The Nehari Manifold for p-Laplacian Equation with Dirichlet Boundary Condition

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Abstract. The Nehari manifold for the equation $-\Delta_p u(x) = \lambda u(x)|u(x)|^{p-2} + b(x)|u(x)|^{\gamma-2}u(x)$ for $x \in \Omega$ together with Dirichlet boundary condition is investigated in the case where $0 < \gamma < p$. Exploiting the relationship between the Nehari manifold and fibrering maps (i.e., maps of the form of $t \to J(tu)$ where J is the Euler functional associated with the equation), we discuss how the Nehari manifold changes as λ changes, and show how existence results for positive solutions of the equation are linked to the properties of Nehari manifold.

Keywords: the p-Laplacian, variational methods, Nehari manifold, fibrering maps.

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1 Introduction

Consider the semilinear boundary value problem

$$\begin{cases} -\Delta_p u(x) = \lambda u(x)\big|u(x)\big|^{p-2} + b(x)\big|u(x)\big|^{\gamma-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded region with smooth boundary in R^N and $b\colon \Omega \to R$ is a smooth function which may change sign.

The study of elliptic equations involving the p-Laplacian and using the fibrering method sees great increase in number of papers published, see [1–3] which have studied the equation with convex-concave linearity. Notice that these results have also generalized to (p, q)-system in the papers such as [4, 5] using the fibrering method.

In this paper we have generalized the article of Brown and Zhang [6] to the p-Laplacian by using fibrering method for $1 < \gamma < p$. This problem when $\gamma > p$ has been studied by Binding et al. [7, 8] by using variational method.

We shall discuss the existence and multiplicity of non-negative solution of (1) from a variational viewpoint making use of the Nehari manifold [9, 10].

Suppose that λ_1 is the principal eigenvalue of the linear problem

$$\begin{cases} -\Delta_p u(x) = \lambda u(x)\big|u(x)\big|^{p-2}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

The direction of bifurcation being determined by the sign of $\int_\Omega b\phi_1^\gamma dx$ where ϕ_1 is the positive principal eigenvalue corresponding to λ_1. We shall show precisely the important role played by $\int_\Omega b\phi_1^\gamma dx$ by investigating the Nehari manifold changes with λ.

The Euler function associated with (1) is

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{\gamma}\int_\Omega b|u|^\gamma dx, \quad u \in W_o^{1,p}(\Omega).$$

By the spectral theorem

$$\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx \geq (\lambda_1 - \lambda)\int_\Omega |u|^p dx \quad \text{for all } u \in W_o^{1,p}(\Omega)$$

and so

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}(\lambda_1 - \lambda)\int_\Omega |u|^p dx - \frac{\overline{b}}{\gamma}\int_\Omega |u|^\gamma dx \\ &\geq \frac{1}{p}(\lambda_1 - \lambda)\int_\Omega |u|^p dx - \frac{\overline{b}}{\gamma}|\Omega|^{1-\frac{\gamma}{p}}\Big(\int_\Omega |u|^p dx\Big)^{\frac{\gamma}{p}}, \end{aligned}$$

where $\overline{b} = \sup_{x\in\Omega} b(x)$. Hence J_λ is bounded below on $W_o^{1,p}(\Omega)$ when $\lambda < \lambda_1$. It is easy to see, however, that, when $\lambda > \lambda_1$, $\lim_{t\to\infty} J_\lambda(t\phi_1) = -\infty$ and so J_λ is no longer bounded below on $W_o^{1,p}(\Omega)$. In order to obtain existence results in this case we introduce the Nehari manifold

$$S(\lambda) = \big\{u \in W_o^{1,p}(\Omega) \colon \big\langle J'_\lambda(u), u\big\rangle = 0\big\},$$

where $\langle\, , \rangle$ the usual duality. Thus $u \in S(\lambda)$ if and only if

$$\int_\Omega |\nabla u|^p dx - \lambda\int_\Omega |u|^p dx - \int_\Omega b|u|^\gamma dx = 0.$$

Clearly $S(\lambda)$ is a much smaller set than $W_o^{1,p}(\Omega)$ and so it is easier to study J_λ on $S(\lambda)$. On $S(\lambda)$ we have that

$$J_\lambda(u) = \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx = \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\int_\Omega b|u|^\gamma dx. \tag{3}$$

The Nehari manifold is closely linked to the behavior of the form $\phi_u : t \to J_\lambda(tu)(t > 0)$. Such maps are known as fibrering maps and were introduce by Drabek and Pohozaev in [11]. If $u \in W_o^{1,p}(\Omega)$, we have

$$\phi_u(t) = \frac{t^p}{p}\int\limits_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx - \frac{t^\gamma}{\gamma}\int\limits_\Omega b|u|^\gamma dx, \tag{4}$$

$$\phi_u'(t) = t^{p-1}\int\limits_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx - t^{\gamma-1}\int\limits_\Omega b|u|^\gamma dx, \tag{5}$$

$$\phi_u''(t) = (p-1)t^{p-2}\int\limits_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx - (\gamma-1)t^{\gamma-2}\int\limits_\Omega b|u|^\gamma dx. \tag{6}$$

It is easy to see that $u \in S(\lambda)$ if and only if $\phi_u'(1) = 0$ and more generally that $\phi_u'(t) = 0$ if and only if $tu \in S(\lambda)$, i.e., elements in $S(\lambda)$ correspond to stationary points of fibrering maps. Thus it is natural to subdivide $S(\lambda)$ into sets corresponding to local minima, local maxima and points of inflection. It follows from (5) and (6) that if $\phi_u'(t) = 0$, then $\phi_u''(t) = (p-\gamma)t^{\gamma-2}\int_\Omega b|u|^\gamma dx$. Thus we define

$$S^+(\lambda) = \Big\{u \in S(\lambda) \colon \int\limits_\Omega b|u|^\gamma dx > 0\Big\},$$

$$S^-(\lambda) = \Big\{u \in S(\lambda) \colon \int\limits_\Omega b|u|^\gamma dx < 0\Big\},$$

$$S^0(\lambda) = \Big\{u \in S(\lambda) \colon \int\limits_\Omega b|u|^\gamma dx = 0\Big\},$$

so that S^+, S^-, S^0 corresponding to minima, maxima and points of inflection respectively.

Let $u \in W_0^{1,p}(\Omega)$. Then

1) if $\int_\Omega(|\nabla u|^p - \lambda|u|^p)dx$ and $\int_\Omega b|u|^\gamma dx$ have the same sign, ϕ_u has a unique turning point at

$$t(u) = \left[\frac{\int_\Omega b|u|^\gamma dx}{\int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx}\right]^{\frac{1}{p-\gamma}}$$

this turning point is a local minimum(maximum) so that $t(u)u \in S^+(\lambda)((S^-(\lambda))$ if and only if $\int_\Omega b|u|^\gamma dx > 0(< 0)$;

2) if $\int_\Omega(|\nabla u|^p - \lambda|u|^p)dx$ and $\int_\Omega b|u|^\gamma dx$ have different sign, then ϕ_u has no turning points and so no multiples of u lie in $S(\lambda)$.

Thus, if we define

$$L_+(\lambda) = \Big\{ u \in W_0^{1,p}(\Omega) \colon \|u\| = 1 \ \text{ and } \int_\Omega \big(|\nabla u|^p - \lambda |u|^p\big) dx > 0 \Big\},$$

$$B_+ = \Big\{ u \in W_0^{1,p}(\Omega) \colon \|u\| = 1 \ \text{ and } \int_\Omega b|u|^\gamma dx > 0 \Big\},$$

where the norm of $W_0^{1,p}(\Omega)$ is defined as $\|u\| = \{\int_\Omega |\nabla|^p dx\}^{\frac{1}{p}}$ and analogously define $L_-(\lambda)$, $L_0(\lambda)$, B_-, B_0 by replacing $' > 0'$ by $' < 0'$ or $' = 0'$ respectively.

Thus, if $u \in L_+(\lambda) \cap B_+$, $\phi_u(t) < 0$ for t small and negative, $\phi_u(t) \to \infty$ as $t \to \infty$ and $\phi_u(t)$ has a unique minimum at $t(u)$ so that $t(u)u \in S^+(\lambda)$. Similarly if $u \in L_-(\lambda) \cap B_-$, $\phi_u(t) > 0$ for t small and positive, $\phi_u(t) \to -\infty$ as $t \to \infty$ and $\phi_u(t)$ has a unique maximum at $t(u)$ so that $t(u)u \in S^-(\lambda)$. Finally if $u \in L_+(\lambda) \cap B_-$ (resp. $u \in L_-(\lambda) \cap B_+$), $\phi_u(t)$ is strictly decreasing (resp. increasing) for all $(t > 0)$.

Thus we have

1) if $u \in L_+(\lambda) \cap B_+$, then $t \to \phi_u(t)$ has a local minimum at $t = t(u)$ and $t(u)u \in S^+(\lambda)$;

2) if $u \in L_-(\lambda) \cap B_-$, then $t \to \phi_u(t)$ has a local maximum at $t = t(u)$ and $t(u)u \in S^-(\lambda)$;

3) if $u \in L_+(\lambda) \cap B_-$, then $t \to \phi_u(t)$ is strictly increasing and no multiple of u lies in $S(\lambda)$;

4) if $u \in L_-(\lambda) \cap B_+$, then $t \to \phi_u(t)$ is strictly decreasing and no multiple of u lies in $S(\lambda)$.

The Euler functional changes sign in $S(\lambda)$, it is positive in $S^-(\lambda)$ and is negative in $S^+(\lambda)$. We shall prove the existence of solutions of (1) by investigating the existence of minimizers on $S(\lambda)$. Although $S(\lambda)$ is only a small subset of $W_o^{1,p}(\Omega)$, it turns out that minimizers of $J(\lambda)$ on $S(\lambda)$ are generically also critical points of $J(\lambda)$ on $W_o^{1,p}(\Omega)$. We have

Lemma 1. *Suppose that u_0 is a local maximum or minimum for $J(\lambda)$ on $S(\lambda)$. If $u_0 \notin S^0$, then u_0 is a critical point of $J(\lambda)$.*

Proof. If u_0 is a local minimizer for J on $S(\lambda)$, then u_0 is a solution of the optimization problem

$$\text{minimizer } \ J(u) \ \text{ subject to } \ \gamma(u) = 0,$$

where $\gamma(u) = \int_\Omega (|\nabla u|^p - \lambda |u|^p - b|u|^\gamma) dx$. Hence, by the theory of Lagrange multipliers, there exists $\mu \in R$ such that $J'(u_0) = \mu\gamma'(u_0)$. Thus

$$\big\langle J'(u_0), u_0 \big\rangle = \mu \big\langle \gamma'(u_0), u_0 \big\rangle. \tag{1'}$$

Since $u_0 \in S(\lambda)$, $\langle J'(u_0), u_0\rangle = 0$ and so $\int_\Omega |\nabla u_0|^p dx = \int(\lambda|u_0|^p + b|u_0|^\gamma)dx$. Hence

$$\langle \gamma'(u_0), u_0\rangle = p\int_\Omega \big(|\nabla u_0|^p - \lambda|u_0|^p\big)dx - \gamma\int b|u_0|^\gamma dx = (p-\gamma)\int b|u_0|^\gamma dx.$$

Thus if $u_0 \notin S^0(\lambda)$, $\langle \gamma'(u_0), u_0\rangle \neq 0$ and so by (1.1) $\mu = 0$. Hence the proof is complete. □

The plane of the paper is as follows. In Section 2 we show the importance of the condition $L_-(\lambda) \subseteq B_-$ in determining the nature of the Nehari manifold, in Section 3 we prove results about the existence of minimizers on the Nehari manifold and in Section 4 we discuss how the previous results yield information about non-negative solutions of (1) as λ changes and in particular about bifurcation from infinity. In Section 5 we investigate the nature of the Nehari manifold in cases where it is known that no non-trivial non-negative solutions of (1) exist.

Finally, it should be noted that our results hold only in the cases where the nonlinearity is a homogeneous function. This ensures that the fibrering maps involve only power of t and the simplicity of our proof rely heavily on this fact. The corresponding existence and global bifurcation results hold in much more general or abstract setting and it seems likely that analougous results for Nehari manifolds should also hold in such cases.

2 Properties of the Nehari manifold

When $\lambda < \lambda_1$, $\int_\Omega(|\nabla u|^p - \lambda|u|^p)dx > 0$ for all $u \in W_o^{1,p}(\Omega)$ and so $L_+(\lambda) = \{u \in W_0^{1,p}(\Omega)\colon \|u\| = 1\}$ and $L_-(\lambda), L_0(\lambda) = \emptyset$. When $\lambda = \lambda_1$, we have $L_-(\lambda) = \emptyset$ and $L_0(\lambda) = \{\phi_1\}$ and when λ is greater than λ_1, $L_-(\lambda)$ becomes non-empty and gets bigger as λ increases. In this section we shall discuss the vital role played by the condition $L_-(\lambda) \subset B_-$ in determining the nature of the Nehari manifold. In view of the preceding remarks it is easy to see that this condition is always satisfied when $\lambda < \lambda_1$, may or may not be satisfied when $\lambda > \lambda_1$ and is increasingly likely to be violated as λ increases.

Theorem 1. *Suppose there exists $\widehat{\lambda}$ such that for all $\lambda < \widehat{\lambda}$, $L_-(\lambda) \subset B_-$. Then, for all $\lambda < \widehat{\lambda}$,*

(i) *$L_0(\lambda) \subseteq B_-$ and so $L_0(\lambda) \bigcap B_0 = \emptyset$;*

(ii) *$S^+(\lambda)$ is bounded;*

(iii) *$0 \notin \overline{S^-(\lambda)}$ and $S^-(\lambda)$ is closed;*

(iv) *$\overline{S^+(\lambda)} \bigcap S^-(\lambda) = \emptyset$.*

Proof. (i) Suppose that the result is false. Then there exists $u \in L_0(\lambda)$ such that $u \notin B_-$. If $\lambda < \mu < \widehat{\lambda}$, then $u \in L_-(\mu)$ and so $L_-(\mu) \not\subseteq B_-$ which is a contradiction.

(ii) Suppose that $S^+(\lambda)$ is unbounded. Then there exists $\{u_n\} \subseteq S^+(\lambda)$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. We may assume without loss of generality that $v_n \rightharpoonup v_0$ in $W_o^{1,p}(\Omega)$ and so $v_n \to v_0$ in $L^p(\Omega)$ and in $L^\gamma(\Omega)$. Since $u_n \in S^+(\lambda)$, $\int_\Omega b|v_n|^\gamma dx > 0$ and so $\int_\Omega b|v_0|^\gamma dx \geq 0$.

Since $u_n \in S(\lambda)$,

$$\int_\Omega \big(|\nabla u_n|^p - \lambda |u_n|^p\big)dx = \int_\Omega b|u_n|^\gamma dx$$

and so

$$\int_\Omega \big(|\nabla v_n|^p - \lambda |v_n|^p\big)dx = \int_\Omega b|v_n|^\gamma \frac{1}{\|u_n\|^{p-\gamma}} dx \to 0.$$

Suppose $v_n \not\to v_0$ in $W_0^{1,p}(\Omega)$. Then $\int_\Omega (|\nabla v_0|^p dx < \underline{\lim} \int_\Omega (|\nabla v_n|^p dx$ and so

$$\int_\Omega \big(|\nabla v_0|^p - \lambda |v_0|^p\big)dx < \lim_{n\to\infty} \int_\Omega \big(|\nabla v_n|^p - \lambda |v_n|^p\big)dx = 0.$$

Thus $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \subset B_-$ which is impossible as $\int_\Omega b|v_0|^\gamma dx \geq 0$.

Hence $v_n \to v_0$ in $W_o^{1,p}(\Omega)$. Thus $\|v_0\| = 1$ and

$$\int_\Omega \big(|\nabla v_0|^p - \lambda |v_0|^p\big)dx = \lim_{n\to\infty} \int_\Omega \big(|\nabla v_n|^p - \lambda |v_n|^p\big)dx = 0.$$

Thus $v_0 \in L_0(\lambda) \subseteq B_-$ which is again impossible. Hence $S^+(\lambda)$ is bounded.

(iii) Suppose $0 \in \overline{S^-(\lambda)}$. Then there exists $\{u_n\} \subseteq S^-(\lambda)$ such that $\lim_{n\to\infty} u_n = 0$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then we may assume that $v_n \rightharpoonup v_0$ in $W_o^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$.

Since $u_n \in S^-(\lambda)$, we have

$$\int_\Omega \big(|\nabla v_n|^p - \lambda |v_n|^p\big)dx = \frac{1}{\|u_n\|^{p-\gamma}} \int_\Omega b|v_n|^\gamma dx \leq 0.$$

Since the left hand side is bounded, it follows that $\lim_{n\to\infty} \int_\Omega b|v_n|^\gamma dx = 0$ and so $\int_\Omega b|v_0|^\gamma dx = 0$.

Suppose $v_n \to v_0$. Then $\|v_0\| = 1$ and so $v_0 \in B_0$. Moreover

$$\int_\Omega \big(|\nabla v_0|^p - \lambda |v_0|^p\big)dx = \lim_{n\to\infty} \int_\Omega \big(|\nabla v_n|^p - \lambda |v_n|^p\big)dx \leq 0$$

and so $v_0 \in L_0(\lambda)$ or $L_-(\lambda)$. Hence $v_0 \in B_-$ and this is impossible.

Thus we must have that $v_n \not\to v_0$ in $W_o^{1,p}(\Omega)$. Then

$$\int_\Omega \big(|\nabla v_0|^p - \lambda|v_0|^p\big)dx < \lim_{n\to\infty}\int_\Omega \big(|\nabla v_n|^p - \lambda|v_n|^p\big)dx \le 0.$$

Hence $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \cap B_0$ which is impossible and so $0 \notin \overline{S^-(\lambda)}$.

We now prove that $S^-(\lambda)$ is closed. Suppose $\{u_n\} \subseteq S^-(\lambda)$ and $u_n \to u$ in $W_o^{1,p}(\Omega)$. Then $u \in \overline{S^-(\lambda)}$ and so $u \neq 0$. Moreover,

$$\int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx = \int_\Omega b|u|^\gamma dx \le 0.$$

If both integrals equal 0, than $\frac{u}{\|u\|} \in L_0(\lambda) \cup B_0$ which is contradicts (i). Hence both integrals must be negative and so $u \in S^-(\lambda)$. Thus $S^-(\lambda)$ is closed.

(iv) Let $u \in \overline{S^+(\lambda)} \cap S^-(\lambda)$. As $u \in S^-(\lambda)$, $u \neq 0$. Moreover it is clear that

$$\int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx = \int_\Omega b|u|^\gamma dx = 0.$$

and so $\frac{u}{\|u\|} \in L_0(\lambda) \bigcap B_0$ which is impossible.

We can also deduce important results about the behaviour of J_λ on $S^+(\lambda)$ and $S^-(\lambda)$. By considering fibrering maps it is clear that $J_\lambda(u) > 0$ on $S^-(\lambda)$ and $J_\lambda(u) < 0$ on $S^+(\lambda)$. Moreover

Theorem 2. *Suppose the same hypotheses are satisfied as in Theorem* 1. *Then*

(i) J_λ *is bounded below on* $S^+(\lambda)$*;*

(ii) $\inf_{u\in S^-(\lambda)} J_\lambda(u) > 0$ *provided* $S^-(\lambda)$ *is non-empty.*

Proof. (i) is an immediate consequence of the boundedness of $S^+(\lambda)$.

(ii) Suppose $\inf_{u\in S^-(\lambda)} J_\lambda(u) = 0$. Then there exists $\{u_n\} \subseteq S^-(\lambda)$ such that $\lim_{n\to\infty} J_\lambda(u_n) = 0$. Then it is clear from (3) that

$$\int_\Omega \big(|\nabla u_n|^p - \lambda|u_n|^p\big)dx \to 0 \quad \text{and} \quad \int_\Omega b|u_n|^\gamma dx \to 0 \quad \text{as} \quad n\to\infty.$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Since $0 \notin \overline{S^-(\lambda)}$, $\{\|u_n\|\}$ is bounded away from 0. Hence

$$\lim_{n\to\infty}\int_\Omega \big(|\nabla v_n|^p - \lambda|v_n|^p\big)dx = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_\Omega b|v_n|^\gamma dx = 0.$$

We may assume that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$. Then $\int_\Omega b|v_0|^\gamma dx = 0$.

If $v_n \to v_0$, we have $\|v_0\| = 1$ and $\int_\Omega (|\nabla v_0|^p - \lambda |v_0|^p)dx = 0$, i.e., $v_0 \in L_0(\lambda)$, whereas, if $v_n \not\to v_0$, $\int_\Omega (|\nabla v_0|^p - \lambda |v_0|^p)dx < 0$, i.e., $\frac{v_0}{\|v_0\|} \in L_-(\lambda)$. In both cases, however, we must also have $\frac{v_0}{\|v_0\|} \in B_0$ and this contradiction. Hence $\inf_{u \in S^-(\lambda)} J_\lambda(u) > 0$.

Lemma 2. *Suppose $L_-(\lambda) \cap B_+ \neq \emptyset$. Then there exists $k > 0$ such that for every $\varepsilon > 0$ there exists $u_\varepsilon \in L_+(\lambda) \cap B_+$ such that*

$$\int_\Omega \left(|\nabla u_\varepsilon|^2 - \lambda |u_\varepsilon|^2\right)dx < \varepsilon \quad \textit{and} \quad \int_\Omega b|u_\varepsilon|^\gamma dx > k.$$

3 The existence of minimizers

Theorem 3. *Suppose $L_-(\lambda) \subseteq B_-$ for all $\lambda < \widehat{\lambda}$. Then, for all $\lambda < \widehat{\lambda}$,*

(i) *there exists a minimizer for J_λ on $S^+(\lambda)$;*

(ii) *there exists a minimizer for J_λ on $S^-(\lambda)$ provided that $L_-(\lambda)$ is non-empty.*

Proof. By Theorem 2 J_λ is bounded below on $S^+(\lambda)$. Let $\{u_n\} \subseteq S^+(\lambda)$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in S^+(\lambda)} J_\lambda(u) < 0.$$

Since $S^+(\lambda)$ is bounded, we may assume that $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and $u_n \to u_0$ in $L^p(\Omega)$. Since $J_\lambda(u_n) = (\frac{1}{p} - \frac{1}{\gamma}) \int_\Omega b|u_n|^\gamma dx$, it follows that

$$\int_\Omega b|u_0|^\gamma dx = \lim_{n\to\infty} \int_\Omega b|u_n|^\gamma dx > 0$$

and so $\frac{u_0}{\|u_0\|} \in B_+$. Hence by Theorem 1, $\frac{u_0}{\|u_0\|} \in L_+(\lambda)$ and so the fibrering map ϕ_{u_0} has a unique minimum at $t(u_0)$ such that $t(u_0)u_0 \in S^+(\lambda)$.

Suppose $u_n \not\to u_0$ in $W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\int_\Omega \left(|\nabla u_0|^p - \lambda |u_0|^p\right)dx &< \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^p - \lambda |u_n|^p)dx \\ &= \lim_{n\to\infty} \int_\Omega b|u_n|^\gamma dx = \int_\Omega b|u_0|^\gamma dx\end{aligned}$$

and so $t(u_0) > 1$. Hence

$$J_\lambda(t(u_0)u_0) < J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in S^+(\lambda)} J_\lambda(u)$$

which is impossible.

Hence $u_n \to u_0$ and so $u_0 \in S(\lambda)$. It now follows easily that u_0 is a minimizer for J_λ on $S^+(\lambda)$.

(ii) Let $\{u_n\}$ be a minimizing sequence for J_λ on $S^-(\lambda)$. Then by Theorem 2 we must have

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-(\lambda)} J_\lambda(u) > 0.$$

Suppose that $\{u_n\}$ is unbounded, we may suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Since $\{J_\lambda(u_n)\}$ is bounded, it follows that $\{\int_\Omega(|\nabla u_n|^p - \lambda|u_n|^p)dx\}$ and $\{\int_\Omega b|u_n|^\gamma dx\}$ are bounded and so

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla v_n|^p - \lambda|v_n|^p\right)dx = \lim_{n\to\infty}\int_\Omega b|v_n|^\gamma dx = 0.$$

Since $\{v_n\}$ is bounded, we may assume that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ so that $\int_\Omega b|v_0|^\gamma dx = 0$.

If $v_n \to v_0$ in $W_0^{1,p}(\Omega)$, it is easy to see that $v_0 \in L_0(\lambda) \cap B_0$ which is impossible because of Theorem 1(i).

Hence $v_n \not\to v_0$ in $W_0^{1,p}(\Omega)$ and so

$$\int_\Omega \left(|\nabla v_0|^p - \lambda|v_0|^p\right)dx < \lim_{n\to\infty}\int_\Omega \left(|\nabla v_n|^p - \lambda|v_n|^p\right)dx = 0.$$

Hence $v_0 \neq 0$ and $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \cap B_0$ which is again impossible.

Thus $\{u_n\}$ is bounded and so we may assume that $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and $u_n \to u_0$ in $L^p(\Omega)$. Suppose $u_n \not\to u_0$ in $W_0^{1,p}(\Omega)$. Then

$$\int_\Omega b|u_0|^\gamma dx = \lim_{n\to\infty}\int_\Omega b|u_n|^\gamma dx = \left(\frac{1}{p} - \frac{1}{\gamma}\right)^{-1}\lim_{n\to\infty} J_\lambda(u_n) < 0$$

and

$$\begin{aligned}\int_\Omega \left(|\nabla u_0|^p - \lambda|u_0|^p\right)dx &< \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^p - \lambda|u_n|^p\right)dx \\ &= \lim_{n\to\infty}\int_\Omega b|u_n|^\gamma dx = \int_\Omega b|u_0|^\gamma dx.\end{aligned}$$

Hence $\frac{u_0}{\|u_0\|} \in L_-(\lambda) \cap B_-$ and so $t(u_0)u_0 \in S^-(\lambda)$ where

$$t(u_0) = \left[\frac{\int_\Omega b|u_0|^\gamma dx}{\int_\Omega(|\nabla u_0|^p - \lambda|u_0|^p)dx}\right]^{\frac{1}{p-\gamma}} < 1.$$

Moreover $t(u_0)u_n \rightharpoonup t(u_0)u_0$ but $t(u_0)u_n \not\to t(u_0)u_0$ and so

$$J_\lambda(t(u_0)u_0) < \underline{\lim}_{n\to\infty} J_\lambda\big(t(u_0)u_n\big).$$

Since the map $t \to J_\lambda(tu_n)$ attains its maximum at $t = 1$,

$$\underline{\lim}_{n\to\infty} J_\lambda(t(u_0)u_n) \le \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-(\lambda)} J_\lambda(u).$$

Hence $J_\lambda(t(u_0)u_0) < \inf_{u\in S^-(\lambda)} J_\lambda(u)$ which is impossible.

Thus $u_n \to u_0$ and it follows easily u_0 is a minimizer for J_λ on $S^-(\lambda)$.

The existence of above minimizers implies the existence of corresponding non-negative solution of (1). Suppose, for example, that u_0 is a minimizer for J_λ on $S^-(\lambda)$. Since $J_\lambda(u) = J_\lambda(|u|)$, we may assume that u_0 is non-negative in Ω. Since $S^-(\lambda)$ is closed, u_0 is a local minimum for J_λ on $S(\lambda)$. It follows from Lemma 1 that u_0 is a minimizers on $S^+,(\lambda)$, $J_\lambda(u_0) < 0$. Thus u_0 must be a local minimizer on $S(\lambda)$ and so again corresponds to a classical solution of (1). So the positive solutions are saddle points of the Euler functional and are characterized as local minimum of Euler functional restricted to $S^+(\lambda)$ and $S^-(\lambda)$.

4 Bifurcation from infinity

It can be shown using bifurcation theory arguments that bifurcation from infinity occurs at $\lambda = \lambda_1$ and that the direction of this bifurcation is determined by the sign of $\int_\Omega b\phi_1^\gamma dx$. In this section we show how these facts are related to properties of the Nehari manifold for the problem.

Since $L_-(\lambda)$ is empty for $\lambda < \lambda_1$, it follows from Theorem 3 that J_λ has a minimizer on $S^+(\lambda)$ whenever $\lambda < \lambda_1$.

Our next result corresponds to the fact that a branch of positive solutions bifurcates from infinity to the left at $\lambda = \lambda_1$ when $\int_\Omega b\phi_1^\gamma dx > 0$.

Theorem 4. *Suppose* $\int_\Omega b\phi_1^\gamma dx > 0$. *Then*

$$\lim_{\lambda\to\lambda_1^-} \inf_{u\in S^+(\lambda)} J_\lambda(u) = -\infty.$$

Proof. Since $\int_\Omega b\phi_1^\gamma dx > 0$ and $\int_\Omega |\nabla\phi_1|^p - \lambda|\phi_1|^p dx = (\lambda_1 - \lambda)\int_\Omega |\phi_1|^p dx$, we have that $\phi_1 \in L_+(\lambda) \cap B_+$ for all $\lambda < \lambda_1$. Hence $t(\phi_1)\phi_1 \in S^+(\lambda)$ and

$$\begin{aligned} J_\lambda(t(\phi_1)\phi_1) &= \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\big|t(\phi_1)\big|^p \int_\Omega \big(|\nabla\phi_1|^p - \lambda|\phi_1|^p\big)dx \\ &= \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\left[\frac{\int_\Omega b|\phi_1|^\gamma dx}{\int_\Omega \big(|\nabla\phi_1|^p - \lambda|\phi_1|^p\big)dx}\right]^{\frac{p}{p-\gamma}} \int_\Omega \big(|\nabla\phi_1|^p - \lambda|\phi_1|^p\big)dx \end{aligned}$$

$$= \Big(\frac{1}{p} - \frac{1}{\gamma}\Big) \frac{\big[\int_\Omega b|\phi_1|^\gamma dx\big]^{\frac{p}{p-\gamma}}}{\big[\int_\Omega \big(|\nabla\phi_1|^p - \lambda|\phi_1|^p\big)dx\big]^{\frac{\gamma}{p-\gamma}}}$$
$$= \Big(\frac{1}{p} - \frac{1}{\gamma}\Big) \frac{1}{(\lambda_1-\lambda)^{\frac{\gamma}{p-\gamma}}} \frac{\big[\int_\Omega b|\phi_1|^\gamma dx\big]^{\frac{p}{p-\gamma}}}{\big[\int_\Omega |\phi_1|^\gamma dx\big]^{\frac{\gamma}{p-\gamma}}}.$$

Thus $\inf_{u\in S^+(\lambda)} J_\lambda(u) \le J_\lambda(t(\phi_1)\phi_1) \to -\infty$ as $\lambda \to \lambda_1^-$.

Corollary 1. *Suppose $\int_\Omega b|\phi_1|^\gamma dx > 0$. Then for every $\lambda < \lambda_1$ there exists a minimizer u_λ on $S^+(\lambda)$ such that $\lim_{\lambda\to\lambda_1^-} \|u_\lambda\| = \infty$.*

We now turn our attention to the case where $\int_\Omega b|\phi_1|^\gamma dx < 0$. In this case the hypotheses of Theorem 1 hold some way to the right of $\lambda = \lambda_1$. Moreover precisely

Lemma 3. *Suppose $\int_\Omega b|\phi_1|^\gamma dx < 0$. Then exist $\delta_1, \delta_2 > 0$ such that $u \in L_-(\lambda) \Rightarrow \int_\Omega bu^\gamma dx \le -\delta_2$ whenever $\lambda_1 \le \lambda \le \lambda_1 + \delta_1$.*

The result can be proved by a straightforward contradiction argument.

Corollary 2. *Suppose $\int_\Omega b|\phi_1|^\gamma dx < 0$ and δ_1 is as in Lemma 3. Then whenever $\lambda_1 \le \lambda \le \lambda_1 + \delta_1$, there exist minimizers u_λ and v_λ of J_λ on $S^+(\lambda)$ and $S^-(\lambda)$ respectively.*

Proof. Clearly $\phi_1 \in L_-(\lambda)$ and so $L_-(\lambda)$ is non-empty whenever $\lambda \ge \lambda_1$. By Lemma 3 the hypotheses of Theorem 3 are satisfied with $\widehat{\lambda} = \lambda_1 + \delta_1$ and so the result follows. □

The next results show that when $\int_\Omega b|\phi_1|^\gamma dx < 0$, bifurcation from infinity occurs to the right at $\lambda = \lambda_1$.

Theorem 5. *Suppose $\int_\Omega b\phi_1^\gamma dx < 0$. As $\lambda \to \lambda_1^+$, v_λ becomes unbounded in $W_0^{1,p}(\Omega)$.*

Proof. Let $v \in S^-(\lambda)$. Then $v = t(u)u$ for some $u \in L_-(\lambda)\cap B_-$. Now $\int_\Omega b|u|^\gamma dx < -\delta_2$ provided $\lambda_1 \le \lambda \le \lambda_1 + \delta_1$ and

$$0 > \int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx \ge \Big(1 - \frac{\lambda}{\lambda_1}\Big)\int_\Omega |\nabla u|^p dx = \frac{\lambda_1-\lambda}{\lambda_1},$$

so that $|\int_\Omega(|\nabla u|^p - \lambda|u|^p)dx| \le \frac{\lambda-\lambda_1}{\lambda_1}$. Hence

$$J_\lambda(v) = J_\lambda\big(t(u)u\big) = \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\big[t(u)\big]^p \int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx$$
$$= \Big(\frac{1}{\gamma} - \frac{1}{p}\Big) \frac{\big|\int_\Omega b|u|^\gamma\big|^{\frac{\gamma}{p-\gamma}}}{\big|\int_\Omega \big(|\nabla u|^p - \lambda_1|u|^p\big)dx\big|^{\frac{\gamma}{p-\gamma}}} \ge \Big(\frac{1}{\gamma} - \frac{1}{p}\Big)\frac{\lambda_1^{\frac{\gamma}{p-\gamma}}\delta_2^{\frac{\gamma}{p-\gamma}}}{(\lambda-\lambda_1)^{\frac{\gamma}{p-\gamma}}}.$$

Hence $\inf_{v\in S^-(\lambda)} J_\lambda(v) \to \infty$ as $\lambda \to \lambda_1$ and so v_λ is unbounded as $\lambda \to \lambda_1^+$.

5 The case of non-existence

Finally we show that under hypotheses where it is known that no positive solutions to (1) exist then J_λ is not bounded below on $S(\lambda)$.

Lemma 4. *J_λ is unbounded below on $S(\lambda)$ whenever $L_-(\lambda) \cap B_+ \neq \emptyset$.*

Proof. Suppose $u_0 \in L_-(\lambda) \cap B_+$. It follows from Lemma 2 that there exists $k > 0$ and a sequence $\{u_n\} \subseteq L_-(\lambda) \cap B_+$ such that $\int_\Omega b|u_n|^\gamma dx \geq k$ and $0 < \int_\Omega (|\nabla u|^p - \lambda|u|^p)dx < \frac{1}{n}$. Then,using the same computation as in the proof of Theorem 5, we have

$$\begin{aligned} J_\lambda\big(t(u_n)u_n\big) &= \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\frac{\big(\int_\Omega b|u_n|^\gamma dx\big)^{\frac{p}{p-\gamma}}}{\big(\int_\Omega \big(|\nabla u|^p - \lambda|u|^p\big)dx\big)^{\frac{\gamma}{p-\gamma}}} \\ &\leq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)n^{\frac{\gamma}{p-\gamma}}k^{\frac{p}{p-\gamma}} \to -\infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence J_λ is unbounded below on $S(\lambda)$.

The following non-existence results for (1) are well-known. For completeness we give their simple proofs.

Theorem 6. (i) *Suppose $\int_\Omega b|\phi_1|^\gamma dx > 0$. Then* (1) *has no positive solutions whenever $\lambda > \lambda_1$.*

(ii) *Equation* (1) *has no positive solutions when $\lambda > \widehat{\lambda}$ where $\widehat{\lambda}$ is the principal eigenvalue of*

$$\begin{aligned} &-\Delta_p u(x) = \lambda u(x)\big|u(x)\big|^{p-2} \ \textit{for} \ x \in \Omega^+, \quad u(x) = 0 \ \textit{for} \ x \in \partial\Omega^+, \\ &\Omega^+ = \big\{x \in \Omega\colon \ b(x) > 0\big\}. \end{aligned} \tag{7}$$

Proof. (i) Suppose $\int_\Omega b|\phi_1|^\gamma dx > 0$ and that (1) has a positive solution u. Multiplying (1) by ϕ_1, (2) by u and subtracting gives

$$\begin{aligned} &-\Delta_p u(x)\phi_1(x) + u(x)\Delta_p\phi_1(x) \\ &\qquad = (\lambda - \lambda_1)u(x)\phi_1(x) + b(x)\big|u(x)\big|^{\gamma-p}u(x)\phi_1(x) \end{aligned}$$

and so

$$\int\limits_\Omega \Big(\frac{\phi_1}{u}\Big)^{\gamma-p+1}(-\Delta u\phi_1 + u\Delta\phi_1)dx = \int\limits_\Omega (\lambda - \lambda_1)\phi_1^\gamma u^{p-\gamma}dx + \int\limits_\Omega b|\phi_1|^\gamma dx.$$

By Picone's identity, the left hand side is negative. Hence we must have $\lambda < \lambda_1$ and so (1) has no positive solutions when $\lambda > \lambda_1$.

(ii) Suppose that (1) has a positive solution. Then $u(x) \geq o$ on Ω^+ and

$$-\Delta u(x) = \lambda u(x) + b(x)|u|^{\gamma-p}u \geq \lambda u \ \text{ on } \ \Omega^+; \quad u(x) \geq 0 \ \text{ on } \ \partial\Omega^+.$$

It follows from the maximum principle that $\lambda \leq \overline{\lambda}$.

Finally we observe that in each of the cases above J_λ is not bounded below on $S(\lambda)$.

Theorem 7. *J_λ is not bounded below on $S(\lambda)$ when either of the following condition hold:*

(i) $\int_\Omega b|\phi_1|^\gamma dx > 0$ *and* $\lambda > \lambda_1$;

(ii) $\lambda > \overline{\lambda}$ *where* $\overline{\lambda}$ *is as in Theorem* 6.

Proof. By Lemma 4 it is sufficient to show that $L_-(\lambda) \cap B_+ \neq \emptyset$. If (i) holds, then $\phi_1 \in L_-(\lambda) \cap B_+$ and, if (ii) holds, then $\psi \in L_-(\lambda) \cap B_+$ where

$$\psi(x) = \begin{cases} \text{positive principal eigenfunction of (7) on } \Omega^+, \\ 0 \text{ if } x \in \Omega/\Omega^+. \end{cases}$$

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