Nonlinear Spectra for Parameter Dependent Ordinary Differential Equations

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Abstract. Eigenvalue problems of the form $x'' = -\lambda f(x) + \mu g(x)$, (i), $x(0) = 0$, $x(1) = 0$ (ii) are considered. We are looking for (λ, μ) such that the problem (i), (ii) has a nontrivial solution. This problem generalizes the famous Fuchik problem for piece-wise linear equations. In our considerations functions f and g may be super-, sub- and quasi-linear in various combinations. The spectra obtained under the normalization condition (otherwise problems may have continuous spectra) structurally are similar to usual Fuchik spectrum for the Dirichlet problem. We provide explicit formulas for Fuchik spectra for super and super, super and sub, sub and super, sub and sub cases, where superlinear and sublinear parts of equations are of the form $|x|^{2\alpha}\,x$ and $|x|^{\frac{1}{2\beta+1}}$ respectively ($\alpha > 0,\ \beta > 0$.)

Keywords: nonlinear spectra, jumping nonlinearity, asymptotically asymmetric nonlinearities, Fuchik spectrum.
AMS classifications: 34B15.

1 Introduction

In this paper we consider boundary value problems of the form

$$x'' = -\lambda f(x) + \mu g(x), \tag{1}$$

$$x(0) = 0, \quad x(1) = 0, \tag{2}$$

where λ and μ are non-negative parameters and f and g are continuous functions such that $f(x) > 0$ for $x > 0$ and $f = 0$ for $x < 0$ and, respectively, $g(x) > 0$ for $x < 0$ and $g = 0$ for $x > 0$. It can be written also as

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0, \\ \ \ \mu g(x), & \text{if} \quad x < 0. \end{cases} \tag{3}$$

Any nontrivial solution $x(t)$ of equation (1) (or, which is the same, of (3)) satisfies the condition $x(t)x''(t) \leq 0$ for any t. Therefore behavior of solutions is rather oscillatory.

In our research we are motivated by the Fuchik equation

$$x'' = -\lambda x^+ + \mu x^-, \tag{4}$$

where $x^+ = \max\{x, 0\},\ x^- = \max\{-x, 0\}$.

This equation may be written also as

$$x'' = \begin{cases} -\lambda x, & \text{if} \quad x \geq 0, \\ -\mu x, & \text{if} \quad x < 0. \end{cases} \tag{5}$$

Equation (4) contains a piece-wise linear function in the right side which possesses some important properties of the linear functions. For example, the positive homogeneity property holds, that is $F(\alpha x) = \alpha F(x),\ \alpha > 0$, where $F(x)$ stands for the right side in (4). Formally equation (4) is nonlinear and the additivity property fails to hold, that is the sum of two solutions $x_1(t)$ and $x_2(t)$ of (4) need not to be a solution. It was the idea of Fuchik [1] to modify a linear equation in this way and to consider nonlinear ("almost" linear) equations of the form (4).

The Fuchik spectrum for the problem (4), (2) is defined as a set of all pairs (λ, μ), for which the problem has a nontrivial solution. This spectrum is well known [2, § 35] and is depicted in Fig. 1.

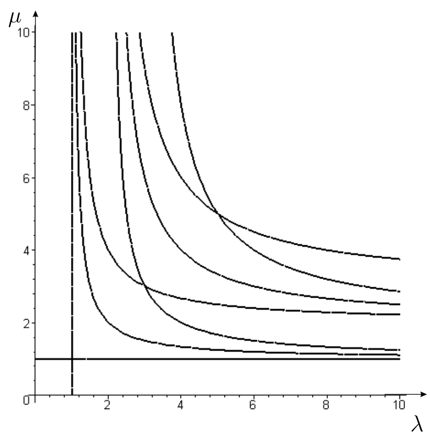

Fig. 1. Fuchik spectrum for the Dirichlet problem (4), (2).

The Fuchik spectrum is useful in the study of the so called "jumping nonlinearities". Imagine equation of the type

$$x'' + g(x) = f(t, x, x'), \tag{6}$$

where $g(x)$ is a "principal" nonlinearity which behaves like a linear function at infinity and f is bounded (nonlinear) function. More preciously, let $g(x)$ satisfy the conditions

$$\begin{aligned} g(x)/x \to \lambda \quad &\text{as} \quad x \to +\infty, \\ g(x)/x \to \mu \quad &\text{as} \quad x \to -\infty. \end{aligned} \tag{7}$$

It appears that "large amplitude" solutions of (6) behave like respective solutions of the Fuchik equation (4). It is supposed, of course, that λ and μ are the same in (4) and (6).

Nonlinearities g of this kind often are referred to as "asymptotically asymmetric" ones.

Asymmetric equations of the form (6) were studied intensively together with the Dirichlet boundary conditions and others. Consider also the linear equation

$$x'' + \Lambda x = 0 \tag{8}$$

along with the boundary conditions (2). Let $\Lambda_1, \Lambda_2, \ldots$ be the eigenvalues. It is used to say that nonlinearity $g(x)$ "crosses several eigenvalues" of the problem (8), (2) if some of Λ_i fall within the interval (μ, λ). One may consult [3–5] for more details.

Another (practical) motivation to study asymptotically asymmetric equations is that these equations appear in the theory of suspension bridges.

Suspension bridges have a roadway that hangs from steel cables supported by two high towers. Suspension bridge cables are not directly connected to the towers. The cables of a suspension bridge are not connected to the bridge - the cables pass through holes in the top of the towers. A suspension bridge has at least two main cables. These cables extend from one end of the bridge to the other. Suspender cables hang from these main cables. The other end of the suspender attaches to the roadway. Schematically suspension bridge is depicted in Fig. 2.

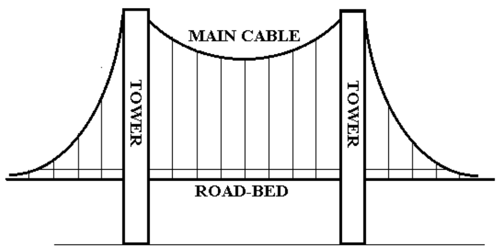

Fig. 2. One-dimensional model of a suspension bridge.

The largest suspension bridges in the world, according to the web information (August 2006), are

1. Akashi-Kaikyo Bridge (Japan) 1991 m (length of the center span) – 1998;
2. Great Belt Bridge (Denmark) 1624 m – 1998;
3. Runyang Bridge (China) 1490 m – 2005;
4. Humber Bridge (England) 1410 m – 1981 (the largest from 1981 until 1998);
5. Jiangyin Suspension Bridge (China) 1385 m – 1997;
6. Tsing Ma Bridge (Hong Kong) 1377 m – 1997;

7. Verrazano Narrows Bridge (USA) 1298 m – 1964 (the largest from 1964 until 1981);
8. Golden Gate Bridge (USA) 1280 m – 1937 (the largest from 1937 until 1964);
9. Hoga Kusten Bridge (Sweden) 1210 m – 1997;
10. Mackinac Bridge (USA) 1158 m – 1958.

However, the most famous is the Tacoma Narrows suspension bridge, which collapsed in 1940 as a result of dramatic large-scale oscillations. The standard explanation of the large oscillations of the bridge attribute the bridge's collapse to the phenomenon of resonance. In the case of so the explanation goes, a suspension bridge oscillates at its own natural frequency. The wind blowing past the bridge generated a train of vortexes that produced a fluctuating force in tune with the bridge's natural frequency, steadily increasing the amplitude of its oscillations until the structure finally collapsed.

Further research showed that this explanation, however, is incomplete and flawed. R.H. Scanlan of Johns Hopkins University in Baltimore and K. Yusuf of Princeton University presented their own engineering report in the article [6]. They focused on the idea that mechanism responsible for large oscillations is self-excitation-an interaction between the bridge's motion and the vortexes produced by that motion – rather than forced resonance.

P. J. McKenna of the University of Connecticut provided his own explanation after spending significant time developing alternative mathematical models to account for the undulations and gyrations shown by the Tacoma bridge. His main idea is that "what distinguishes suspension bridges . . . is their fundamental nonlinearity". This nonlinearity is like "jumping nonlinearity" and it appears in a respective mathematical model because "the restoring force due to a cable is such that it strongly resists expansion, but does not resist compression. Thus, the simplest function to model the restoring force of the stays in the bridge would be a constant times x, the expansion, if x is positive, but zero, if x is negative, corresponding to compression." McKenna asserted, that one of the reasons, explaining strange behavior of suspension bridges under the influence of slow wind, lies in the behavior of the vertical strands of wire, or stays, connecting the roadbed to a bridge's main cable. Civil engineers usually assume that the stays always remain in tension under a bridge's weight, in effect acting as stiff springs. That allows them to use relatively simple, linear differential equations to model the bridge's behavior. When a bridge starts to oscillate, however, the stays begin alternately loosening and tightening. That produces a nonlinear effect, changing the nature of the force acting on the bridge. When the stays are loose, they exert no force, and only gravity acts on the roadbed. When the stays are tight, they pull on the bridge, countering the effect of gravity. Solutions of the nonlinear differential equations that correspond to such an asymmetric situation suggest that, for a wide range of initial conditions, a given push can produce either small or large oscillations. Lazer and McKenna went on to argue that the alternate slackening and tightening of cables might also explain the large twisting oscillations experienced by a suspension bridge.

The whole story of mathematical explanation of behavior of suspension bridges can be traced by the following references [4, 7–11].

2 One parameter problems

Due to definition of equation (1) one have to consider the nonlinear eigenvalue problem

$$x'' = -\lambda f(x), \quad x(0) = 0,\ x(1) = 0, \tag{9}$$

looking for positive solutions of (1), (2) (similarly, one has to consider the problem $x'' = \mu g(x), \quad x(0) = 0,\ x(1) = 0$, looking for negative solutions of (1), (2)).

The problem (9) was considered in [12, 13]. It is known that any positive solution $x(t)$ of (9) is symmetric with respect to the middle point $t = \frac{1}{2}$, where the maximal value is attained.

We assume that $f(x)$ satisfies the following condition:

(A1) A first zero $t_1(\gamma)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(0) = 0,\ u'(0) = \gamma \tag{10}$$

exists for any $\gamma > 0$.

Similar property can be assigned to $g(x)$. We assume that $g(x)$ satisfies the condition:

(A2) A first zero $\tau_1(\delta)$ of a solution to the Cauchy problem

$$v'' = g(v), \quad v(0) = 0,\ v'(0) = -\delta \tag{11}$$

exists for any $\delta > 0$.

Simple examples of $f(x)$ possessing the property **(A1)** are the functions $f(x) = x^3$ ($t_1(\gamma)$ decreases from $+\infty$ to zero as γ increases from zero to $+\infty$) and $f(x) = x^{\frac{1}{3}}$ ($t_1(\gamma)$ increases from zero to $+\infty$ as γ increases from zero to $+\infty$). This can be verified by direct calculation.

Proposition 1. *Suppose that $f(x)$ satisfies the condition* **(A1)** *and $t_1(\gamma)$ maps $(0, +\infty)$ onto $(0, +\infty)$ continuously. Then the problem* (9) *has a continuous spectrum.*

Proof. Fix $\lambda > 0$ and consider a solution $u(t;\gamma)$ of the Cauchy problem (10). This solution has its first positive zero at $t_1(\gamma)$. Consider a function $X(t) := u(\sqrt{\lambda}\, t; \gamma)$. This function solves the equation in (9). Moreover, $X(0) = 0$ and $X(\frac{t_1(\gamma)}{\sqrt{\lambda}}) = 0$. In view of properties of the function $t_1(\gamma)$ for fixed λ a value $\gamma_0 > 0$ exists such that $\frac{t_1(\gamma_0)}{\sqrt{\lambda}} = 1$. □

Example 1. Consider the boundary value problem $x'' = -\lambda x^3, \quad x(0) = 0,\ x(1) = 0,$ $x(t) > 0, \forall t \in (0, 1)$.

The value $\max_{[0,1]} x(t) := \|x\|$ and λ relate as

$$\|x\| \cdot \lambda = 2\sqrt{2} \cdot \int_0^1 \frac{dx}{\sqrt{1 - x^4}}.$$

The problem has continuous spectrum therefore, that is, for any positive λ there exists a unique positive solution of the problem.

Similarly the problem

$$x'' = \mu g(x), \quad x(0) = 0,\ x(1) = 0, \quad x(t) < 0 \text{ in } (0,1) \tag{12}$$

also has continuous spectrum.

A solution of the problem (9) under the condition **(A1)** (and (12) under the condition **(A2)**) is unique however, if the normalization condition $x'(0) = 1$ (resp.: $x'(0) = -1$) is imposed.

3 Two-parameter problems

3.1 Nonlinear spectra for Fuchik type problems

Consider

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0, \\ \mu g(x), & \text{if} \quad x < 0, \end{cases} \quad x(0) = x(1) = 0, \tag{13}$$

where $f(x)$ and $g(x)$ are positive valued continuous functions described in Introduction. Suppose that f and g satisfy the conditions **(A1)** and **(A2)** respectively.

Example 2. Consider the eigenvalue problem

$$x'' = \begin{cases} -\lambda x^2, & \text{if} \quad x \geq 0, \\ \mu x^4, & \text{if} \quad x < 0, \end{cases} \tag{14}$$

$$x(0) = x(1) = 0, \quad x(t) \text{ has exactly one zero in } (0,1). \tag{15}$$

Let us show that this problem has a continuous spectrum. Indeed, let us look for nontrivial solutions which vanish exactly once in the interval $(0,1)$. Let $\tau \in (0,1)$. There exists a continuum of solutions $x(t;\lambda,\tau)$, which are positive in $(0,\tau)$ and vanish at the ends of the interval $(0,\tau)$. The derivative $x'(\tau;\lambda,\tau)$ is a monotonic continuous function of λ with the range of values $(-\infty,0)$. Similarly there exists a continuum of solutions $x(t;\mu,\tau)$ of the equation $x'' = \mu x^4$, which are negative in $(\tau,1)$ and vanish at the ends of the interval $(\tau,1)$. The derivative $x'(\tau;\mu,\tau)$ is a monotonic continuous function of μ with the range of values $(-\infty,0)$. Thus for any $\tau \in (0,1)$ and for any $\lambda > 0$ there exists $\mu(\lambda)$ such that a solution $x(t)$ of the problem (14) has exactly one zero and is C^1-smooth (it follows that in fact it is C^2-smooth).

One is led thus to the conclusion that in order to have reasonable nonlinear eigenvalue problems normalized solutions should be considered.

Consider

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0, \\ \mu g(x), & \text{if} \quad x < 0, \end{cases} \quad x(0) = x(1) = 0, \quad \left|x'(0)\right| = 1. \tag{16}$$

Let us state our main result.

Theorem 1. *Let the conditions* **(A1)** *and* **(A2)** *hold with the functions* $t_1(\gamma)$ *and* $\tau_1(\delta)$. *The Fuchik spectrum for the problem* (16) *is given by the relations (*$i = 1, 2, \ldots$*):*

$$F_0^+ = \Big\{(\lambda;\mu):\quad \lambda \text{ is a solution of } \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1, \quad \mu \geq 0\Big\}, \tag{17}$$

$$F_0^- = \Big\{(\lambda;\mu):\quad \lambda \geq 0,\ \mu \text{ is a solution of } \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}, \tag{18}$$

$$F_{2i-1}^+ = \Big\{(\lambda;\mu):\quad i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}, \tag{19}$$

$$F_{2i-1}^- = \Big\{(\lambda;\mu):\quad i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Big\}, \tag{20}$$

$$F_{2i}^+ = \Big\{(\lambda;\mu):\quad (i+1)\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}, \tag{21}$$

$$F_{2i}^- = \Big\{(\lambda;\mu):\quad (i+1)\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\Big\}. \tag{22}$$

Proof. Consider first solutions of (16), which are positive in $(0,1)$. These solutions, if any, are solutions of the problem (9). Let us find appropriate λ. Consider a solution $u(t;\gamma)$ of the Cauchy problem (10). A function $X(t) := u(\sqrt{\lambda}\,t;\gamma)$ solves equation $X'' = -\lambda f(X)$ and satisfies the conditions $X(0) = 0,\ \frac{dX}{dt}|_{t=0} = \frac{du(\xi)}{d\xi}|_{\xi=0}\sqrt{\lambda} = \gamma\sqrt{\lambda}$. Then $X_t'(0) = 1$ if $\gamma = \frac{1}{\sqrt{\lambda}}$. Since $u(t;\gamma)$ has its first positive zero at $t_1(\gamma)$ the function $X(t) = u(\sqrt{\lambda}\,t;\gamma)$ has the first zero at $\frac{1}{\sqrt{\lambda}}\, t_1(\gamma) = \frac{1}{\sqrt{\lambda}}\, t_1(\frac{1}{\sqrt{\lambda}})$. If λ is such that $\dfrac{1}{\sqrt{\lambda}}\, t_1(\dfrac{1}{\sqrt{\lambda}}) = 1$, then $X(t)$ is a solution to the problem (9) and to the problem (16) also. Hence (17).

Similarly a solution $Y(t) := v(\sqrt{\mu}\,t;\delta)$ with $\delta = \frac{1}{\sqrt{\mu}}$ solves equation $Y'' = \mu g(Y)$ and satisfies the conditions $Y(0) = 0,\ \frac{dY}{dt}|_{t=0} = -1$. This solution has its first zero at $t = \frac{1}{\sqrt{\mu}}\, \tau_1(\frac{1}{\sqrt{\mu}})$. If μ is such that $\dfrac{1}{\sqrt{\mu}}\, \tau_1(\dfrac{1}{\sqrt{\mu}}) = 1$, then $Y(t)$ is a solution to the problem (12) and to the problem (16) also. Hence (18).

Consider now solutions of (16) (if any), which have exactly one zero in $(0,1)$. We have to distinguish between solutions which are first positive and then negative, and solutions, which are first negative and then positive. Consider the first case. Let $X(t) = u(\sqrt{\lambda}\,t;\gamma)$, where $\gamma = \frac{1}{\sqrt{\lambda}}$. This solution is positive in the interval $(0, T_\lambda)$, where $T_\lambda = \frac{1}{\sqrt{\lambda}}\, t_1(\frac{1}{\sqrt{\lambda}})$, vanishes at the end points and satisfy $X'(0) = 1,\ X'(T_\lambda) = -1$. Similarly $Y(t) = v(\sqrt{\mu}\,t;\delta)$, where $\delta = \frac{1}{\sqrt{\mu}}$, is negative in the interval $(0, T_\mu)$, where $T_\mu = \frac{1}{\sqrt{\mu}}\, \tau_1(\frac{1}{\sqrt{\mu}})$, vanishes at the end points and satisfy $Y'(0) = -1,\ Y'(T_\mu) = 1$. Since equation (12) is autonomous, a function $Y(t - T_\lambda)$ solves it also, and can be combined with $X(t)$ in order to get a solution of equation in (16). By construction the function

$$Z(t) = \begin{cases} X(t), & \text{if} \quad t \in [0, T_\lambda], \\ Y(t - T_\lambda), & \text{if} \quad t \in [T_\lambda, T_\lambda + T_\mu] \end{cases} \tag{23}$$

satisfies the conditions

$$\begin{aligned} Z(0) = 0, \quad Z(T_\lambda) = 0, \qquad Z(T_\lambda + T_\mu) = 0, \\ Z'(0) = 1, \quad Z'(T_\lambda) = -1, \quad Z'(T_\lambda + T_\mu) = 1. \end{aligned}$$

If the condition

$$T_\lambda + T_\mu = \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1$$

holds then $Z(t)$ is a solution of the problem (16). Hence (19) for $i = 1$.

Similarly (20) for $i = 1$ can be obtained from the relation $T_\mu + T_\lambda = 1$, considering first $Y(t)$ and combining it with $X(t - T_\mu)$.

The rest of (19) and (20) (for $i = 2, 3, \ldots$) can be obtained using respectively the relations

$$(T_\lambda + T_\mu) + \ldots + (T_\lambda + T_\mu), \qquad i \ \text{ pairs } \ (T_\lambda + T_\mu)$$

and

$$(T_\mu + T_\lambda) + \ldots + (T_\mu + T_\lambda), \qquad i \ \text{ pairs } \ (T_\mu + T_\lambda).$$

Then the case of solutions of the problem (16), which have odd number zeros in $(0, 1)$, is exhausted.

The relations (19) and (20) define the same set (for i fixed).

This is not the case for solutions of (16), which have even number of zeros in $(0, 1)$. For i fixed the relations (21) and (22) generally define different sets. $\square$

3.2 Samples

3.2.1 Superlinear+superlinear

Consider the boundary value problem

$$x'' = \begin{cases} -\lambda |x|^{2\alpha}\, x, & \text{if} \quad x \ge 0, \\ -\mu |x|^{2\beta}\, x, & \text{if} \quad x < 0, \end{cases} \qquad x(0) = x(1) = 0, \qquad \big|x'(0)\big| = 1, \tag{24}$$

where $\alpha > 0$ and $\beta > 0$. Both “positive” and “negative” parts of equation are superlinear. Computations show that the Fuchik spectrum for the problem (24) consists of two straight lines

$$\begin{aligned} F_0^+ &= \Big\{ \big((2A_\alpha)^{2\alpha+2}(\alpha+1); \mu\big) : \ \ \mu \ge 0 \Big\}, \\ F_0^- &= \Big\{ \big(\lambda; (2A_\beta)^{2\beta+2}(\beta+1)\big) : \ \ \lambda \ge 0 \Big\}, \end{aligned}$$

and a set of curves

$$F_{2i-1}^+ = \Bigg\{ (\lambda; \mu) : \ \ i\frac{2A_\alpha (\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + i\frac{2A_\beta (\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1 \Bigg\},$$

$$F^-_{2i-1} = \left\{ (\lambda;\mu) :\quad i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} = 1 \right\},$$

$$F^+_{2i} = \left\{ (\lambda;\mu) :\quad (i+1)\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1 \right\},$$

$$F^-_{2i} = \left\{ (\lambda;\mu) :\quad (i+1)\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} = 1 \right\},$$

where

$$A_\alpha = \int\limits_0^1 \frac{ds}{\sqrt{1-s^{2\alpha+2}}}, \quad A_\beta = \int\limits_0^1 \frac{ds}{\sqrt{1-s^{2\beta+2}}}.$$

The respective Fuchik spectrum is depicted in Fig. 3.

Remark 1. *The even-numbered branches of the spectrum cannot intersect at the bisectrix unless $\alpha = \beta$. Indeed, consider the branches F^+_{2i} and F^-_{2i}. Suppose that $\lambda = \mu$, that is, the branches intersect at the bisectrix. Then*

$$(i+1)\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\lambda^{\frac{1}{2\beta+2}}} = 1$$

and

$$i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + (i+1)\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\lambda^{\frac{1}{2\beta+2}}} = 1.$$

Therefore one gets comparing the above two lines that

$$\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} = \frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\lambda^{\frac{1}{2\beta+2}}}. \tag{25}$$

The left (resp.: right) side of (25) is a value $t_1(\alpha)$ (resp.: $t_1(\beta)$) of the first zero of a solution to the Cauchy problem

$$x'' = -\lambda|x|^{2\alpha}x \quad (\textit{resp.: } x'' = -\lambda|x|^{2\beta}x), \quad x(0)=0,\ x'(0)=1.$$

Since the function $t_1(z)$ is monotone, equality $t_1(\alpha) = t_1(\beta)$ implies $\alpha = \beta$.

3.2.2 Superlinear+sublinear

Consider the boundary value problem

$$x'' = \begin{cases} -\lambda|x|^{2\alpha}x, & \text{if} \quad x \ge 0, \\ \mu|x|^{\frac{1}{2\beta+1}}, & \text{if} \quad x < 0, \end{cases} \quad x(0)=x(1)=0, \quad \left|x'(0)\right| = 1, \tag{26}$$

where $\alpha > 0$ and $\beta > 0$. "Negative" part of equation is now sublinear. Computations show that the Fuchik spectrum for the problem (26) consists of two straight lines

$$F_0^+ = \Big\{ \big((2A_\alpha)^{2\alpha+2}(\alpha+1); \mu\big): \quad \mu \geq 0 \Big\},$$

$$F_0^- = \Big\{ \big(\lambda; (2A_\beta)^{\frac{2\beta+2}{2\beta+1}} \frac{\beta+1}{2\beta+1}\big): \quad \lambda \geq 0 \Big\},$$

and curves

$$F_{2i-1}^+ = \Bigg\{ (\lambda;\mu): \quad i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + i\frac{2A_\beta(\frac{\beta+1}{2\beta+1})^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} = 1 \Bigg\},$$

$$F_{2i-1}^- = \Bigg\{ (\lambda;\mu): \quad i\frac{2A_\beta(\frac{\beta+1}{2\beta+1})^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} + i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} = 1 \Bigg\},$$

$$F_{2i}^+ = \Bigg\{ (\lambda;\mu): \quad (i+1)\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} + i\frac{2A_\beta(\frac{\beta+1}{2\beta+1})^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} = 1 \Bigg\},$$

$$F_{2i}^- = \Bigg\{ (\lambda;\mu): \quad (i+1)\frac{2A_\beta(\frac{\beta+1}{2\beta+1})^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} + i\frac{2A_\alpha(\alpha+1)^{\frac{1}{2\alpha+2}}}{\lambda^{\frac{1}{2\alpha+2}}} = 1 \Bigg\},$$

where meaning of A_α and A_β is the same as above.

The respective Fuchik spectrum is depicted in Fig. 4.

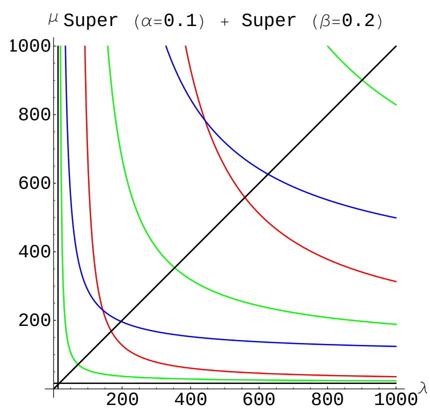

Fig. 3. Fuchik spectrum for the case super+super, the first six pairs of branches.

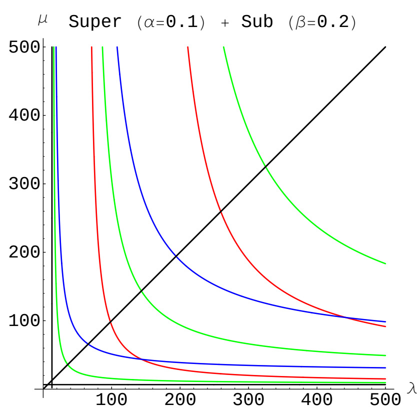

Fig. 4. Fuchik spectrum for the case super+sub, the first six pairs of branches.

3.2.3 Sublinear+superlinear

Consider the boundary value problem

$$x'' = \begin{cases} -\lambda |x|^{\frac{1}{2\alpha+1}}, & \text{if} \quad x \geq 0, \\ -\mu |x|^{2\beta}\, x, & \text{if} \quad x < 0, \end{cases} \quad x(0) = x(1) = 0, \quad \big|x'(0)\big| = 1, \tag{27}$$

where $\alpha > 0$ and $\beta > 0$. "Positive" part of equation is sublinear. Computations show that the Fuchik spectrum for the problem (26) consists of two straight lines

$$F_0^+ = \Big\{ \big((2A_\alpha)^{\frac{2\alpha+2}{2\alpha+1}}\, \frac{\alpha+1}{2\alpha+1}; \mu \big) : \quad \mu \geq 0 \Big\},$$
$$F_0^- = \Big\{ \big(\lambda; (2A_\beta)^{2\beta+2}(\beta+1) \big) : \quad \lambda \geq 0 \Big\},$$

and curves

$$F_{2i-1}^+ = \Big\{ (\lambda;\mu) : \quad i\frac{2A_\alpha (\frac{\alpha+1}{2\alpha+1})^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} + i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1 \Big\},$$
$$F_{2i-1}^- = \Big\{ (\lambda;\mu) : \quad i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\frac{2A_\alpha (\frac{\alpha+1}{2\alpha+1})^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} = 1 \Big\},$$
$$F_{2i}^+ = \Big\{ (\lambda;\mu) : \quad (i+1)\frac{2A_\alpha (\frac{\alpha+1}{2\alpha+1})^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} + i\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1 \Big\},$$
$$F_{2i}^- = \Big\{ (\lambda;\mu) : \quad (i+1)\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\frac{2A_\alpha (\frac{\alpha+1}{2\alpha+1})^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} = 1 \Big\},$$

where A_α and A_β are as above.

The respective Fuchik spectrum is depicted in Fig. 5.

3.2.4 Sublinear+sublinear

Consider the boundary value problem

$$x'' = \begin{cases} -\lambda |x|^{\frac{1}{2\alpha+1}}, & \text{if} \quad x \geq 0, \\ \mu |x|^{\frac{1}{2\beta+1}}, & \text{if} \quad x < 0, \end{cases} \quad x(0) = x(1) = 0, \quad \big|x'(0)\big| = 1, \tag{28}$$

where $\alpha > 0$ and $\beta > 0$. Both "positive" and "negative" parts of equation are sublinear. Computations show that the Fuchik spectrum for the problem (28) consists of two straight lines

$$F_0^+ = \Big\{ \big((2A_\alpha)^{\frac{2\alpha+2}{2\alpha+1}}\, \frac{\alpha+1}{2\alpha+1}; \mu \big) : \quad \mu \geq 0 \Big\},$$

$$F_0^- = \Big\{ \big(\lambda; (2A_\beta)^{\frac{2\beta+2}{2\beta+1}}\, \frac{\beta+1}{2\beta+1} \big) : \quad \lambda \geq 0 \Big\},$$

and curves

$$F_{2i-1}^{+} = \left\{(\lambda;\mu):\ \ i\frac{2A_\alpha\left(\frac{\alpha+1}{2\alpha+1}\right)^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} + i\frac{2A_\beta\left(\frac{\beta+1}{2\beta+1}\right)^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} = 1\right\},$$

$$F_{2i-1}^{-} = \left\{(\lambda;\mu):\ \ i\frac{2A_\beta\left(\frac{\beta+1}{2\beta+1}\right)^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} + i\frac{2A_\alpha\left(\frac{\alpha+1}{2\alpha+1}\right)^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} = 1\right\},$$

$$F_{2i}^{+} = \left\{(\lambda;\mu):\ \ (i+1)\frac{2A_\alpha\left(\frac{\alpha+1}{2\alpha+1}\right)^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} + i\frac{2A_\beta\left(\frac{\beta+1}{2\beta+1}\right)^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} = 1\right\},$$

$$F_{2i}^{-} = \left\{(\lambda;\mu):\ \ (i+1)\frac{2A_\beta\left(\frac{\beta+1}{2\beta+1}\right)^{\frac{2\beta+1}{2\beta+2}}}{\mu^{\frac{2\beta+1}{2\beta+2}}} + i\frac{2A_\alpha\left(\frac{\alpha+1}{2\alpha+1}\right)^{\frac{2\alpha+1}{2\alpha+2}}}{\lambda^{\frac{2\alpha+1}{2\alpha+2}}} = 1\right\},$$

where A_α and A_β are as above.

The respective Fuchik spectrum is depicted in Fig. 6.

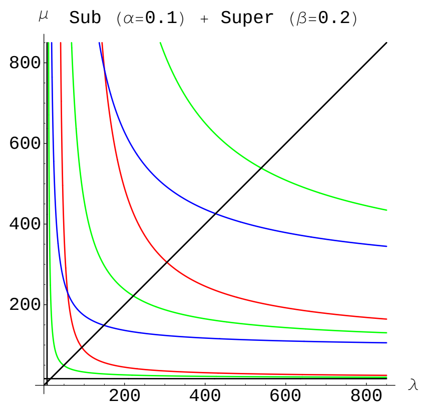

Fig. 5. Fuchik spectrum for the case sub+super, the first six pairs of branches.

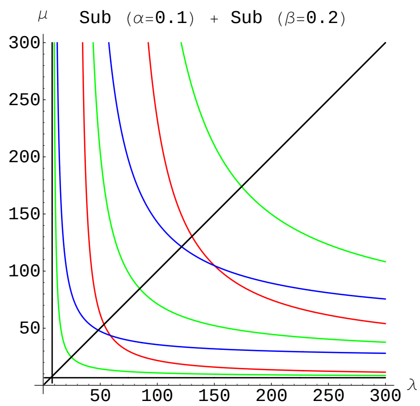

Fig. 6. Fuchik spectrum for the case sub+sub, the first six pairs of branches.

Remark 2. *The even-numbered branches of the spectrum cannot intersect at the bisectrix unless $\alpha = \beta$ (see Subsection* 3.2.1 *for explanation).*

4 Semilinear spectra for Fuchik type problems

Consider semilinear problems, where equation is linear for x positive, and superlinear for x negative. Let equation be of the form

$$x'' = \begin{cases} -\lambda x, & \text{if} \quad x \geq 0, \\ -\mu|x|^{2\beta}x, & \text{if} \quad x < 0, \end{cases} \qquad x(0) = x(1) = 0, \tag{29}$$

where $\lambda \geq 0$ and $\mu \geq 0$, $\beta > 0$.

Let us look for solutions, normalized by the condition $|x'(0)| = 1$. Computations show that the Fuchik spectrum for the problem (29) consists of two straight lines

$$F_0^+ = \Big\{(\pi^2; \mu): \quad \mu \geq 0\Big\},$$

$$F_0^- = \Big\{\big(\lambda; (2A_\beta)^{2\beta+2}(\beta+1)\big): \quad \lambda \geq 0\Big\},$$

where $A_\beta = \int\limits_0^1 \dfrac{dt}{\sqrt{1 - t^{2\beta+2}}}$ and curves

$$F_{2i-1}^+ = \left\{(\lambda; \mu): \quad i\,\frac{\pi}{\sqrt{\lambda}} + i\,\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1\right\},$$

$$F_{2i-1}^- = \left\{(\lambda; \mu): \quad i\,\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\,\frac{\pi}{\sqrt{\lambda}} = 1\right\},$$

$$F_{2i}^+ = \left\{(\lambda; \mu): \quad (i+1)\,\frac{\pi}{\sqrt{\lambda}} + i\,\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} = 1\right\},$$

$$F_{2i}^- = \left\{(\lambda; \mu): \quad (i+1)\,\frac{2A_\beta(\beta+1)^{\frac{1}{2\beta+2}}}{\mu^{\frac{1}{2\beta+2}}} + i\,\frac{\pi}{\sqrt{\lambda}} = 1\right\}.$$

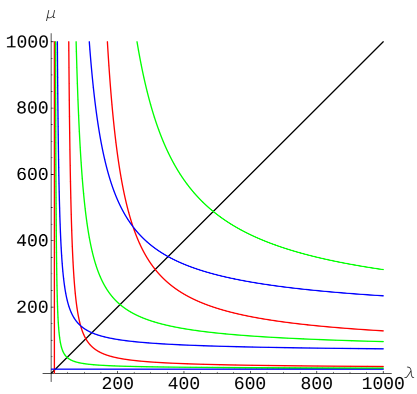

Fig. 7. The first branches of the Fuchik spectrum for the problem (29), $\beta = 0.1$.

4.1 Semilinear boundary value problems

At the end we consider the boundary value problem for equation

$$x'' + g(x) = f(t, x, x'), \quad g \in C^1, \; f, f_x, f_{x'} \in C^1; \tag{30}$$

where g is the principal (semilinear) term, and f is bounded.

Suppose that $g(x)$ satisfies the conditions:

(C1) $g(x)/x \to \omega^2$ as $x \to +\infty$ ($g(x)$ is "almost linear" at $+\infty$);

(C2) $g(x) < K|x|^p x$, $x < -N$, where $K > 0$, $N > 0$ and $p > 0$ are constants.

A sample equation might be

$$x'' = \begin{cases} -\omega^2 x, & \text{if} \quad x \geq 0, \\ -Kx^3, & \text{if} \quad x < 0. \end{cases}$$

In the related literature often the linear eigenvalue problem

$$x'' + k^2 x = 0, \tag{31}$$

$$x(0) = 0, \quad x(1) = 0 \tag{32}$$

is considered for comparison. The interval $(-\infty, \omega)$ is compared with the eigenvalues $k_1 = \pi, k_2 = 2\pi, \ldots$ of the problem (31), (32). If

$$-\infty < k_1 < \ldots < k_i < \omega < k_{i+1}, \tag{33}$$

then $g(x)$ is referred to as "nonlinearity crossing several eigenvalues."

The conditions **(C1)** and **(C2)** are insufficient to make conclusions on the number of solutions to the problem (30), (32). An extra condition is needed.

Introduce the additional condition in the spirit of the work [14], where motivation and reasoning can be found:

(C3) there exists the trivial solution $x \equiv 0$ to the problem (30), (32) and a solution $y(t)$ of the respective equation of variations

$$y'' + g_x(0)y = f_x(t, 0, 0)y + f_{x'}(t, 0, 0)y', \tag{34}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{35}$$

has exactly m zeros in $(0, 1)$ and $y(1) \neq 0$.

Theorem 2. *Let the conditions* **(C1)** *to* **(C3)** *hold. Suppose that the condition* (33) *holds for some* $i = 1, 2, \ldots,$ *where* k_i *are eigenvalues of the linear problem* (31), (32)*. Then the problem* (30), (32) *has at least* $|m - 2i| + |m - (2i + 1)|$ *solutions.*

The proof is not simple, but analogous to that of the main result in [14]. It can be carried out considering solutions $x(t; \gamma)$ of the Cauchy problem (30), $x(0) = 0$, $x'(0) = \gamma$. For γ small in modulus solutions $x(t; \gamma)$ behave like $y(t)$. For γ positive and large $x(t; \gamma)$ behave like solutions of $z'' + \omega^2 z = 0$, $z(0) = 0$, $z'(0) = 1$. For γ negative and large $x(t; \gamma)$ behave like solutions of superlinear equation with $t_1(\gamma)$ (the first zero) tending to zero as γ tends to $-\infty$. "Behaves like" means here "has the same number of zeros in $(0, 1)$." We omit the proof.

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