# A Prey-Predator Model with a Reserved Area

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**Abstract.** In this paper, a mathematical model is proposed and analysed to study the dynamics of a prey-predator model. It is assumed that the habitat is divided into two zones, namely free zone and reserved zone. Predators are not allowed to enter into the reserved zone. Criteria for the coexistence of predator-prey are obtained. The role of reserved zone is investigated and it is shown that the reserve zone has a stabilizing effect on predator-prey interactions.

**Keywords:** prey-predator, reserve zone, stability.

## 1 Introduction

The biosphere is an important zone for biological activities that are mainly responsible for the changes in ecology and environment. The co-existence of interacting biological species has been of great interest in the past few decades and has been studied extensively using mathematical models by several researchers [1–10]. Many biological species have been driven to extinction and many others are at the verge of extinction due to several external forces such as overexploitation, over predation, environmental pollution, mismanagement of the habitat, etc. In order to protect these species, appropriate measures such as restriction on harvesting, creating reserved zones/refuges, etc. should be adopted that will decrease the interaction of these species with external forces. The role of reserve zones/refuges in predator-prey dynamics has received considerable attention and has also been investigated by several researchers [11–21]. In particular, Collings [11] studied the nonlinear behavior of predator-prey model with refuge protecting a constant proportion of prey and wit temperature dependent parameters chosen appropriately for a mite interaction on fruit species. He showed the existence of a temperature interval in which increasing the amount of refuge dynamically destabilizes the system; and on part of this interval the interaction is less likely to persist in that predator and prey minimum population densities are lower than when no refuge is available. Krivan [12] proposed a mathematical model and investigated the effects of optimal antipredator behavior of prey in predator-prey system. He showed that optimal antipredator behavior of prey leads to persistence and reduction of oscillations in population densities. Chattopadhyay et al. [13]

studied a prey-predator model with some cover on prey species. They observed that global stability of the system around positive equilibrium does not necessarily imply the permanence of the system. Recently, Kar [18] proposed a predator-prey model incorporating a prey refuge and independent harvesting on either species. He showed that using the harvesting efforts as control, it is possible to break the cyclic behavior of the system.

In the above investigations, the dynamics of predator living in unreserved zone together with prey has not been studied explicitly. The reserve zone plays a vital role in aquatic environment for the protection of fishery resources from its overexploitation [22–26]. In particular, Dubey et al. [22] proposed and analyzed a mathematical model to study the dynamics of a fishery resource system in an aquatic environment consisting of two zones, namely a free fishing zone and a reserve zone where fishing is strictly prohibited. It was suggested that even if fishery is exploited continuously in the unreserved zone, fish populations can be maintained at an appropriate equilibrium level in the habitat. The model presented in this paper will be of great use in a National Park where prey-predator are living together. The prey species which are to be conserved can be protected from predators by creating an artificial boundary or shelter that will divide the habitat into two zones – one reserved and other unreserved. The entry of predators into reserved zone can be restricted by the artificial boundary that may be in the form of fencing of suitable mesh size through which prey can pass but predators can not. The model studied in Section 4 (when predator is partially dependent on the prey) can also be used in fishery resources where fisherman can be thought of as predator (in fact, generalist predator) and fishing is not permitted in a particular zone, called the reserved zone.

Keeping this in view, we consider a habitat consisting of two zones: an unreserved zone where prey and predator can move freely and a reserved zone where prey can live but predators are not allowed to enter inside. We consider the two cases: one when the predator is wholly dependent on the prey and other when the predator is partially dependent on the prey in the unreserved zone. In fact, we consider the model developed in [22] by incorporating an additional equation for predator in the unreserved zone. Then we study the coexistence and stability behavior of predator-prey system in the habitat.

## 2 Mathematical model

Consider a habitat where prey and predator species are living together. It is assumed that the habitat is divided into two zones, namely, reserved and unreserved zones. It is assumed that predator species are not allowed to enter inside the reserved zone whereas the free mixing of prey species from reserved to unreserved zone and vice-versa is permissible.

Let $x(t)$ be the density of prey species in unreserved zone, $y(t)$ the density of prey species in reserved zone and $z(t)$ the density of the predator species at any time $t \geq 0$. Let $\sigma_1$ be the migration rate coefficient of the prey species from unreserved to reserved zone and $\sigma_2$ the migration rate coefficient of prey species from reserved to unreserved zone. It is assumed that the prey species in both zones are growing logistically.

Keeping these in view and following Dubey et al. [22], the dynamics of system may be governed by the following system of ordinary differential equations:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \beta_1 xz,\\
\frac{dy}{dt} &= sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y,\\
\frac{dz}{dt} &= Q(z) - \beta_0 z,\\
x(0) &\geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0.
\end{aligned} \tag{1}
$$

In model (1), $r$ and $s$ are intrinsic growth rate coefficients of prey species in unreserved and reserved zones respectively; $K$ and $L$ are their respective carrying capacities. $\beta_1$ is the depletion rate coefficient of the prey species due to the predator, and $\beta_0$ is the natural death rate coefficient of the predator species.

In model (1), the function $Q(z)$ represents the growth rate of predator. The model (1) is analyzed in two different cases, namely,

$$\text{(i)}\ \ Q(z) = \beta_2 xz, \tag{2}$$

i.e. when predator is wholly dependent on the prey species;

$$\text{(ii)}\ \ Q(z) = bz\left(1 - \frac{z}{M_0}\right) + \beta_2 xz \tag{3}$$

i.e. when the predator is partially dependent on the prey. In this case, the prey species of density $x(t)$ can be thought of as an alternative resource for the predator.

By denoting $a = b - \beta_0 > 0,\ \ M = M_0(b - \beta_0)/b$ we note that the third equation of model (1) can be written as

$$\frac{dz}{dt} = az\left(1 - \frac{z}{M}\right) + \beta_2 xz. \tag{4}$$

In model system (1)–(4), $r, s, \sigma_1, \sigma_2, \beta_1, \beta_2, \beta_0$ and $a$ are assumed to be positive constants.

Now we present the analysis of model (1) in two cases (2) and (3) by using stability theory of ordinary differential equations [27].

## 3 Case I: when predator is wholly dependent on the prey

In this case, $Q(z)$ satisfies equation (2).

### 3.1 Existence of equilibria

It can be checked that model (1), when $Q(z)$ satisfies (2), has only three nonnegative equilibria, namely $E_0(0,0,0), E_1(\widehat{x}, \widehat{y}, 0)$ and $\overline{E}(\overline{x}, \overline{y}, \overline{z})$. The equilibrium $E_0$ exists obviously and we shall show the existence of $E_1$ and $\overline{E}$ as follows:

### *3.1.1 Existence of* $E_1(\widehat{x}, \widehat{y}, 0)$

Here $\widehat{x}$ and $\widehat{y}$ are the positive solutions of the following algebraic equations:

$$rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y = 0, \tag{5a}$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0. \tag{5b}$$

From equation (5a), we have

$$y = \frac{1}{\sigma_2}\left[\frac{rx^2}{K} - (r - \sigma_1)x\right]. \tag{6}$$

Substituting the value of $y$ from equation (6) into equation (5b), a little algebraic manipulation yields

$$ax^3 + bx^2 + cx + d = 0, \tag{7}$$

where

$$a = \frac{sr^2}{L\sigma_2^2 K^2}, \quad b = \frac{-2rs(r - \sigma_1)}{KL\sigma_2^2},$$
$$c = \frac{s(r - \sigma_1)^2}{L\sigma_2^2} - \frac{(s - \sigma_2)r}{\sigma_2 K}, \quad d = \frac{(r - \sigma_1)(s - \sigma_2)}{\sigma_2} - \sigma_1.$$

It may be noted that equation (7) has a unique positive solution $x = x^*$ if the following inequalities hold:

$$\frac{s(r - \sigma_1)^2}{L\sigma_2} < \frac{(s - \sigma_2)r}{K}, \tag{8a}$$

$$(r - \sigma_1)(s - \sigma_2) < \sigma_1\sigma_2. \tag{8b}$$

From the model system (1) we note that if there is no migration of the prey species from reserved to unreserved zone (i.e. $\sigma_2 = 0$) and $r - \sigma_1 < 0$, then $\frac{dx}{dt} < 0$. Similarly if there is no migration from of the prey species from unreserved to reserved zone (i.e. $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\frac{dy}{dt} < 0$. Hence it is natural to assume that

$$r > \sigma_1 \quad \text{and} \quad s > \sigma_2. \tag{8c}$$

Knowing the value of $\widehat{x}$, the value of $\widehat{y}$ can be computed from equation (6). It may also be noted that for $\widehat{y}$ to be positive, we must have

$$\widehat{x} > \frac{K}{r}(r - \sigma_1). \tag{9}$$

### *3.1.2 Existence of* $\overline{E}(\overline{x}, \overline{y}, \overline{z})$

Here $\overline{x},\ \overline{y},\ \overline{z}$ are the positive solutions of the following algebraic equations:

$$rx\left(1-\frac{x}{K}\right)-\sigma_1 x+\sigma_2 y-\beta_1 xz=0,$$
$$sy\left(1-\frac{y}{L}\right)+\sigma_1 x-\sigma_2 y=0,$$
$$\beta_2 xz-\beta_0 z=0.$$

Solving the above equations, we get,

$$\overline{x}=\frac{\beta_0}{\beta_2}, \tag{10a}$$
$$\overline{y}=\frac{1}{2s\beta_2}\left[(s-\sigma_2)+\sqrt{(s-\sigma_2)^2+4s\sigma_1 L\beta_0\beta_2}\right], \tag{10b}$$
$$\overline{z}=\frac{\beta_2}{\beta_0\beta_1}\left[\sigma_2\overline{y}+(r-\sigma_1)\frac{\beta_0}{\beta_2}-\frac{r\beta_0^2}{K\beta_2^2}\right]. \tag{10c}$$

For $\overline{z}$ to be positive, we must have

$$\sigma_2\overline{y}+(r-\sigma_1)\frac{\beta_0}{\beta_2}>\frac{r\beta_0^2}{K\beta_2^2}. \tag{11}$$

Equation (11) gives a threshold value of the carrying capacity of the free access zone for the survival of predators.

In the following lemma, we show that all solutions of model (1) are nonnegative and bounded.

**Lemma 1.** *The set*

$$\Omega=\left\{(x,y,z)\in\Re_3^+:\quad 0<w=x+y+z\le\frac{\mu}{\eta}\right\}$$

*is a region of the attraction for all solutions initiating in the interior of the positive orthant, where* $\eta$ *is a constant such that*

$$0<\eta<\beta_0,\quad \mu=\frac{K}{4r}(r+\eta)^2+\frac{L}{4s}(s+\eta)^2,\quad \beta_1\ge\beta_2.$$

*Proof.* Let $w(t)=x(t)+y(t)+z(t)$ and $\eta>0$ be a constant. Then

$$\frac{dw}{dt}+\eta W=(r+\eta)x-\frac{rx^2}{K}+(s+\eta)y-\frac{sy^2}{L}-(\beta_1-\beta_2)xz-(\beta_0-\eta)z. \tag{12}$$

Since $\beta_1$ is the depletion rate coefficient of prey due to its intake by the predator and $\beta_2$ is the growth rate coefficient of predator due to its interaction with their prey, and hence it is natural to assume that $\beta_1\ge\beta_2$.

Now choose $\eta$ such that $0 < \eta < \beta_0$.
Then equation (12) can be written as

$$\begin{aligned}\frac{dW}{dt} + \eta w &\leq (r+\eta)x - \frac{rx^2}{K} + (s+\eta)y - \frac{sy^2}{L} \\ &= \frac{K}{4r}(r+\eta)^2 - \frac{r}{K}\left\{x - \frac{K}{2r}(r+\eta)\right\}^2 \\ &\quad + \frac{L}{4s}(s+\eta)^2 - \frac{s}{L}\left\{y - \frac{L}{2s}(s+\eta)\right\}^2 \\ &\leq \frac{K}{4r}(r+\eta)^2 + \frac{L}{4s}(s+\eta)^2 = \mu(say).\end{aligned}$$

By using the differential inequality [28], we obtain

$$0 < w\big(x(t), y(t), z(t)\big) \leq \frac{\mu}{\eta}(1 - e^{-\eta t}) + \big(x(0), y(0), z(0)\big)e^{-\eta t}.$$

Taking limit when $t \to \infty$, we have, $0 < w(t) \leq \frac{\mu}{\eta}$, proving the lemma. □

### 3.2 Stability analysis

By computing the variational matrices corresponding to each equilibrium, we note the following:

1. $E_0$ is a saddle point with stable manifold locally in the $z$-direction.
2. If $\beta_2\widehat{x} > \beta_0$ then $E_1$ is a saddle point with stable manifold locally in the $xy$-plane and with unstable manifold locally in the $z$-direction.
3. If $\beta_2\widehat{x} < \beta_0$ then $E_1$ is locally asymptotically stable.

In the following theorem, we show that the model system (1) does not have any closed trajectory in the interior of the positive quadrant of the $xy$-plane.

**Theorem 1.** *The model system* (1) *under the assumption* (2) *can not have any periodic solution in the interior of the positive quadrant of the $xy$-plane.*

*Proof.* Let $H(x,y) = \frac{1}{xy}$. Clearly $H(x,y)$ is positive in the interior of the positive quadrant of the $xy$-plane. Let

$$h_1(x,y) = rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y,$$
$$h_2(x,y) = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y.$$

Then

$$\Delta(x,y) = \frac{\partial}{\partial x}(h_1 H) + \frac{\partial}{\partial y}(h_2 H) = -\frac{1}{y}\left(\frac{r}{K} + \frac{\sigma_2 y}{x^2}\right) - \frac{1}{x}\left(\frac{s}{L} + \frac{\sigma_1 x}{y^2}\right) < 0.$$

From the above equation, we note that $\Delta(x, y)$ does not change sign and is not identically zero in the interior of the positive quadrant of the $xy$-plane. By Dulac-Bendixon criteria, it follows that there is no closed trajectory in the interior of the positive quadrant of the $xy$-plane, and hence the theorem follows. □

In the following theorem, we show that $\overline{E}$ is locally asymptotically stable.

**Theorem 2.** *The interior equilibrium $\overline{E}$ is locally asymptotically stable.*

*Proof.* In order to prove this theorem, we first linearize model (1) by taking the following transformations:

$$x = \overline{x} + X, \quad y = \overline{y} + Y, \quad z = \overline{z} + Z.$$

Now we consider the following positive definite function:

$$V(t) = \frac{1}{2}X^2 + \frac{1}{2}c_1 Y^2 + \frac{1}{2}c_2 Z^2,$$

where $c_1$ and $c_2$ are positive constants to be chosen suitably.

Now differentiating $V$ with respect to time $t$ along the linear version of model (1), we get

$$\begin{aligned}\frac{dV}{dt} = &-\left(\frac{r\overline{x}}{K} + \frac{\sigma_2\overline{y}}{\overline{x}}\right)X^2 - c_1\left(\frac{s\overline{y}}{L} + \frac{\sigma_1\overline{x}}{\overline{y}}\right)Y^2 \\ &+ XY(\sigma_2 + c_1\sigma_1) + XZ(c_2\beta_2\overline{z} - \beta_1\overline{x}).\end{aligned}$$

Choosing $c_2 = \frac{\beta_1\overline{x}}{\beta_2\overline{z}}$ we note that $\dot{V}$ is negative definite if

$$(\sigma_2 + c_1\sigma_1)^2 < 4c_1\left(\frac{r\overline{x}}{K} + \frac{\sigma_2\overline{y}}{\overline{x}}\right)\left(\frac{s\overline{y}}{L} + \frac{\sigma_1\overline{x}}{\overline{y}}\right).$$

The above equation can further be written as

$$(\sigma_2 - c_1\sigma_1)^2 + 4c_1\sigma_1\sigma_2 < 4c_1\left(\frac{r\overline{x}}{K} + \frac{\sigma_2\overline{y}}{\overline{x}}\right)\left(\frac{s\overline{y}}{L} + \frac{\sigma_1\overline{x}}{\overline{y}}\right).$$

It may be noted that if we choose $c_1 = \frac{\sigma_2}{\sigma_1}$ then the above condition is automatically satisfied. This shows that $V$ is a Liapunov function [27], and hence the theorem follows. □

In the following theorem, we are able to show that $\overline{E}$ is globally asymptotically stable.

**Theorem 3.** *The interior equilibrium $\overline{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

*Proof.* Consider the following positive definite function about $\overline{E}$,

$$W(t) = \left(x - \overline{x} - \overline{x}\ln\frac{x}{\overline{x}}\right) + c_1\left(y - \overline{y} - \overline{y}\ln\frac{y}{\overline{y}}\right) + c_2\left(z - \overline{z} - \overline{z}\ln\frac{z}{\overline{z}}\right).$$

Differentiating $W$ with respect to time $t$ along the solutions of model (1), we get

$$\begin{aligned}\frac{dW}{dt} =& -\frac{r}{K}(x-\overline{x})^2 - \frac{c_1 s}{L}(y-\overline{y})^2 + (x-\overline{x})(z-\overline{z})(c_2\beta_2 - \beta_1) \\ &+ \sigma_2(x-\overline{x})\left(\frac{\overline{x}y - x\overline{y}}{x\overline{x}}\right) + c_1\sigma_1(y-\overline{y})\left(\frac{x\overline{y} - \overline{x}y}{y\overline{y}}\right).\end{aligned}$$

Choosing $c_1 = \frac{\overline{y}\sigma_2}{\overline{x}\sigma_1}$ and $c_2 = \frac{\beta_1}{\beta_2}$, $\frac{dW}{dt}$ can further be written as

$$\frac{dW}{dt} = -\frac{r}{K}(x-\overline{x})^2 - \frac{\overline{y}\sigma_2 s}{x\sigma_1 L}(y-\overline{y})^2 - \frac{\sigma_2}{\overline{x}xy}(\overline{x}y - x\overline{y})^2,$$

which is negative definite. Hence $W$ is a Liapunov function [27] with respect to $\overline{E}$ whose domain contains the region of attraction $\Omega$, proving the theorem. ☐

## 4 Case II: when the predator is partially dependent on the prey

In this case $Q(z)$ satisfies equation (3) and the prey can be thought of as an alternative food for the predator.

### 4.1 Existence of equilibria

When $Q(z)$ satisfies equation (3), then the third equation of model (1) can be replaced by equation (4). Then it can be checked that model (1) has four nonnegative equilibria, namely, $F_0(0,0,0), F_1(0,0,M), F_2(\widetilde{x},\widetilde{y},0), F^*(x^*,y^*,z^*)$.

The equilibriums $F_0$ and $F_1$ obviously exist. As in Case I, equilibrium $F_2(\widetilde{x},\widetilde{y},0)$ exists if the inequalities (8a) and (8b) are satisfied. Further, for $\widetilde{x}$ to be positive, we must have

$$\widetilde{x} > \frac{K}{r}(r - \sigma_1). \tag{13}$$

To see the existence of $F^*$, we note that $x^*, y^*, z^*$ are the positive solutions of the following algebraic equations:

$$rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \beta_1 xz = 0, \tag{14a}$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0, \tag{14b}$$

$$z = \frac{M}{a}(a + \beta_2 x). \tag{14c}$$

Solving the above system of algebraic equations, we get

$$Ax^3 + Bx^2 + Cx + D = 0, \tag{15}$$

where

$$\begin{aligned}
A &= \frac{s}{L\sigma_2^2}\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right),\\
B &= -\frac{2s}{L\sigma_2^2}\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right)(r - \sigma_1 - \beta_1 M),\\
C &= \frac{s}{L\sigma_2^2}(r - \sigma_1 - \beta_1 M)^2 - \frac{s - \sigma_2}{\sigma_2}\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right),\\
D &= \frac{s - \sigma_2}{\sigma_2}(r - \sigma_1 - \beta_1 M) - \sigma_1.
\end{aligned}$$

We note that the equation (15) has a real positive root $x = x^*$ if the following conditions are satisfied:

$$s(r - \sigma_1 - \beta_1 M)^2 < L\sigma_2(s - \sigma_2)\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right), \tag{16a}$$

$$(r - \sigma_1 - \beta_1 M)(s - \sigma_2) < \sigma_1\sigma_2, \tag{16b}$$

$$r - \sigma_1 - \beta_1 M > 0. \tag{16c}$$

Knowing the value of $x^*$, the value of $z^*$ can be computed from equation (14c) and the value of $y^*$ can be computed from the equation given below:

$$y^* = \frac{1}{\sigma_2}\left[\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right)x^{*2} - (r - \sigma_1 - \beta_1 M)x^*\right]. \tag{17}$$

For $y^*$ to be positive, we must have

$$\left(\frac{r}{K} + \frac{\beta_1\beta_2 M}{a}\right)x^* > (r - \sigma_1 - \beta_1 M). \tag{18}$$

In the following lemma, we show that the model system (1) is biologically well behaved. The proof of this lemma is similar to that of Lemma 1, and hence omitted.

**Lemma 2.** *The set*

$$\Omega_1 = \left\{(x, y, z)\colon \;\; w(t) = x(t) + y(t) + z(t),\;\; 0 < w(t) \le \frac{\mu^*}{\eta^*}\right\}$$

*attracts all solutions initiating in the interior of the positive orthant, where*

$$\mu^* = \frac{K}{4r}(r + \eta^*)^2 + \frac{L}{4s}(s + \eta^*)^2 + \frac{M}{4a}(a + \eta^*)^2,$$

*and $\eta^*$ is a positive constant.*

### 4.2 Stability analysis

In order to study the local stability behavior of $F^*$, we compute the variational matrices corresponding to each equilibrium. From these matrices, we note the following:

1. $F_0$ is an unstable equilibrium point.
2. $F_1$ is a saddle point with stable manifold locally in the $z$-direction and with unstable manifold locally in the $xy$-plane.
3. $F_2$ is also a saddle point whose stable manifold is locally in the $xy$-plane and unstable manifold locally in the $z$-direction.

**Remark.** *It may be noted that Theorem* 1 *will remain valid in the case when predator is partially dependent on the prey.*

In the following theorems, local and global stability behavior of $F^*$ have been studied. The proof of Theorem 4 is similar to that of Theorem 2, and the proof of Theorem 5 is similar to that of Theorem 3. Hence we omit the proofs of these theorems.

**Theorem 4.** *The interior equilibrium $F^*$ is locally asymptotically stable.*

**Theorem 5.** *The interior equilibrium $F^*$ is globally asymptotically stable with respect to all solutions initiating in the interior of positive orthant.*

## 5 Numerical simulation

In this section we present numerical simulation to illustrate the results obtained in previous sections. We choose the following values of parameters in model (1):

$$\begin{aligned} &a = 3, \quad r = 4, \quad s = 3.5, \quad K = 40, \quad L = 50, \quad M = 30, \\ &\beta_0 = 3, \quad \beta_1 = 2, \quad \beta_2 = 1, \quad \sigma_1 = 2.5, \quad \sigma_2 = 1.5. \end{aligned} \tag{19}$$

With the above values of parameters, we note that conditions (8) and (9) are satisfied. This shows that equilibrium exists, and it is given by

$$\widehat{x} = 36.7429, \quad \widehat{y} = 53.2598. \tag{20}$$

When predator is wholly dependent on the prey, it is noted that the positive equilibrium $\overline{E}(\overline{x}, \overline{y}, \overline{z})$ exists and it is given by

$$\overline{x} = 3, \quad \overline{y} = 10.6406, \quad \overline{z} = 3.2602. \tag{21}$$

Further, when the predator is partially dependent on the prey, it is seen that the positive equilibrium $F^*(x^*, y^*, z^*)$ exists, and it is given by

$$x^* = 10.4939, \quad y^* = 5.5363, \quad z^* = 31.0494. \tag{22}$$

From (20)–(22), we note the following:

1. When the predator is at zero equilibrium level ($z = 0$), the total density of the prey species at equilibrium level is $90.0027$ $(36.7429 + 53.2598)$.
2. When the predator is completely dependent on the prey, then density of the predator is $3.2602$ while the total density of the prey has decreased from $90.0027$ to $13.6406$.
3. Comparing (21) and (22), it is noted that when the predator is partially dependent on the prey, then density of the predator has increased from $3.2602$ to $31.0494$, and prey density has also increased from $13.6406$ to $16.0302$.

This suggests that an alternative food for the predator leads an increase in the density of the prey as well as predator.

Figs. 1–5 correspond to model (1) when the predator is wholly dependent on the

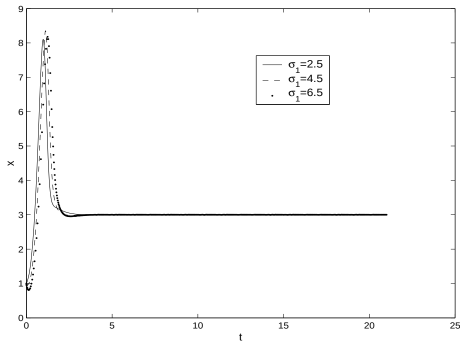


Fig. 1. Case I: graph of $x$ verses $t$ for different value of $\sigma_1$ obtained using parameters: $s = 3.5$, $K = 40$, $L = 50$, $\beta_0 = 3$, $\beta_1 = 2$, $\beta_2 = 1$, $\sigma_2 = 1.5$.

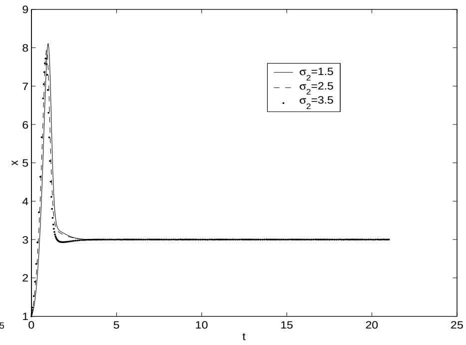


Fig. 2. Case I: graph of $x$ verses $t$ for different value of $\sigma_2$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 1.

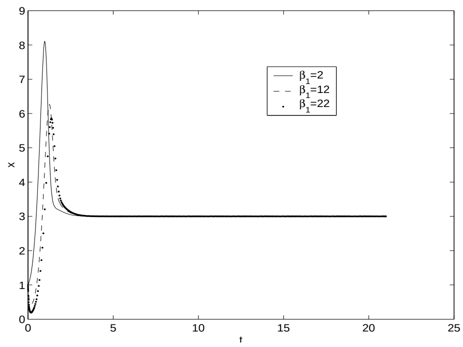


Fig. 3. Case I: graph of $x$ verses $t$ for different value of $\beta_1$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 1.

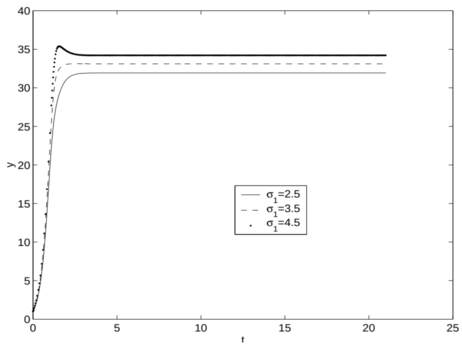


Fig. 4. Case I: graph of $y$ verses $t$ for different value of $\sigma_1$ obtained using the same values of parameters as in Fig. 1.

prey. Fig. 1 shows the behavior of $x$ with time for different values of $\sigma_1$. This figure shows that initially $x$ increases for some time, then it starts decreasing and finally attains its equilibrium level. We also note that initially $x$ decreases as $\sigma_1$ increases but after certain time this behavior is just reversal and finally $x$ settles down at its equilibrium level. Fig. 2 shows the behavior of $x$ with time $t$ for different values of $\sigma_2$. From this figure, we note that initially $x$ increases as $\sigma_2$ increases, after certain time $x$ decreases with $\sigma_2$ and finally attains its equilibrium level. From Fig. 3, we note that behavior of $x$ with time $t$ is similar to that of Fig. 1. Fig. 4 shows the behavior of prey species in reserved area w.r.t. time $t$. This figure shows that initially $y$ increases with time and after certain period of time, it attains its equilibrium level. We also note that $y$ increases as $\sigma_1$ increases. Fig. 5 shows that $y$ increases with time and $y$ decreases as $\sigma_2$ increases, and finally settles down at its equilibrium level.

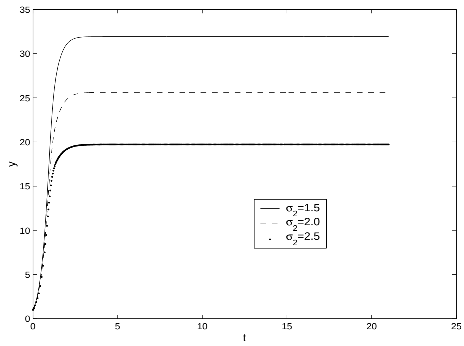


Fig. 5. Case I: graph of $y$ verses $t$ for different value of $\sigma_2$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 1.

Figs. 6–10 correspond to model (1) when predator is partially dependent on the prey. Figs. 6–8 show the behavior of prey species in unreserved area with respect to time $t$. Fig. 6 shows that behavior of $x$ with time when predator is partially dependent on the prey. It is noted that $x$ exhibits periodic behavior for some time and finally it settles down at its equilibrium level. It is also observed that initially $x$ increases as $\sigma_1$ increases and after certain time $x$ decreases as $\sigma_1$ increases, and finally obtains its equilibrium level. From Fig. 7 we note that $x$ has oscillatory behavior for certain time, and then it settles down at its equilibrium level. It is also noted that initially $x$ increases as $\sigma_2$ increases, but after certain time this behavior is just reversed. Fig. 8 shows the behavior of $x$ w.r.t. time $t$ for different values of $\beta_1$. It is noted that if $\beta_1$ is small, then initially $x$ increases and then exhibits oscillatory behavior and finally obtains its equilibrium level. But if $\beta_1$ is larger than a threshold values, then initially $x$ decreases, then after a slight increase it obtains its equilibrium level. It is also observed that $x$ decreases as $\beta_1$ increases. Fig. 9 and Fig. 10 show the behavior of prey species in reserved area w.r.t. time $t$. From these figures it is noted that y increases with time and finally settles down at its equilibrium level. It is also noted that $y$ increases as $\sigma_1$ increases whereas $y$ decreases as $\sigma_2$ increases. It is observed that the prey species in reserved zone do not exhibit periodic behavior.

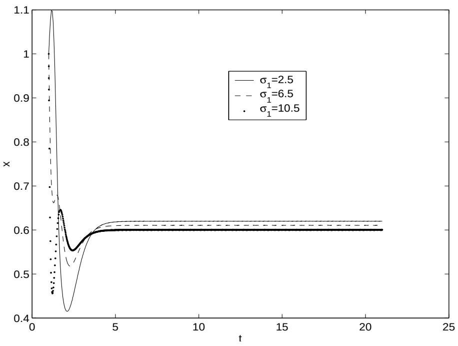


Fig. 6. Case II: graph of $x$ verses $t$ for different value of $\sigma_1$ obtained using parameters: $a = 3$, $s = 3.5$, $K = 40$, $L = 50$, $M = 30$, $\beta_1 = 2$, $\beta_2 = 1$, $\sigma_2 = 1.5$.

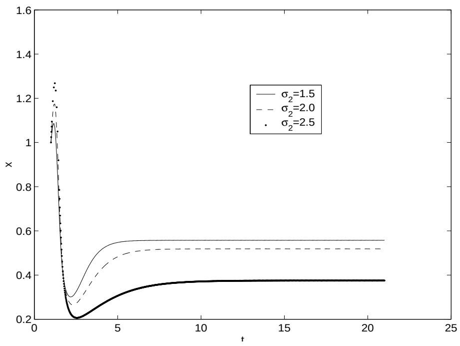


Fig. 7. Case II: graph of $x$ verses $t$ for different value of $\sigma_2$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 6.

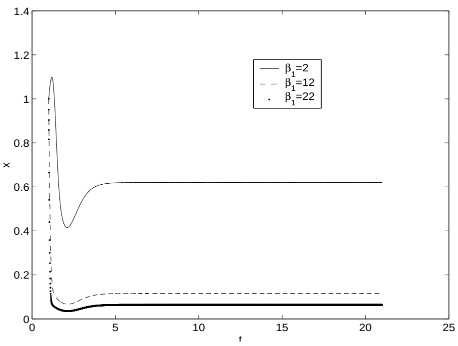


Fig. 8. Case II: graph of $x$ verses $t$ for different value of $\beta_1$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 6.

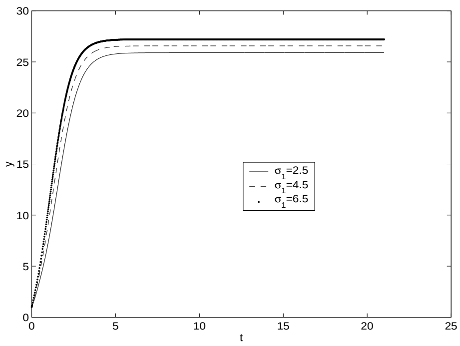


Fig. 9. Case II: graph of $y$ verses $t$ for different value of $\sigma_1$ obtained using the same values of parameters as in Fig. 6.

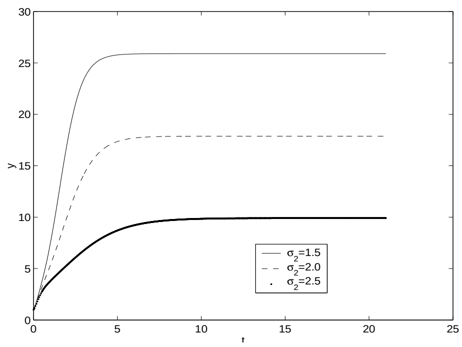


Fig. 10. Case II: graph of $y$ verses $t$ for different value of $\sigma_2$ obtained with $\sigma_1 = 2.5$ and other values of parameters are same as in Fig. 6.

## 6 Conclusions

In this paper, a mathematical model has been proposed and analyzed to study the role of a reserved zone on the dynamics of predator-prey system. The model has been analyzed in two cases: first when predator species are wholly dependent on the prey and second when predator species are partially dependent on the prey in the unreserved zone. In both cases, computer simulations with MATLAB have been performed to study the effects of various parameters on the dynamics of the system. By analytical and numerical simulations, the following observations have been made:

1. In the absence of predator, the density of prey is maximum in reserved as well as unreserved zone.

2. In the case when predators are wholly dependent on the prey, then cumulative density of prey decreases in comparison to the case 1.

3. In the case when predators are partially dependent on the prey and alternative food is also made available to predators in unreserved zone, then the cumulative density of the prey decreases in comparison to case 1, but it increases in comparison to the case 2 and density of predator also increases in comparison to the case 2.

This shows that an alternative resource for the predator is better suited in comparison to the wholly dependent case as it leads an increase in the density of the prey and predator both that ensures the survival of prey and predator in a better way. In both cases, it has been found that prey species has oscillatory behavior in the unreserved zone where as oscillatory behavior has not been observed for prey species in the reserved zone.

By using stability theory of ordinary differential equations, it has been shown that the positive equilibrium, whenever exists, is always globally asymptotically stable in both the cases, namely predators are wholly or partially dependent on the prey species. This shows that reserve zone has a stabilizing effect on the predator-prey system. This study suggests that the role of reserved zone is an important integrating concept in ecology and evolution. By creating reserved zones in the habitat where predator have no access or chance of settling, the prey species can grow without any external disturbances and hence the prey species can be maintained at an appropriate level.

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