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Fučik Spectrum for the Second Order BVP with Nonlocal Boundary Condition*

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Abstract. We construct the Fučik spectrum for some second order boundary value problem with nonlocal boundary condition. This spectrum differs essentially from the known Fučik spectra. We apply this result to the second order differential equation $x'' + g(x) = f(t, x, x')$ with the conditions $x(a) = 0$, $\int_a^b x(s)ds = 0$.

Keywords: Fučik spectrum, Fučik problem, eigenvalue, nonlocal condition.

1 Introduction

In this paper we study Fučik spectra for some second order equations with piece-wise linear right sides. The Fučik spectrum is useful in the study of the so called "jumping nonlinearities."

Investigations of Fučik spectra have started in sixtieth of XX century [1]. A number of authors have studied the specific cases. Let us mention the cases of the Dirichlet [1] and the Sturm-Liouville [2] boundary conditions. There are some papers on higher order equations. Habets and Gaudenzi have studied the third order problem with the boundary conditions $x(0) = x'(0) = 0 = x(1)$ in the work [3], where many useful references on the subject can be found. Fučik spectra for the fourth order equations were considered by Kreiči [4] and Pope [5]. The eigenvalue problems for differential equations with nonlocal conditions, except of few separate articles, has been systematically investigated only over the past decade. Eigenvalue problems with nonlocal conditions were considered for example in the work [6]. To the best of our knowledge Fučik spectra for problems with nonlocal boundary conditions were not considered previously.

The paper is organized as follows.

In Section 2 we present results on the Fučik spectrum for the second order problem

$$\begin{aligned} &z'' + \mu^2 z^+ - \lambda^2 z^- = 0, \quad \mu, \lambda > 0, \\ &z^+ = \max\{z, 0\}, \quad z^- = \max\{-z, 0\}, \end{aligned} \tag{1}$$

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with the boundary conditions

$$z(a) = 0; \quad \int_a^b z(s)ds = 0. \tag{2}$$

In Section 3 we compare the well known classical Fučik problem with the problem (1), (2).

The results from Section 2 are employed to investigation of the problem

$$x'' + g(x) = f(t, x, x') \tag{3}$$

with the boundary conditions

$$x(a) = 0; \int_a^b x(s)ds = 0 \tag{4}$$

in Section 4. In this Section we prove the existence theorem.

2 Fučik spectrum for the problem (1), (2)

Consider the second order BVP (1), (2).

Definition 1. *The Fučik spectrum is a set of points* (λ, μ) *such that the problem* (1), (2) *has nontrivial solutions.*

Notice that λ and μ must be nonnegative in order the problem (1), (2) to have a nontrivial solution.

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$ for the problem (1), (2).

Proposition 1. *The Fučik spectrum* $\sum = \bigcup_{i=0}^{+\infty} F_i^{\pm}$ *consists of a set of curves*

$$\begin{aligned} F_i^+ = \big\{(\lambda, \mu)|\ z'(a) > 0, \quad & \textit{the nontrivial solution } z(t) \textit{ of the problem has exactly} \\ & i \textit{ zeroes in } (a, b)\big\}; \\ F_i^- = \big\{(\lambda, \mu)|\ z'(a) < 0, \quad & \textit{the nontrivial solution } z(t) \textit{ of the problem has exactly} \\ & i \textit{ zeroes in } (a, b)\big\}. \end{aligned}$$

Theorem 1. *The Fučik spectrum* $\sum = \bigcup_{i=0}^{+\infty} F_i^{\pm}$ *for the problem* (1), (2) *consists of the branches given by*

$$\begin{aligned} F_{2i-1}^+ = \Bigg\{(\lambda, \mu)|\ & \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda} - \frac{\mu\cos(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0, \\ & \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} \le b - a, \quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b - a\Bigg\}, \end{aligned}$$

$$
\begin{aligned}
F_{2i}^{+} &= \Big\{(\lambda,\mu)|\ \frac{(2i+1)\lambda}{\mu}-\frac{2i\mu}{\lambda}-\frac{\lambda\cos(\mu(b-a)-\frac{\mu\pi i}{\lambda}+\pi i)}{\mu}=0,\\
&\qquad \frac{i\pi}{\mu}+\frac{i\pi}{\lambda}\le b-a,\quad \frac{(i+1)\pi}{\mu}+\frac{i\pi}{\lambda}>b-a\Big\},\\
F_{2i-1}^{-} &= \Big\{(\lambda,\mu)|\ \frac{2i\mu}{\lambda}-\frac{(2i-1)\lambda}{\mu}-\frac{\lambda\cos(\mu(b-a)-\frac{\mu\pi i}{\lambda}+\pi i)}{\mu}=0,\\
&\qquad \frac{(i-1)\pi}{\mu}+\frac{i\pi}{\lambda}\le b-a,\quad \frac{i\pi}{\mu}+\frac{i\pi}{\lambda}>b-a\Big\},\\
F_{2i}^{-} &= \Big\{(\lambda,\mu)|\ \frac{(2i+1)\mu}{\lambda}-\frac{2i\lambda}{\mu}-\frac{\mu\cos(\lambda(b-a)-\frac{\lambda\pi i}{\mu}+\pi i)}{\lambda}=0,\\
&\qquad \frac{i\pi}{\mu}+\frac{i\pi}{\lambda}\le b-a,\quad \frac{i\pi}{\mu}+\frac{(i+1)\pi}{\lambda}>b-a\Big\},\quad i=1,2,\dots
\end{aligned}
$$

Proof. Consider the problem (1), (2).

It is clear that $z(t)$ must have zeroes in (a,b). That is why $F_0^{\pm}=\emptyset$.

We will prove the theorem for the case of F_{2i-1}^{+}. Suppose that $(\lambda,\mu)\in F_{2i-1}^{+}$ and let $z(t)$ be a respective nontrivial solution of the problem (1), (2). The solution has $2i-1$ zeroes in (a,b) and $z'(a)>0$. Let these zeroes be denoted by τ_1, τ_2 and so on.

Consider a solution of the problem (1), (2) in the intervals (a,τ_1), (τ_1,τ_2), $\dots$, (τ_{2i-1},b). We obtain that the problem (1), (2) in these intervals reduces to the linear eigenvalue problems. So in the odd intervals we have the problem $z''=-\mu^2 z$ with boundary conditions $z(a)=z(\tau_1)=0$ in the first such interval and with boundary conditions $z(\tau_{2i-2})=z(\tau_{2i-1})=0$ in other ones, but in the even intervals we have the problem $z''=-\lambda^2 z$ with boundary condition $z(\tau_{2i-3})=z(\tau_{2i-2})=0$ in each such interval but for the last one the only condition is $z(\tau_{2i-1})=0$. In view of (2) a solution $z(t)$ must satisfy the condition

$$
\int_a^{\tau_1} z(s)ds+\int_{\tau_2}^{\tau_3} z(s)ds+\dots+\int_{\tau_{2i-2}}^{\tau_{2i-1}} z(s)ds=\left|\int_{\tau_1}^{\tau_2} z(s)ds+\int_{\tau_3}^{\tau_4} z(s)ds+\dots+\int_{\tau_{2i-1}}^{b} z(s)ds\right|. \quad (5)
$$

Since $z(t)=A\sin(\mu t-\mu a)$ $(A>0)$ and $z(\tau_1)=0$ we obtain $\tau_1=\frac{\pi}{\mu}+a$. Analogously we obtain for the other zeroes

$$
\begin{aligned}
\tau_2 &= \frac{\pi}{\mu}+\frac{\pi}{\lambda}+a,\\
\tau_3 &= 2\frac{\pi}{\mu}+\frac{\pi}{\lambda}+a,\\
&\dots\dots\dots\dots\dots\dots\\
\tau_{2i-2} &= (i-1)\frac{\pi}{\mu}+(i-1)\frac{\pi}{\lambda}+a,\\
\tau_{2i-1} &= i\frac{\pi}{\mu}+(i-1)\frac{\pi}{\lambda}+a.
\end{aligned}
$$

In view of these facts it is easy to get that $\int_a^{\tau_1} z(s)ds = \frac{A}{\mu}(1-\cos\mu(\tau_1-a)) = \frac{2A}{\mu}$.
Analogously

$$\int_{\tau_2}^{\tau_3} z(s)ds = \int_{\tau_4}^{\tau_5} z(s)ds = \ldots = \int_{\tau_{2i-2}}^{\tau_{2i-1}} z(s)ds = \frac{2A}{\mu}.$$

Therefore

$$\int_{a}^{\tau_1} z(s)ds + \int_{\tau_2}^{\tau_3} z(s)ds + \ldots + \int_{\tau_{2i-2}}^{\tau_{2i-1}} z(s)ds = i\frac{2A}{\mu}.$$

We have also

$$z'(\tau_1) = z'(\tau_3) = \ldots = z'(\tau_{2i-1}) = -\mu A. \tag{6}$$

Now we consider a solution of the problem (1), (2) in the remaining intervals. Since $z(t) = -B\sin(\lambda t - \lambda\tau_1)$ $(B > 0)$ in (τ_1, τ_2) we obtain $\int_{\tau_1}^{\tau_2} z(s)ds = \frac{B}{\lambda}\big(\cos\lambda(\tau_2 - \tau_1) - 1\big) = -\frac{2B}{\lambda}$.
Analogously

$$\int_{\tau_3}^{\tau_4} z(s)ds = \int_{\tau_5}^{\tau_6} z(s)ds = \ldots = \int_{\tau_{2i-3}}^{\tau_{2i-2}} z(s)ds = -\frac{2B}{\lambda}.$$

But in the last interval (τ_{2i-1}, b) we obtain

$$\int_{\tau_{2i-1}}^{b} z(s)ds = -\frac{B}{\lambda}\left(1 + \cos\left(\lambda(b-a) - i\frac{\lambda\pi}{\mu} + i\pi\right)\right).$$

It follows from the last two lines that

$$\Big|\int_{\tau_1}^{\tau_2} z(s)ds + \int_{\tau_3}^{\tau_4} z(s)ds + \ldots + \int_{\tau_{2i-3}}^{\tau_{2i-2}} z(s)ds + \int_{\tau_{2i-1}}^{b} z(s)ds\Big|$$
$$= (i-1)\frac{2B}{\lambda} + \frac{B}{\lambda}\left(1 + \cos\left(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\right)\right).$$

We have also that

$$z'(\tau_1) = z'(\tau_3) = \ldots = z'(\tau_{2i-1}) = -\lambda B. \tag{7}$$

It follows from (6) and (7) that $A = \frac{\lambda B}{\mu}$.
In view of the last equality and (5) we obtain

$$i\frac{2\lambda B}{\mu^2} = (2i-1)\frac{B}{\lambda} + \frac{B}{\lambda}\cos\left(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\right).$$

Dividing it by B and multiplying by μ, we obtain

$$2i\frac{\lambda}{\mu} - (2i-1)\frac{\mu}{\lambda} - \frac{\mu}{\lambda}\cos\left(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\right) = 0. \tag{8}$$

Considering the solution of the problem (1), (2) it is easy to prove that $\tau_{2i-1} \le b < \tau_{2i}$ or $\frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} \le b - a < \frac{i\pi}{\mu} + \frac{i\pi}{\lambda}$.

This result and (8) prove the theorem for the case of F^+_{2i-1}. The proof for other branches is analogous. $\square$

Visualization of the spectrum to the problem (1), (2) in the case of $a = 0$, $b = 1$ is given in Fig. 1.

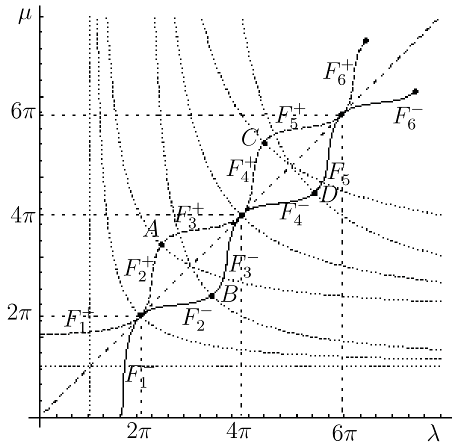

Fig. 1. The Fučik spectrum for the problem (1), (2).

Remark 1. *The point A (B) is a point at which the branches F_2^+ and F_3^+ (F_2^- and F_3^-) intersect. Analogously the branches F_4^+ and F_5^+ (F_4^- and F_5^-) are intersected at the point C (D). And so on.*

Computations shows that the point A is at $((\sqrt{2}+1)\pi; (\sqrt{2}+2)\pi$, the point C is at $((\sqrt{6}+2)\pi; (\sqrt{6}+3)\pi)$. So the branches F^+_{2i} and F^+_{2i+1} intersect at the point

$$\Big(\big(\sqrt{i(i+1)} + i\big)\pi; \big(\sqrt{i(i+1)} + i + 1\big)\pi\Big), \quad i = 1, 2, \ldots.$$

(The points of intersections of negative branches are symmetric to those of the positive branches with respect of the bisectrix.)

3 Comparison

Now we consider the equation (1) with boundary conditions

$$z(a) = 0, \;\; (1-\alpha)z(b) + \alpha \int_a^b z(s)ds = 0, \quad \alpha \in [0,1]. \tag{9}$$

Theorem 2. *The Fučik spectrum* $\sum_\alpha = \bigcup_{i=0}^{+\infty} F_i^\pm$ *for the problem* (1), (9)*, where meaning of the notation is the same as earlier, consists of the branches given by (where* $i = 1, 2, \ldots$*)*

$$\begin{aligned}
F_{2i-1}^+ = \Big\{(\lambda,\mu)| \;\; & \frac{2i\lambda}{\mu}\alpha - \frac{(2i-1)\mu}{\lambda}\alpha - \frac{\mu\alpha\cos\left(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\right)}{\lambda} \\
& + \mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) \\
& - \alpha\mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) = 0, \\
& \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} \le b-a, \quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b-a\Big\}, \\
F_{2i}^+ \;\;\; = \Big\{(\lambda,\mu)| \;\; & \frac{(2i+1)\lambda}{\mu}\alpha - \frac{2i\mu}{\lambda}\alpha\frac{\lambda\alpha\cos\left(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\right)}{\mu} \\
& + \lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) \\
& - \alpha\lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) = 0, \\
& \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \le b-a, \quad \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > b-a\Big\}, \\
F_{2i-1}^- = \Big\{(\lambda,\mu)\Big| \;\; & \frac{2i\mu}{\lambda}\alpha - \frac{(2i-1)\lambda}{\mu}\alpha - \frac{\lambda\alpha\cos\left(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\right)}{\mu} \\
& + \lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) \\
& - \alpha\lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) = 0, \\
& \frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} \le b-a, \quad \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b-a\Big\}, \\
F_{2i}^- \;\;\; = \Big\{(\lambda,\mu)| \;\; & \frac{(2i+1)\mu}{\lambda}\alpha - \frac{2i\lambda}{\mu}\alpha - \frac{\mu\alpha\cos\left(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\right)}{\lambda} \\
& + \mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) \\
& - \alpha\mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) = 0, \\
& \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \le b-a, \quad \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > b-a\Big\}.
\end{aligned}$$

Proof. The proof of Theorem is similar to that of Theorem 1. □

Remark 2. *If* $\alpha = 0$ *we obtain the problem*

$$z'' + \mu^2 z^+ - \lambda^2 z^- = 0, \quad z(a) = 0, \;\; z(b) = 0. \tag{10}$$

This problem is the classical Fučik problem, which was investigated in the work [1]. The spectrum of this problem is well known.

In case of $\alpha = 1$ *we have the problem* (1), (2).

The branches $F_1^{\pm}$ *to* $F_5^{\pm}$ *of the spectrum for the problem* (1), (9) *for several values of* α *are depicted in Figs.* 2 *and* 3 *in the case of* $a = 0$, $b = 1$.

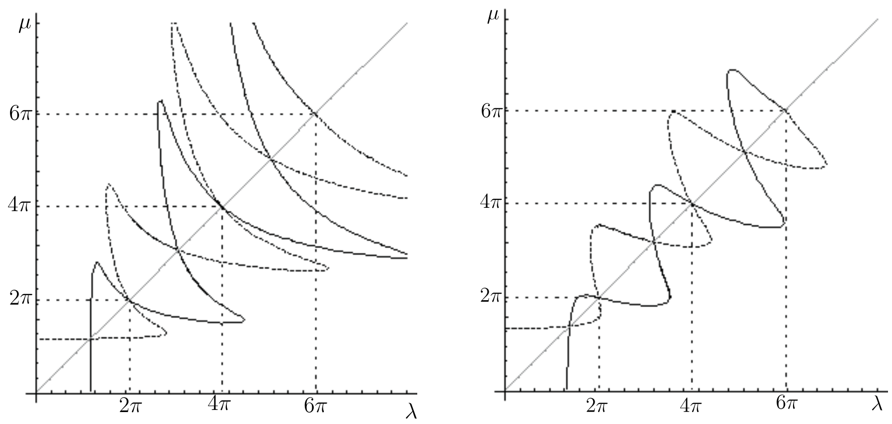

Fig. 2. The Fučik spectrum of the problem (1), (9) for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

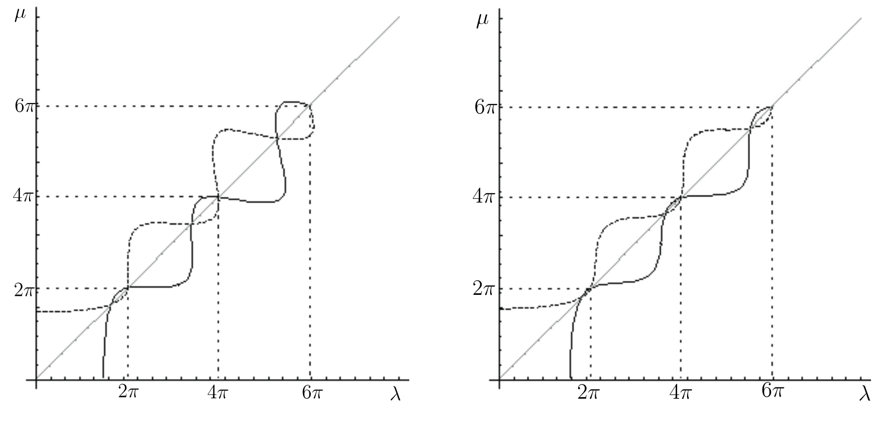

Fig. 3. The Fučik spectrum of the problem (1), (9) for $\alpha = \frac{8}{9}$ and $\alpha = \frac{19}{20}$.

4 Application

Consider the second order BVP (3), (4), where $g(x)$ is a “principal” nonlinearity which behaves like a linear function at infinity and $f(x)$ is a bounded (nonlinear) continuous function which satisfies the Lipschitz condition with respect to x. More precisely, let $g(x) = g_+x^+ - g_-x^- + \epsilon(x)$ where $\epsilon(x)$ is a continuous function in $[a, b]$ and $\epsilon(x)$ tends to 0 as x tends to $\pm\infty$ and $g(x)$ satisfies the conditions

$$\begin{aligned} &g(x)/x \to g_+ \quad \text{as} \quad x \to +\infty; \\ &g(x)/x \to g_- \quad \text{as} \quad x \to -\infty, \end{aligned} \tag{11}$$

where $g_-, g_+ \in (0, +\infty)$.

Theorem 3 (the existence theorem). *Suppose that*

(A1) *equation* (3) *has a solution* $\xi(t)$ *such that* $\xi(a) = 0$, $\int_a^b \xi(s)ds > 0$*;*

(A2) *equation* $z'' + g_+z^+ - g_-z^- = 0$ *with the conditions* $z(a) = 0$, $z'(a) = 1$ *has a solution* $z_+(t)$ *such that* $\int_a^b z_+(s)ds < 0$*;*

(A3) *equation* $z'' + g_+z^+ - g_-z^- = 0$ *with the conditions* $z(a) = 0$, $z'(a) = -1$ *has a solution* $z_-(t)$ *such that* $\int_a^b z_-(s)ds < 0$*.*

Let the conditions (A1) *to* (A3) *hold. Then, if* $g(x)$ *satisfies the condition* $(\sqrt{g_-}, \sqrt{g_+}) \notin \sum$*, the problem* (3), (4) *has at least two solutions.*

In the proof of Theorem 3 we use the following Lemma.

Lemma 1. *Suppose that the conditions (A0) to (A3) hold. Then functions* $\frac{1}{|\gamma|}x(t, \gamma)$ *tend to* $z_+(t)$ *if* $\gamma \to +\infty$ *(or tend to* $z_-(t)$ *if* $\gamma \to -\infty$*), where* $x(t, \gamma)$ *is a solution of the Cauchy problem*

$$x'' + g(x) = f(t, x, x'), \quad x(a) = 0, \;\; x'(a) = \gamma. \tag{12}$$

Proof. Consider the Cauchy problem (12). We has that the equation (3) can be written in the form

$$x'' + g_+x^+ - g_-x^- + \epsilon(x) = f(t, x, x') \tag{13}$$

or

$$\left(\frac{1}{|\gamma|}x\right)'' + g_+\left(\frac{x}{|\gamma|}\right)^+ - g_-\left(\frac{x}{|\gamma|}\right)^- + \frac{\epsilon(x)}{|\gamma|} = \frac{f(t, x, x')}{|\gamma|}.$$

Let $u(t, \gamma) := \frac{1}{|\gamma|}x(t, \gamma)$. Then

$$u'' + g_+u^+ - g_-u^- = \frac{1}{\gamma}\big(f(t, |\gamma|u, |\gamma|u') - \epsilon(|\gamma|u)\big). \tag{14}$$

This means that the problem (12) transforms to the problem

$$u'' + g_+u^+ - g_-u^- = F_{|\gamma|}(t,u,u'), \quad u(a) = 0, \; u'(a) = 1. \tag{15}$$

By the continuous dependence of solutions of (15) on the right-hand side [7, p.178], and taking into account that $F_{|\gamma|}(t,u,u') \to 0$ as $|\gamma| \to +\infty$ uniformly in t we obtain, that $u(t,\gamma)$ tends to the $z_+(t)$ uniformly in t.

The proof of Lemma for $\gamma \to -\infty$ is analogous. □

Proof of Theorem 3. Consider the Cauchy problem (12).

If $\gamma = \xi'(a) \in \mathbb{R}$ then $x(t,\gamma) \equiv \xi(t)$. It follows from the condition (A1) that

$$\int_a^b x\big(s,\xi'(a)\big)ds > 0. \tag{16}$$

Let γ tend to the $+\infty$.

From Lemma we obtain that $\frac{x(t,\gamma)}{|\gamma|}$ tends to a solution $z_+(t)$ which is mentioned in the condition (A2) of the theorem. It means that $\int_a^b \frac{x(s,\gamma)}{|\gamma|}ds < 0$ or

$$\int_a^b x(s,\gamma)ds < 0. \tag{17}$$

Let $I(\gamma) = \int_a^b x(s,\gamma)ds$. It follows from the unique solvability of the Cauchy problems that $x(t,\gamma)$ continuously depend on γ and, therefore, the function $I(\gamma)$ is a continuous function of γ. We obtain $I(\gamma) > 0$, if $\gamma = \xi'(a)$, and $I(\gamma) < 0$, if $\gamma \to +\infty$. Then there exists $\gamma_1 \in (\xi'(a), +\infty)$ such that $I(\gamma_1) = 0$. It follows that $x(t,\gamma_1)$ is a solution of the problem (3), (4).

Similarly for $\gamma \to -\infty$ we obtain that there exists $\gamma_2 \in (-\infty, \xi'(a))$ such that $I(\gamma_2) = 0$. This means that $x(t,\gamma_2)$ is a solution of the problem (3), (4). □

Example 1. Consider the problem

$$x'' + g(x) = 10, \tag{18}$$

$$x(0) = 0, \quad \int_0^1 x(s)ds = 0, \tag{19}$$

where

$$g(x) = \begin{cases} (2\pi)^2 x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The function $\xi(t) = \frac{10}{(2\pi)^2}(1 - \cos 2\pi t)$ is a solution of the equation (18) and $\xi(0) = 0$, $\int_0^1 \xi(s)ds > 0$.

The function

$$z_+(t) = \begin{cases} \frac{1}{2\pi}\sin 2\pi t, & \text{if } 0 \le t < \frac{1}{2}, \\ \frac{1}{2} - t, & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

is a solution of the Cauchy problem $z'' + (2\pi)^2 z^+ = 0$, $z(0) = 0$, $z'(0) = 1$ and $\int_0^1 z_+(s)ds < 0$.

The function $z_-(t) = -t$ is a solution of the Cauchy problem $z'' + (2\pi)^2 z^+ = 0$, $z(0) = 0$, $z'(0) = -1$ and $\int_0^1 z_-(s)ds < 0$.

The graphs of these functions are depicted ir Fig. 4.

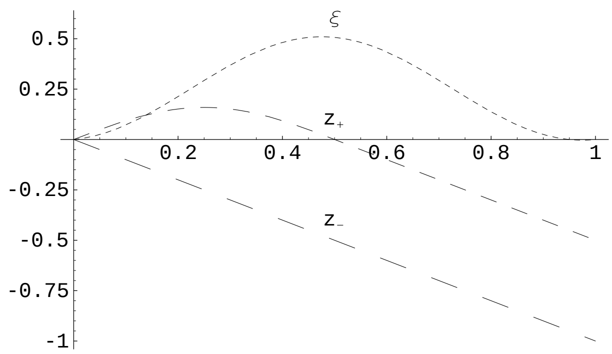

Fig. 4. Graphs of solutions.

Thus, all conditions of Theorem 3 are fulfilled, and this means that the problem (18), (19) has at least two solutions.

Remark 3. *The conditions* (A2), (A3) *are essential for the existence of two solutions of the problem. Indeed, consider the problem*

$$x'' + g(x) = 0, \tag{20}$$

$$x(0) = 0, \quad \int_0^1 x(s)ds = 0, \tag{21}$$

where

$$g(x) = \begin{cases} \pi^2 x, & \textit{if } x \ge 0, \\ 0, & \textit{if } x < 0. \end{cases}$$

The problem has exactly one solution (the trivial one). This is because the condition (A2) *is not satisfied. Indeed, all solutions of the Cauchy problem* $z'' + \pi^2 z^+ - 0z^- = 0$, $z(0) = 0$, $z'(0) = 1$ *satisfy the condition* $\int_0^1 z_+(s)ds > 0$.

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