_u^N}^{U^N} \kappa(x, (N-1)\tau, s) ds dx \\
 &\leq \sum_{j=1}^{N-1} \int_{\Omega} \int_0^{U^j} (\kappa(x, (j-1)\tau, s) - \kappa(x, j\tau, s)) ds dx \\
 &\quad + \int_{\Omega} \int_0^{\Psi_u^N} \kappa(x, (N-1)\tau, s) ds dx - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx \\
 &\quad + \int_{\Omega} \int_{\Psi_u^N}^{U^N} \kappa(x, N\tau, s) ds dx - \int_{\Omega} \int_{\Psi_u^N}^{U^N} (\kappa(x, N\tau, s) - \kappa(x, (N-1)\tau, s)) ds dx,
 \end{aligned}$$

however, there exists  $\tilde{c} \in ]\Psi_u^N, U^N[$  such that

$$\begin{aligned}
 &\int_{\Omega} \int_{\Psi_u^N}^{U^N} \kappa(x, N\tau, s) ds dx - \int_{\Omega} \kappa(x, N\tau, U^N)(U^N - \Psi_u^N) dx \\
 &= \int_{\Omega} (\kappa(x, N\tau, \tilde{c}) - \kappa(x, N\tau, U^N))(U^N - \Psi_u^N) dx,
 \end{aligned}$$

which is nonnegative, due to the monotonicity of  $\kappa$ .

Now, using the fact that  $\kappa$  is Lipschitz with respect to the second term and estimate

(8), we get

$$\begin{aligned}
& \left| \sum_{j=1}^{N-1} \int_{\Omega} \int_0^{U^j} (\kappa(x, (j-1)\tau, s) - \kappa(x, j\tau, s)) ds dx \right| \\
& \leq \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)}^2 + \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)}, \\
& \left| \int_{\Omega} \int_{\Psi_u^N}^{U^N} (\kappa(x, N\tau, s) - \kappa(x, (N-1)\tau, s)) ds dx \right| \\
& \leq \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)}^2 + \frac{C}{6K_n} \|\Psi_u^N\|_{L^2(\Omega)}^2 + \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)} + \frac{C}{6K_n} \|\Psi_u^N\|_{L^2(\Omega)},
\end{aligned}$$

on the other hand

$$\int_{\Omega} \int_0^{\Psi_u^N} \kappa(x, (N-1)\tau, s) ds dx \leq C \|\Psi_u^N\|_{L^2(\Omega)}^2 + C \|\Psi_u^N\|_{L^2(\Omega)},$$

it follows that

$$J_1 \leq \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)}^2 + \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)} + C \|\Psi_u^N\|_{L^2(\Omega)}^2 + C \|\Psi_u^N\|_{L^2(\Omega)}.$$

The term  $J_2$  is treated as follows

$$\begin{aligned}
J_2 &= \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1}) (\Psi_u^{j-1} - \Psi_u^j) dx \\
&\quad - \int_{\Omega} \kappa(x, 0, U^0) \Psi_u^0 dx + \int_{\Omega} \kappa(x, 0, U^0) U^0 dx \\
&\leq C\tau \sum_{j=1}^N \int_{\Omega} \kappa(x, (j-1)\tau, U^{j-1}) dx + \|\kappa(x, 0, U^0)\|_{L^2(\Omega)} \|U^0\|_{L^2(\Omega)} \\
&\leq \frac{C}{6K_n} \|U_\tau\|_{L^2(Q)}^2 + C \|U_\tau\|_{L^2(Q)} + \|\kappa(x, 0, U^0)\|_{L^2(\Omega)} \|U^0\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, by substituting the above estimates into (7) and using embedding inequalities, we

obtain

$$\begin{aligned}
 \|\nabla U_\tau\|_{L^2(Q)}^2 &\leq C\|\chi_\tau\|_{L^2(Q)}(\|\nabla U_\tau\|_{L^2(Q)}\|\nabla \Psi_u\|_{L^2(Q)}) \\
 &\quad + C\|\nabla U_\tau\|_{L^2(Q)}\|\nabla \Psi_u\|_{L^2(Q)} + \frac{C}{6K_n}\|\nabla U_\tau t\|_{L^2(Q)}^2 \\
 &\quad + C\|\nabla U_\tau\|_{L^2(Q)} + C\|\Psi_u^N\|_{L^2(\Omega)}^2 \\
 &\quad + C\|\Psi_u^N\|_{L^2(\Omega)} + \|\kappa(x, 0, U^0)\|_{L^2(\Omega)}\|U^0\|_{L^2(\Omega)} \\
 &\leq \frac{C}{6\lambda K_n^2}\|\nabla U_\tau\|_{L^2(Q)}^2 + C\|\nabla U_\tau\|_{L^2(Q)} + C.
 \end{aligned}$$

Hence, if  $1 - \frac{C}{6K_n^2} > 0$  (which is verified for a big enough  $K_n$ ), it follows that  $\|\nabla U_\tau\|_{L^2(Q)}^2 \leq C$  which achieves the proof of the proposition.  $\square$

**Proposition 2.** *There exists a constant  $C$  independent on  $\tau$  such that*

$$\|\kappa(\cdot, n\tau, U^n)\|_{L^2(\Omega)} \leq C, \quad n = 1, 2, \dots, N.$$

*Proof.* By choosing  $v = (\kappa(x, j\tau, U^j) - \kappa(x, j\tau, \Psi_u^j))\tau$  in  $(\mathcal{P}_u)_V^\tau$  and summing on  $j$  from 1 to  $n$ , we get

$$\begin{aligned}
 &\sum_{j=1}^n \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1}))(\kappa(x, j\tau, U^j) - \kappa(x, j\tau, \Psi_u^j)) dx \\
 &\quad + \tau \sum_{j=1}^n \int_{\Omega} h_j^3(x) \nabla U^j (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx \\
 &\quad - \tau \sum_{j=1}^n \int_{\Omega} \chi(x, j\tau, U^j) (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx = 0,
 \end{aligned}$$

using inequality  $-\frac{y^2}{2} - \frac{z^2}{2} \leq -yz$ , yields

$$\begin{aligned}
 &\frac{1}{2} \int_{\Omega} (\kappa(x, n\tau, U^n))^2 - \frac{1}{2} \int_{\Omega} (\kappa(x, 0, U^0))^2 dx \\
 &= \frac{1}{2} \sum_{j=1}^n \int_{\Omega} ((\kappa(x, j\tau, U^j))^2 - (\kappa(x, (j-1)\tau, U^{j-1}))^2) dx \\
 &\leq \sum_{j=1}^n \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1}))\kappa(x, j\tau, \Psi_u^j) dx \\
 &\quad - \tau \sum_{j=1}^n \int_{\Omega} h_j^3(x) \nabla U^j (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx \\
 &\quad + \tau \sum_{j=1}^n \int_{\Omega} \chi(x, j\tau, U^j) (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx.
 \end{aligned} \tag{10}$$

Now, we estimate successively the last three terms of (10).

With the help of the Cauchy-Schwarz inequality we get

$$\begin{aligned}
& \sum_{j=1}^n \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})) \kappa(x, j\tau, \Psi_u^j) dx \\
&= \int_{\Omega} \kappa(x, n\tau, U^n) \kappa(x, n\tau, \Psi_u^n) dx - \int_{\Omega} \kappa(x, 0, U^0) \kappa(x, \tau, \Psi_u^1) dx \\
&\quad - \sum_{j=1}^{n-1} \int_{\Omega} (\kappa(x, (j+1)\tau, \Psi_u^{j+1}) - \kappa(x, j\tau, \Psi_u^j)) \kappa(x, j\tau, U^j) dx \\
&\leq \|\kappa(\cdot, n\tau, U^n)\|_{L^2(\Omega)} \|\kappa(\cdot, n\tau, \Psi_u^n)\|_{L^2(\Omega)} \\
&\quad + \left( \tau \int_{\Omega} \sum_{j=1}^{n-1} \left( \frac{\kappa(x, (j+1)\tau, \Psi_u^{j+1}) - \kappa(x, j\tau, \Psi_u^j)}{\tau} \right) dx \right)^{\frac{1}{2}} \\
&\quad \times \left( \tau \int_{\Omega} \sum_{j=1}^{n-1} (\kappa(x, j\tau, U^j)) dx \right)^{\frac{1}{2}}.
\end{aligned}$$

From the definition of  $\kappa$ , we have  $\kappa(\cdot, j\tau, \Psi_u^j) \in H^1(Q)$  and

$$\tau \int_{\Omega} \sum_{j=1}^{n-1} \left( \frac{\kappa(x, (j+1)\tau, \Psi_u^{j+1}) - \kappa(x, j\tau, \Psi_u^j)}{\tau} \right) dx \leq \left\| \frac{\partial \kappa(x, t, \Psi_u^j)}{\partial t} \right\|_{L^2(Q)},$$

it follows that

$$\begin{aligned}
& \sum_{j=1}^n \int_{\Omega} (\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})) \kappa(x, j\tau, \Psi_u^j) dx \\
&\leq \left\| \frac{\partial \kappa(x, t, \Psi_u^j)}{\partial t} \right\|_{L^2(Q)} \|\kappa_{\tau}\|_{L^2(Q)} + \|\kappa(\cdot, n\tau, U^n)\|_{L^2(\Omega)} \|\kappa(\cdot, n\tau, \Psi_u^n)\|_{L^2(\Omega)},
\end{aligned}$$

hence

$$\|\kappa_{\tau}\|_{L^2(Q)} \leq C \|U_{\tau}\|_{L^2(\Omega)} + C,$$

and from the previous proposition, we gain

$$\|\kappa_{\tau}\|_{L^2(Q)} \leq C.$$

On the other hand, using the Cauchy-Schwarz inequality in the elliptic term of (10) gives

$$\begin{aligned}
 & -\tau \sum_{j=1}^n \int_{\Omega} h_j^3(x) \nabla U^j (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx \\
 & \leq C \left( \tau \sum_{j=1}^n \int_{\Omega} (\nabla U^j)^2 dx \right)^{\frac{1}{2}} \left( \tau \sum_{j=1}^n \int_{\Omega} (\nabla \kappa(x, j\tau, U^j))^2 dx \right)^{\frac{1}{2}} \\
 & \quad + C \left( \tau \sum_{j=1}^n \int_{\Omega} (\nabla U^j)^2 dx \right)^{\frac{1}{2}} \left( \tau \sum_{j=1}^n \int_{\Omega} (\nabla \kappa(x, j\tau, \Psi_u^j))^2 dx \right)^{\frac{1}{2}} \\
 & \leq C \|\nabla U_{\tau}\|_{L^2(Q)} (\|\nabla \kappa_{\tau}\|_{L^2(Q)} + \|\nabla \kappa(\cdot, \cdot, \Psi_u^j)\|_{L^2(Q)}).
 \end{aligned}$$

The same calculation for the convective term of (10) leads to

$$\begin{aligned}
 & \tau \sum_{j=1}^n \int_{\Omega} \chi(x, j\tau, U^j) \cdot (\nabla \kappa(x, j\tau, U^j) - \nabla \kappa(x, j\tau, \Psi_u^j)) dx \\
 & \leq C \|\chi_{\tau}\|_{L^2(Q)} (\|\nabla \kappa_{\tau}\|_{L^2(Q)} + \|\nabla \kappa(\cdot, \cdot, \Psi_u^j)\|_{L^2(Q)}).
 \end{aligned}$$

However, from (8), (9) and Proposition 1, it is easy to see that

$$\|\chi_{\tau}\|_{L^2(Q)} \leq C, \quad (11)$$

by substituting the above estimates into (10), we obtain using the previous proposition

$$\|\kappa(\cdot, n\tau, U^n)\|_{L^2(\Omega)}^2 \leq C \|\kappa(\cdot, n\tau, U^n)\|_{L^2(\Omega)} + C,$$

which give the desired estimate.  $\square$

Here, we give a strong convergence of  $\kappa(x, t, U_{\tau})$  in  $L^1(Q)$ .

**Proposition 3.** *There exists  $u \in L^2(0, T, H^1(\Omega))$  such that  $\kappa(x, t, U_{\tau})$  strongly converges to  $\kappa(x, t, u)$  in  $L^1(Q)$  for  $\tau$  tends to 0.*

*Proof.* From Proposition 1 there exists  $u \in L^2(0, T, H^1(\Omega))$  such that  $U_{\tau}$  weakly converges to  $u$  in  $L^2(0, T, H^1(\Omega))$ . Since  $\kappa$  is Lipschitz with respect to the second variable and according to [12, Lemma 1.9], it suffices to show that there exists a constant  $C$  such that, for any  $\delta > 0$

$$\begin{aligned}
 & \frac{1}{\delta} \int_0^{T-\delta} \int_{\Omega} (\kappa(x, t + \delta, U_{\tau}(x, t + \delta)) - \kappa(x, t, U_{\tau}(x, t))) \\
 & \quad \times (U_{\tau}(x, t + \delta) - U_{\tau}(x, t)) dx dt \leq C.
 \end{aligned} \quad (12)$$

Let  $l$  be fixed ( $1 \leq l \leq N$ ) and  $m \in \{1, \dots, N-l\}$ , summing on  $j$  from  $m+1$  to  $m+l$  in  $(\mathcal{P}_u)_V^r$  leads to

$$\begin{aligned} & \int_{\Omega} (\kappa(x, (m+l)\tau, U^{m+l}) - \kappa(x, m\tau, U^m)) v \, dx \\ &= -\tau \sum_{j=m+1}^{m+l} \int_{\Omega} h_j^3(x) \nabla U^j \nabla v \, dx + \tau \sum_{j=m+1}^{m+l} \int_{\Omega} \chi(x, j\tau, U^j) \nabla v \, dx, \\ & \forall v \in H_0^1(\Omega). \end{aligned}$$

Taking  $v = (U^{m+l} - \Psi_u^{m+l}) - (U^m - \Psi_u^m)$  and summing from  $m = 1$  to  $N-l$ , yields

$$\begin{aligned} & \sum_{m=1}^{N-l} \int_{\Omega} (\kappa(x, (m+l)\tau, U^{m+l}) - \kappa(x, m\tau, U^m)) (U^{m+l} - U^m) \, dx \\ &= \sum_{m=1}^{N-l} \int_{\Omega} (\kappa(x, (m+l)\tau, U^{m+l}) - \kappa(x, m\tau, U^m)) (\Psi_u^{m+l} - \Psi_u^m) \, dx \\ &\quad - \tau \sum_{m=1}^{N-l} \sum_{j=m+1}^{m+l} \int_{\Omega} h_j^3(x) \nabla U^j (\nabla U^{m+l} - \nabla U^m) \, dx \\ &\quad + \tau \sum_{m=1}^{N-l} \sum_{j=m+1}^{m+l} \int_{\Omega} h_j^3(x) \nabla U^j (\nabla \Psi_u^{m+l} - \nabla \Psi_u^m) \, dx \\ &\quad + \tau \sum_{m=1}^{N-l} \sum_{j=m+1}^{m+l} \int_{\Omega} \chi(x, j\tau, U^j) (\nabla U^{m+l} - \nabla U^m) \, dx \\ &\quad - \tau \sum_{m=1}^{N-l} \sum_{j=m+1}^{m+l} \int_{\Omega} \chi(x, j\tau, U^j) (\nabla \Psi_u^{m+l} - \nabla \Psi_u^m) \, dx. \end{aligned} \tag{13}$$

The left-hand side of the above equation is equal to

$$\begin{aligned} & \sum_{m=1}^{N-l} \int_{\Omega} (\kappa(x, (m+l)\tau, U^{m+l}) - \kappa(x, m\tau, U^m)) (U^{m+l} - U^m) \, dx \\ &= \frac{1}{\tau} \int_0^{T-l\tau} \int_{\Omega} (\kappa(x, t+l\tau, U_{\tau}(x, t+l\tau)) - \kappa(x, t, U_{\tau}(x, t))) \\ &\quad \times (U_{\tau}(x, t+l\tau) - U_{\tau}(x, t)) \, dx dt. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \left| \sum_{m=1}^{N-l} \int_{\Omega} (\kappa(x, (m+l)\tau, U^{m+l}) - \kappa(x, m\tau, U^m)) (\Psi_u^{m+l} - \Psi_u^m) dx \right| \\
 & \leq Cl\tau \sum_{m=1}^{N-l} \int_{\Omega} (|\kappa(x, (m+l)\tau, U^{m+l})| + |\kappa(x, m\tau, U^m)|) dx \\
 & \leq Cl\tau \sum_{j=1}^N \int_{\Omega} |\kappa(x, j\tau, U^j)| dx \leq Cl\tau \sum_{j=1}^N \int_{\Omega} |U^j| dx \leq Cl \|U_{\tau}\|_{L^2(Q)}.
 \end{aligned}$$

Similarly we prove that all the other terms in the right-hand side of (13) are bounded by  $l\tau$ . Then we obtain the estimate (12) for  $\delta = l\tau$ . Consequently, the estimation (12) is valid and the strong convergence can be deduced.  $\square$

## 5 Existence of a weak solution

According to propositions of the previous section, we are now ready to prove the main theorem.

**Theorem 2.** *Under assumptions  $(A_1)$ , (6) and for a big enough value of  $K_n$ , the problem  $(\mathcal{P}_u)$  admits a weak solution in the sense of Definition 1.*

*Proof.* We have, from  $(\mathcal{P}_u)_{\tau}^{\tau}$ , that

$$\begin{aligned}
 & \int_{\Omega} \left( \frac{\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})}{\tau} \right) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) v dx \\
 & + \int_{\Omega} (h_j^3(x) \nabla U^j) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) \nabla v dx \\
 & - \int_{\Omega} \chi(x, j\tau, U^j) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) \nabla v dx = 0, \quad \forall v \in H_0^1(\Omega).
 \end{aligned}$$

Summing the above equation, on  $j$  from 1 to  $N$  leads to

$$\begin{aligned}
 & \sum_{j=1}^N \int_{\Omega} \left( \frac{\kappa(x, j\tau, U^j) - \kappa(x, (j-1)\tau, U^{j-1})}{\tau} \right) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) v dx \\
 & + \sum_{j=1}^N \int_{\Omega} (h_j^3(x) \nabla U^j) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) \nabla v dx \\
 & - \sum_{j=1}^N \int_{\Omega} \chi(x, j\tau, U^j) \mathbb{I}_{[(j-1)\tau, j\tau]}(t) \nabla v dx = 0, \quad \forall v \in H_0^1(\Omega).
 \end{aligned} \tag{14}$$

Putting  $\tilde{\kappa}_\tau(x, t, U_\tau) = \sum_{j=1}^N \left( \frac{\kappa(x, j\tau, U^j) - \kappa^{j-1}(x, (j-1)\tau, U^{j-1})}{\tau} \right) \mathbb{1}_{[(j-1)\tau, j\tau]}(t)$ , we obtain

$$\begin{aligned} & \int_0^T \int_\Omega \tilde{\kappa}_\tau(x, t, U_\tau) v \, dx dt + \int_0^T \int_\Omega h_\tau^3(x) \nabla U_\tau \nabla v \, dx dt \\ & - \int_0^T \int_\Omega \chi_\tau(x, t, U_\tau) \nabla v \, dx dt = 0, \quad \forall v \in L^2(0, T, H_0^1(\Omega)). \end{aligned}$$

Using (11) and Proposition 1, yields

$$\left| \int_0^T \int_\Omega \tilde{\kappa}_\tau(x, t, U_\tau) v \, dx dt \right| \leq C \|v\|_{L^2(0, T, H_0^1(\Omega))}, \quad \forall v \in L^2(0, T, H_0^1(\Omega)).$$

Then there is  $\tilde{\kappa} \in L^2(0, T, H^{-1}(\Omega))$  such that  $\tilde{\kappa}_\tau \rightharpoonup \tilde{\kappa}$  weakly in  $L^2(0, T, H^{-1}(\Omega))$  for  $\tau \rightarrow 0$ .

On the other hand, due to Proposition 1, there exists  $u \in L^2(0, T, H^1(\Omega))$  such that  $U_\tau$  weakly converges to  $u$  in  $L^2(0, T, H^1(\Omega))$ .

Next, we will prove that  $\tilde{\kappa} = \frac{\partial \kappa(\dots, u)}{\partial t}$  in  $\mathcal{D}'(\Omega \times [0, T[)$ , indeed, we have

$$\begin{aligned} & \int_0^T \int_\Omega \tilde{\kappa}_\tau(x, t, U_\tau) \psi \, dx dt = - \int_0^{T-\tau} \int_\Omega \kappa_\tau(x, t, U_\tau) \frac{\psi(x, t+\tau) - \psi(x, t)}{\tau} \, dx dt \\ & + \frac{1}{\tau} \int_\Omega \kappa(x, N\tau, U^N) \, dx \int_{T-\tau}^T \psi(x, t) \, dx dt \\ & - \frac{1}{\tau} \int_\Omega \kappa(x, 0, U^0) \, dx \int_0^\tau \psi(x, t) \, dx dt, \quad \forall \psi \in \mathcal{D}(\Omega \times [0, T[), \end{aligned} \tag{15}$$

hence,

$$\begin{aligned} & \int_0^{T-\tau} \int_\Omega \kappa_\tau(x, t, U_\tau) \frac{\psi(x, t+\tau) - \psi(x, t)}{\tau} \, dx dt - \int_0^T \int_\Omega \kappa(x, t, u) \frac{\partial \psi}{\partial t} \, dx dt \\ & = - \int_{T-\tau}^T \int_\Omega \kappa(x, t, u) \frac{\partial \psi}{\partial t} \, dx dt \\ & + \int_{T-\tau}^T \int_\Omega \left( \frac{\psi(x, t+\tau) - \psi(x, t)}{\tau} - \frac{\partial \psi}{\partial t} \right) \kappa_\tau(x, t, U_\tau) \, dx dt \\ & + \int_0^{T-\tau} \int_\Omega (\kappa_\tau(x, t, U_\tau) - \kappa(x, t, u)) \, dx dt, \end{aligned}$$



it is easy to see that  $\int_{T-\tau}^T \int_{\Omega} \kappa(x, t, u) \frac{\partial \psi}{\partial t} dx dt$  converges to 0 when  $\tau$  tends to 0, and due to the boundedness of  $\kappa_{\tau}$ ,  $\int_0^{T-\tau} \int_{\Omega} \left( \frac{\psi(x, t+\tau) - \psi(x, t)}{\tau} - \frac{\partial \psi}{\partial t} \right) \kappa_{\tau}(x, t, U_{\tau}) dx dt$  tends also to 0.

However,

$$\begin{aligned} & \int_0^{T-\tau} \int_{\Omega} (\kappa_{\tau}(x, t, U_{\tau}) - \kappa(x, t, u)) dx dt \\ & \leq \sum_{j=1}^N \int_{\Omega} \int_{(j-1)\tau}^{j\tau} |\kappa(x, j\tau, U^j) - \kappa(x, t, U^j)| \left| \frac{\partial \psi}{\partial t} \right| dx dt \\ & \quad + \sum_{j=1}^N \int_{\Omega} \int_{(j-1)\tau}^{j\tau} |\kappa(x, t, U^j) - \kappa(x, t, u)| \left| \frac{\partial \psi}{\partial t} \right| dx dt \\ & \leq C\tau \int_{\Omega} \sum_{j=1}^N |U^j| \left( \int_{(j-1)\tau}^{j\tau} \left| \frac{\partial \psi}{\partial t} \right| dt \right) dx + C \| \kappa(\cdot, \cdot, U_{\tau}) - \kappa(\cdot, \cdot, u) \|_{L^1(Q)}, \end{aligned}$$

which converges to 0 when  $\tau$  tends to 0, and it follows that

$$\int_0^{T-\tau} \int_{\Omega} \kappa_{\tau}(x, t, U_{\tau}) \frac{\psi(x, t+\tau) - \psi(x, t)}{\tau} dx dt \rightarrow \int_0^T \int_{\Omega} \kappa(x, t, u) \frac{\partial \psi}{\partial t} dx dt \text{ for } \tau \rightarrow 0.$$

On the other hand

$$\begin{aligned} & \frac{1}{\tau} \int_{\Omega} \kappa(x, N\tau, U^N) dx \int_{T-\tau}^T \psi(x, t) dx dt - \frac{1}{\tau} \int_{\Omega} \kappa(x, 0, U^0) dx \int_0^{\tau} \psi(x, t) dx dt \\ & \rightarrow - \int_{\Omega} \kappa(x, 0, U^0) \psi(x, 0) dx. \end{aligned}$$

While passing to the limit in (15), we get  $\tilde{\kappa} = \frac{\partial \kappa(\cdot, \cdot, u)}{\partial t}$  in  $\mathcal{D}'(\Omega \times [0, T])$ .

In the same way, we can prove that

$$\sum_{j=1}^N \int_{\Omega} \chi(x, j\tau, U^j) \mathbb{1}_{[(j-1)\tau, j\tau]}(t) \nabla v dx \rightarrow \int_0^T \int_{\Omega} \chi(x, t, u) \nabla v dx dt.$$

Finally, we can pass to the limit in (14) and we find that  $u$  satisfy (5).  $\square$

**Remark 3.** Particular domain  $\Omega$  (for which  $C_p$  is small enough) is necessary to get assumptions of Theorem 2 while keeping  $K_n \leq 10$ .

## References

1. W. K. Chan, Y. Sun, Analytical modeling of ultra-thin film bearings, *J. Micromech. Microeng.*, **13**(3), pp. 463–473, 2003.
2. Y. Sun, W. K. Chan, N. Liu, A slip model with molecular dynamics, *J. Micromech. Microeng.*, **12**(3), pp. 316–322, 2002.
3. G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*, Oxford: Clarendon, 1994.
4. A. Beskok, *Simulations an models for gas flows in microgeometries*, Ph.D. Dissertation Princeton university, 1996.
5. R. K. Brunner, J. M. Harker, Gas film lubrication study, Part III: experimental investigation of pivoted slider bearings, *IBM J. Res. Development*, **8**(3), pp. 260–274, 1959.
6. S. Fukui, R. Kaneko, Analysis of ultra-thin gas film lubrication based on linearized Boltzmann equation: first report-derivation of a generalized lubrication equation including thermal creep flow, *ASME J. Tribol.*, **110**, pp. 253–262, 1988.
7. Y. T. Hsia, G. A. Domoto, An experimental investigation of molecular rarefaction effects in gas lubricated bearings at ultra-low clearances, *ASME J. Tribol.*, **105**, pp. 120–130, 1983.
8. K. Ait Hadi, M. H. El Bansami, On the second order slip Reynolds equation with molecular dynamics: existence and uniqueness, *J. Inequ. Pure Appl. Math.*, **5**(2), 2004.
9. K. Ait Hadi, M. H. El Bansami, On a certain lubrication slip model, *Int. J. Math. Math. Sci.*, **2005**(20), pp. 3225–3235, 2005.
10. N. Su, Multidimensional degenerate diffusion problem with evolutionary boundary condition: existence, uniqueness and approximation, *International series of numerical mathematics*, **114**, pp. 165–178, 1993.
11. H. Brezis, *Analyse fonctionnelle Théorie et applications*, Masson, Paris, 1983.
12. H. W. Alt, S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.*, **183**(3), pp. 311–341, 1983.