On the Probabilities of Correlated Defaults: a First Passage Time Approach*

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Abstract. This article investigates the joint probability of correlated defaults in the first passage time approach of credit risk subject to condition that the underlying firms' assets values and the default boundaries follow geometric Brownian motion processes. The exact analytical expression of joint probability of two correlated defaults in the case of stochastic default boundaries is presented. Also, some properties of this solution are provided.

Keywords: correlated defaults, joint probability of default, implied correlation.

1 Introduction

Together with the evaluation of loss given default and expected losses of any defaultable financial claim, there is equally important to estimate the cumulative distribution function of correlated defaults. The knowledge of marginal probabilities is not sufficient to assess the credit risk of corporate bonds portfolio due to existing correlation (or interdependence) between the asset returns. Usually, default correlation is defined by correlation between the Brownian motions driving the individual companies and plays a crucial role in determining the joint probability of default, i.e. the probability of multiple defaults. Also, the default correlations are the important factors in order to calculate portfolio risk.

In this paper we analyze the credit risk of portfolio of correlated defaultable claims. One of the most important measures of credit risk, the probability of default of one claim is investigated extensively in the literature. However analyzing the credit risk of portfolio, it is also important to measure the joint probability of correlated defaults and the general probability of portfolio default. *"Modeling correlated default risk is a new phenomenon currently sweeping through the credit markets. Due to the rapid growth in the credit derivatives market and the increasing importance of measuring and controlling default risks in corporate bonds portfolios, derivatives, and the other securities, the importance*

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of default correlation (or inter-dependence) analysis has been recognized by the financial industry" (Zhou, [1]). The losses on the initial credit portfolio due to the default of the underlying firms depend on the default probability of each firm and the losses given default. In addition, the degree of dependence between the firms' default probability plays an important role on the timing of the firms' defaults and, as a consequence, on the distribution of the portfolio losses. The correlation between several assets is important for estimating general credit risk of portfolio because higher correlation of defaults implies a greater likelihood that losses will wipe out the assets. Conversely, higher general correlation also makes the extreme case of very few defaults more likely.

In general, the structure of default correlation is a crucial issue in pricing multi-name credit derivatives as well as in credit risk management. In addition, the joint probability of correlated defaults is important for assessment systematic risk of whole financial system due to financial contagion (more about the economic and financial importance of default correlation see, for example [2] and [3]).

Many papers analyzing the default correlation are focused on the joint probability of default in reduced form models, but there are fewer publications concerning the joint probability of correlated defaults in structural approach, notably, [2,4–7] and [1]. Drawing on literature, we noticed that one of the biggest disadvantages of structural models is their limited possibilities of calculating the joint probability of correlated defaults. The aim of this paper is to enlarge this possibility for structural approach of credit risk. We derive the joint probability of two correlated defaults in the case when the default thresholds (in special case the values of bonds) are also stochastic.

In many papers (see, for example, Overbeck and Stahl [8], Zhou [1]) the default correlation is defined as the correlation of Bernoulli distributed random variables. In this paper, we generalize the Zhou's model for two correlated defaults by defining the more general structure of default correlation. Unlike Zhou [1], Patras [7], Overbeck and Schmidt [6] approaches, in this paper we assume that the value of i-th default threshold is stochastic without jumps. That means the case when, for example, investors do not have full information about financial markets (notably, some exogenous shocks, depositors panics or other changes) or due to stochastic behavior of interest rates. In fact, by assuming stochastic behavior of default boundary, we also consider other exogenously defined type of financial risk, notably, liquidity and market changes risk. Other economic interpretation is that stochastic default threshold can represent some debt covenant violation. Unlike other structural models with incomplete information, we assume that there are no jumps and that default time of company remains predictable due to the continuity of both stochastic processes and we avoid the further modelling of transformed structural approach to reduced form model with *endogenous* intensity processes. Also, presented model differs from others by the correlation structure of *implied* Brownian motions and by the defined structure of this correlation.

The paper is organized as follows. In Section 2 we give a short overview of related literature. In Section 3 we present a generalized first passage time model. In Section 4 we outline the expression of the probability of single default. The closed form expression of the joint probability of correlated defaults is given in Section 5, and in Section 6 we give some conclusions.

2 Overview of the literature

In practice, it is complicated way to find an analytical expression of cumulative distribution function of correlated defaults. There are three main approaches to estimate the joint probability of correlated defaults: calculations using credit market historical data (Lucas, [3]), reduced form credit risk approach and structural approach of credit risk which is the less developed in this sense.

Calculation using historical data has some important drawbacks because it is hard to determine whether historical fluctuations in default rates are caused by default correlations or by changes in default probability. Also, there are not enough data concerning the default correlation among and between specific industries and defaultable bonds, it does not use firm-specific information and default correlations are time-varying, so past history may not reflect the current reality. Das and Geng [9] calculated joint probabilities of default for U.S. corporations using credit ratings data for *copula functions*. They used a metric that compares alternative specifications of the joint default distribution using three criteria: (a) the level of default risk, (b) the asymmetry in default correlations, and (c) the tail-dependence of joint defaults.

In reduced form (or intensity) approach of credit risk this problem has been investigated in several ways: *models of conditionally independent defaults, models of contagion* and *copula functions*. The intensity models the conditional default arrival rate during some period. These models can incorporate correlations between defaults by allowing hazard rates to be stochastic and correlated with macroeconomic variables. To induce correlation between defaults, one would typically introduce correlation between the intensity processes. However the problems begin when one attempts to estimate them. These problems are, in part, due to the lack of sufficiently adequate default information about the dependence structure of the credit risk of the firms under consideration. Their main disadvantage is that the range of default correlations that can be achieved is limited. Even when there is a perfect correlation between two hazard rates, the corresponding correlation between defaults in any chosen period of time is usually very low (for more details, see [3]). The recent so called second generation models that come under the heading of "intensity based top down models" avoid this problem (see, for example, [10]). Moreover, these models, (for more details, see the papers of Davis and Lo [11] and Giesecke and Goldberg [10]) incorporate the contagion observed in credit markets.

Due to their intuitive simplicity, structural approach is more attractive than reduced form approach but one of the biggest problems of application structural models in assessment the credit risk of corporate bonds portfolio is the complicated way of estimation of the joint probability of correlated defaults. In structural approach, the default correlation between issuers is introduced through asset return correlation. With predictable defaults, however, jumps in bond prices and credit spreads cannot appear at all: prices converge continuously to their default-contingent values. This means that, although the existing structural models provide important insights into the relation between firms' fundamentals and correlated default events as well as practically most valuable tools, they fail to be consistent in particular with the observed contagion phenomena. Zhou [1] obtained the closed form expression of the joint probability of correlated defaults in the case when default boundaries are exponential functions. Zhou derived this formula in both cases: when firm's asset value and default threshold grow at the same and at the different rates. Overbeck and Schmidt [6] derived similar expression in the case where the underlying ability-to-pay process for each bond is a transformation of Brownian motions with default trend. Patras [7] presented a generalized reflection principle and evaluated digital swap on two credit instruments[1] in the case when the default boundaries are deterministic functions in planar Brownian motion and the correlation follows Bessel process. Giesecke [2] derived similar formulas in the case of complete and incomplete information. This paper provides a structural model of correlated default which is consistent with several significant credit spread characteristics: the implied short-term credit spreads are typically non-zero, credit spreads cyclical correlations across firms, and, most importantly, contagion effects are predicted. Giesecke characterized the joint default probabilities and the default dependence structure as assessed by investors, using the modeling of dependence with copulas for stochastic boundaries. In the paper of He, Keirstead and Rebholz [5] the closed form expressions of the joint probability of the maximum and minimum values of two correlated Brownian motions are derived and applied to the valuation of double lookback and barrier (or knockout) options in the case when hitting boundaries are constants. Fouque, Wignall and Zhou [4] extended the first passage time model by defining the default dependence in two directions: by extending to multi-dimension and by incorporating stochastic volatility. They derived analytical approximations for the joint survival probabilities and subsequently for the distribution of number of defaults in a corporate bonds portfolio when default boundaries are defined as exponential functions.

3 Generalized setup of the first passage time approach

Throughout the paper, t denotes the running time variable. Unlike as in Black-Scholes financial market model, there is no risk-free asset generating constant interest rate. In structural approach, the evolution of each rating process of i-th firm is determined by the behavior of i-th firm's asset value [2] process $\{V_i(t),\ t \geq 0\},\ i = 1, 2$. The rating process jumps down to the respective state at the first moment $\{V_i(t),\ t \geq 0\}$ crosses the default barrier. Assume that the both firms' default thresholds represent the financial liabilities, starting from the time $t = 0$. The liabilities of i-th firm mature at deterministic time $T_i > 0,\ i = 1, 2$ and $T := \min\{T_1, T_2\}$. Assume that two Bernoulli binomial random variables $F_1(T)$ and $F_2(T)$ describe the default status of two companies:

$$F_i(T) = \begin{cases} 1, & \text{if } i\text{-th firm defaults by } T, \quad i = 1, 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

[1] Swap is of digital type: the payoff is settled at the maturity date T; it is A (resp. B) if only one of the two instruments defaults (resp. if both default). In particular, when $A = 0$ (resp. $A = B$), we get a pricing formula for a second to default (resp. first to default) digital swap.

[2] More exactly, pre-default total value of the firm's asset.

In what follows, the default correlation is defined as $\mathrm{Corr}[F_1(T), F_2(T)]$. Since $F_1(T)$ and $F_2(T)$ are Bernoulli random variables the probability of both firms defaulting before the maturity T is as follows:

$$\begin{aligned} &P\big(F_1(T) = 1 \text{ and } F_2(T) = 1\big) \\ &\quad = E[F_1(T)] \cdot E[F_2(T)] + \mathrm{Corr}[F_1(T), F_2(T)] \cdot \sqrt{\mathrm{Var}[F_1(T)] \cdot \mathrm{Var}[F_2(T)]}, \end{aligned} \tag{2}$$

where $E[F_i(T)],\ i = 1, 2$ is the probability of default of single company. So, it follows that defining the joint default probability is equivalent to specifying the default event correlation. Let us consider a continuous trading economy with the time interval $[0, T]$. A complete probability space $(\Omega,\ \mathcal{F},\ P)$ satisfies the usual conditions. Assume that the value of i-th firm's asset and the i-th default threshold (in partial case the value of i-th bond) under the probability P for all $t \geq 0$ is given by[3]

$$\begin{cases} \mathrm{d}V_i(t) = V_i(t)\big(\mu_{V,i}\mathrm{d}t + \sigma_{V,i}\mathrm{d}W_{V,i}(t)\big), \\ \mathrm{d}D_i(t) = D_i(t)\big(\mu_{D,i}\mathrm{d}t + \sigma_{D,i}\mathrm{d}W_{D,i}(t)\big), \end{cases} \tag{3}$$

where $\mu_{D,i},\ \mu_{V,i},\ \sigma_{D,i} > 0$ and $\sigma_{V,i} > 0,\ i = 1, 2$ are constants, $\{W_{D,i}(t),\ t \geq 0\}$ and $\{W_{V,i}(t),\ t \geq 0\},\ i = 1, 2$ are correlated real-valued standard Brownian motions with instantaneous correlations $\mathrm{Corr}[W_{D,i}(t), W_{V,i}(t)] = \rho_{i,i}^{D,V}$ for any $t \geq 0,\ i = 1, 2$ and

$$\begin{cases} \rho_{1,2}^{V} = \mathrm{Corr}[W_{V,1}(t), W_{V,2}(t)], \\ \rho_{1,2}^{D} = \mathrm{Corr}[W_{D,1}(t), W_{D,2}(t)], \\ \rho_{1,2}^{D,V} = \mathrm{Corr}[W_{D,1}(t), W_{V,2}(t)], \\ \rho_{2,1}^{D,V} = \mathrm{Corr}[W_{D,2}(t), W_{V,1}(t)]. \end{cases}$$

The most usual practice is to consider default correlations constant through time, similar across firms, and independent of the firms' default probabilities. It is reasonable to assume that if one company defaults, another positively correlated company has a higher likelihood to default because they are both experiencing pressures from the same sources: general economy or pressures from their specific industries or regions and, vice versa, negatively correlated company has a smaller likelihood to default. All possible instantaneous correlations between the Brownian motions $\{W_{V,i}(t),\ t \geq 0\},\ i = 1, 2$ and $\{W_{D,j}(t),\ t \geq 0\},\ j = 1, 2$ mean that the firms' asset is influenced by the developments and perturbations in secondary market of credit derivatives and vice versa, i.e., interest rates and various macroeconomic factors. The correlation $\rho_{1,2}^{V}$ means the inter-companies ties and the shocks from the area of each firm activity (i.e., an external factor that directly impacts multiple companies, either in the same industry, sector, region or related for some other reason), i.e. direct contagion effect, $\rho_{1,2}^{D}$ means the shocks in secondary market of credit derivatives and interest rates and $\rho_{1,2}^{D,V}$ and $\rho_{2,1}^{D,V}$ denote the interdependence between secondary market of credit derivatives and economical sectors. We generalize

[3]These equations assume that the $\log D_i(t)$ and $\log V_i(t),\ i = 1, 2, \ldots, n$ follow a unit root, i.e., non stationary processes.

the definition of default correlation allowing the correlation not only between firms' asset returns but also between the rates of default tresholds and the interdependence between i-th asset return and rate of j-th default boundary.

Let us define the four-dimensional Brownian motion $(W_{V,1}(t), W_{V,2}(t), W_{D,1}(t), W_{D,2}(t))$ with zero mean $E[W_{V,i}(t)] = 0$ and $E[W_{D,i}(t)] = 0,\ t \geq 0,\ i = 1,2$ and the nonnegatively defined correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{1,2}^{V} & \rho_{1,1}^{D,V} & \rho_{1,2}^{D,V} \\ \dots & 1 & \rho_{2,1}^{D,V} & \rho_{2,2}^{V} \\ \dots & \dots & 1 & \rho_{1,2}^{D} \\ \dots & \dots & \dots & 1 \end{pmatrix}.$$

All these parameters must be estimated using historical observations until time $t \geq 0$. The as exact as possible estimation of correlation always should be calculated on the basis of internal information concerning the firm's asset and liabilities. Assume that at initial time the i-th firm's asset value is greater than i-th default threshold, i.e. the condition $V_i(0) > D_i(0),\ i = 1,2$ holds.

Definition 1. *The i-th firm defaults if for any $t \geq 0$ and $i = 1,2$ the value of its asset hits the default threshold. The default time of i-th company is a random variable defined as follows:*

$$\tau_i = \begin{cases} \inf\{t \geq 0 \colon V_i(t) = D_i(t)\}, \\ \infty, \quad \textit{if } V_i(t) \neq D_i(t),\ \ \forall t \geq 0. \end{cases} \tag{4}$$

The event of default should be known for all agents of financial market at any time because we assume a perfect market with a free flow of complete information.

It is clear that the probability of any company defaulting is $P\{\tau_1 \leq T \text{ or } \tau_2 \leq T\} = P\{\tau_1 \leq T\} + P\{\tau_2 \leq T\} - P\{\tau_1 \leq T \text{ and } \tau_2 \leq T\}$ where $P\{\tau_i \leq T\} = E[F_i(T)],\ i = 1,2$ and $P\{\tau_1 \leq T \text{ and } \tau_2 \leq T\} = E[F_1(T) \cdot F_2(T)]$. Clearly, assuming the independence of default events, the joint default probability of the two firms is $P\{\tau_1 \leq T \text{ and } \tau_2 \leq T\} = P\{\tau_1 \leq T\}P\{\tau_2 \leq T\}$ and this probability is easy to calculate. However default correlation plays an important role in determining the joint default probability. After some simple calculations we obtain implied processes $Y_i(t) = D_i(0)\frac{V_i(t)}{D_i(t)}e^{\mu_{D,i}t}$ with initial values $Y_i(0) = V_i(0),\ i = 1,2$ and which are described by the following equation:

$$\mathrm{d}Y_i(t) = Y_i(t)\big(\mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_i(t)\big), \quad i = 1,2, \tag{5}$$

where

$$\begin{aligned} \mu_i &= \mu_{V,i} + \sigma_{D,i}^2 - \rho_{i,i}^{D,V}\sigma_{D,i}\sigma_{V,i}, \\ W_i(t) &= \frac{\sigma_{V,i}W_{V,i}(t) - \sigma_{D,i}W_{D,i}(t)}{\sigma_i}, \\ \sigma_i &= \sqrt{\sigma_{D,i}^2 + \sigma_{V,i}^2 - 2\rho_{i,i}^{D,V}\sigma_{D,i}\sigma_{V,i}} \end{aligned} \tag{6}$$

under the initial conditions $V_i(0) > D_i(0),\ i = 1,2$ and the implied instantaneous correlation $\mathrm{Corr}[W_1(t), W_2(t)] = \rho$, where the correlation coefficient ρ is defined as follows:

$$\rho = \frac{\sigma_{V,1}\sigma_{V,2}\rho^{V}_{1,2} - \sigma_{D,1}\sigma_{V,2}\rho^{D,V}_{1,2} - \sigma_{V,1}\sigma_{D,2}\rho^{D,V}_{2,1} + \sigma_{D,1}\sigma_{D,2}\rho^{D}_{1,2}}{\sigma_1\sigma_2}. \tag{7}$$

Remark 1. *For* $i \neq j,\ i,j = 1,2$

1. $\lim_{\sigma_{V,i}\to\infty} \rho = \frac{\sigma_{V,j}\rho^{V}_{1,2} - \sigma_{D,j}\rho^{D,V}_{j,i}}{\sigma_j}$
2. $\lim_{\sigma_{V,i}\to 0} \rho = \frac{\sigma_{D,1}\sigma_{D,2}\rho^{D,V}_{1,2} - \sigma_{D,i}\sigma_{V,j}\rho^{D,V}_{j,i}}{\sigma_j\sigma_{D,i}}$

   In other words, the implied correlation does not depend from the volatility of the i*-th firm's asset if* $\sigma_{V,i} \to \infty$ *or* $\sigma_{V,i} \to 0$
3. $\lim_{\sigma_{D,i}\to\infty} \rho = \frac{\sigma_{D,j}\rho^{D}_{1,2} - \sigma_{V,j}\rho^{D,V}_{i,j}}{\sigma_j}$
4. $\lim_{\sigma_{D,i}\to 0} \rho = \frac{\sigma_{V,1}\sigma_{V,2}\rho^{V}_{1,2} - \sigma_{D,j}\sigma_{V,i}\rho^{D,V}_{j,i}}{\sigma_j\sigma_{V,i}}$

   In other words, the implied correlation does not depend from the volatility of the i*-th default threshold if* $\sigma_{D,i} \to \infty$ *or* $\sigma_{D,i} \to 0$

The default time of i-th company for all $t \geq 0$ can be rewritten as follows:

$$\tau_i = \inf\{t \geq 0\colon V_i(t) = D_i(t)\} = \inf\left\{t \geq 0\colon Y_i(t) = D_i(0)e^{\mu_{D,i}t}\right\}, \quad i = 1,2.$$

The correlation of implied Brownian motions $\{W_i(t),\ t \geq 0\},\ i = 1,2$, defined using the formula (7), absorbs all type shocks and can be treated as a common (i.e. macroeconomic) shock which in particular case causes financial contagion in banking sector and influences the credit risk of corporate bonds portfolio. The correlation defined in such way is used for calculations of the default probability for implied processes $\{Y_i(t),\ t \geq 0\},\ i = 1,2$ defined above. Gersbach and Lipponer [12] highlighted the main properties of the relationship between the asset returns and the default correlations, illustrating how adverse macroeconomic shocks raise not only the likelihood of defaults, but also the correlations of defaults in the case when firm's asset values are jointly lognormally distributed. Finally, it is possible to use directly results of Zhou [1] for implied correlated processes $\{Y_i(t),\ t \geq 0\},\ i = 1,2$ to obtain formula (9).

4 Probability of single default

In this section, we analyze the probability of single company default and its properties in the previously defined setup.

Proposition 1. *Assume that the value of i-th firm's asset $\{V_i(t),\ t \geq 0\}$ and the i-th default threshold $\{D_i(t),\ t \geq 0\},\ i = 1, 2$ follow dynamics* (3)*. Then*

$$P\{\tau_i \leq T\} = \Phi\left(-\frac{b_i}{\sigma_i\sqrt{T}} - \frac{\nu_i}{\sigma_i}\sqrt{T}\right) + e^{-2\frac{b_i\nu_i}{\sigma_i}}\Phi\left(-\frac{b_i}{\sigma_i\sqrt{T}} + \frac{\nu_i}{\sigma_i}\sqrt{T}\right), \qquad (8)$$

where $\Phi(\cdot)$ denotes the cumulative probability distribution function of a standard normal variable and $b_i = \log\frac{V_i(0)}{D_i(0)},\ \nu_i = \frac{\mu_{V,i} - \mu_{D,i} + \sigma_{D,i}^2 - \rho_{i,i}^{D,V}\sigma_{D,i}\sigma_{V,i}}{\sigma_i} - \frac{\sigma_i}{2},\ i = 1, 2.$

See the proof in Appendix.

This formula can be used not only for estimation the probability of default. It can be useful to define the probability of a change of market implied rating of an entity. It is necessary in such case redefine the stopping time as a time of structural change of any firm's asset leverage ratio to upper or lower level.

The impact of correlation of processes $\{D_i(t),\ t \geq 0\}$ and $\{V_i(t),\ t \geq 0\}$ and the initial conditions $V_i(0) \geq D_i(0),\ i = 1, 2$ to the probability of i-th company default, by Proposition 1, in the case when $\mu_{D,i} = 0.05,\ \mu_{V,i} = 0.1,\ \sigma_{D,i} = \sigma_{V,i} = 2,\ i = 1, 2$ and $T = 1$ is presented in Table 1.

Table 1. The impact of credit quality at initial time $\frac{V_i(0)}{D_i(0)}$ and individual correlation $\rho_{i,i}^{D,V}$ to the probability of i-th company default

Individual correlation $\rho_{i,i}^{D,V}$	Credit quality $V_i(0)/D_i(0)$	Probability of default $P\{\tau_i \leq T\}$
– 0.75	1.5	0.9130
– 0.75	3	0.7676
– 0.75	5	0.6653
– 0.5	1.5	0.9061
– 0.5	3	0.7495
– 0.5	5	0.6402
– 0.25	1.5	0.8971
– 0.25	3	0.7265
– 0.25	5	0.6085
0	1.5	0.8849
0	3	0.6956
0	5	0.5668
0.25	1.5	0.8671
0.25	3	0.6512
0.25	5	0.5082
0.5	1.5	0.8372
0.5	3	0.5795
0.5	5	0.4175
0.75	1.5	0.7704
0.75	3	0.4331
0.75	5	0.2515

The influence of volatilities of processes $\{D_i(t),\ t \geq 0\}$ and $\{V_i(t),\ t \geq 0\}$ to the probability of default, by Proposition 1, in the case when credit quality at initial time is $V_i(0)/D_i(0) = 1.5,\ \mu_{D,i} = 0.05,\ \mu_{V,i} = 0.1,\ \rho_{i,i}^{D,V} = 0,75,\ i = 1,2$ and $T = 1$ is given in Table 2.

Table 2. The impact of volatilities $\sigma_{D,i}$ and $\sigma_{V,i}$ to the probability of i-th company default

Volatility $\sigma_{D,i}$	Volatility $\sigma_{V,i}$	Probability of default $P\{\tau_i \leq T\}$
0	0.25	0.1003
0	0.5	0.4747
0	0.75	0.6866
0.25	0	0.1312
0.25	0.25	0.0667
0.25	0.5	0.3021
0.25	0.75	0.6189
0.5	0	0.3238
0.5	0.25	0.2081
0.5	0.5	0.2255
0.5	0.75	0.5026
0.75	0	0.4514
0.75	0.25	0.3407
0.75	0.5	0.3099
0.75	0.75	0.4143

Next, we give some properties of the probability of single company default.

Remark 2. 1. *Assume that at initial time the value of i-th firm's asset is equal to i-th default threshold, i.e. $D_i(0) = V_i(0),\ i = 1,2$. Then $P\{\tau_i \leq T\} = 1$.*

2. *Assume that both the i-th underlying firm's asset and i-th default threshold grow at the same rate, i.e. $\mu_{D,i} - \frac{\sigma_{D,i}^2}{2} = \mu_{V,i} - \frac{\sigma_{V,i}^2}{2},\ i = 1,2$. Then $P\{\tau_i \leq T\} = 2\Phi(-\frac{b_i}{\sigma_i\sqrt{T}})$.*

3. *Assume that the return i-th of firm's asset is greater than the rate of i-th default threshold, i.e. $\mu_{V,i} - \frac{\sigma_{V,i}^2}{2} > \mu_{D,i} - \frac{\sigma_{D,i}^2}{2},\ i = 1,2$. Then*

$$\lim_{T\to\infty} P\{\tau_i \leq T\} = e^{-2\frac{b_i \nu_i}{\sigma_i}}.$$

4. *Assume that the credibility of i-th company is infinitely high at initial time. Then*

$$\lim_{b_i\to\infty} P\{\tau_i \leq T\} = 0, \quad i = 1,2.$$

5. $$\lim_{\sigma_{D,i}\to\infty} P\{\tau_i \le T\} = \Phi\biggl(-\frac{\sqrt{T}}{2}\biggr) + \frac{D_i(0)}{V_i(0)}\Phi\biggl(\frac{\sqrt{T}}{2}\biggr), \quad i = 1,2.$$

In addition,

$$\lim_{T\to\infty}\lim_{\sigma_{D,i}\to\infty} P\{\tau_i \le T\} = \frac{D_i(0)}{V_i(0)}, \quad i = 1,2.$$

6. $$\lim_{\sigma_{V,i}\to\infty} P\{\tau_i \le T\} = \Phi\biggl(\frac{\sqrt{T}}{2}\biggr) + \frac{V_i(0)}{D_i(0)}\Phi\biggl(-\frac{\sqrt{T}}{2}\biggr), \quad i = 1,2.$$

In addition,

$$\lim_{T\to\infty}\lim_{\sigma_{V,i}\to\infty} P\{\tau_i \le T\} = 1, \quad i = 1,2.$$

Since the closed form expression of the probability of single default in case of stochastic default boundary is known, it's sufficient to find the probability $P\{\tau_1 \le T$ or $\tau_2 \le T\}$ or $P\{\tau_1 \le T$ and $\tau_2 \le T\}$.

5 Joint probability of correlated defaults

In this section, we investigate expression of the probability $P\{\tau_1 \le T$ or $\tau_2 \le T\}$ under the assumptions presented in Section 3.

Proposition 2. *Assume that the value of i-th firm's asset $\{V_i(t),\ t \ge 0\}$ and the value of i-th default threshold $\{D_i(t),\ t \ge 0\},\ i = 1,2$ follow dynamics* (3)*. The cumulative distribution function of correlated default subject to initial conditions $V_i(0) > D_i(0)$, $i = 1,2$ is*

$$P\{\tau_1 \le T \text{ or } \tau_2 \le T\} = 1 - \frac{2}{\alpha T} e^{-\frac{r_0^2}{2T}} e^{a_1 d_1 + a_2 d_2 + a_3 T} \sum_{n=1}^{\infty} \sin\biggl(\frac{n\pi\theta_0}{\alpha}\biggr) \int_0^{\alpha} \sin\biggl(\frac{n\pi\theta}{\alpha}\biggr) g_n(\theta)\,\mathrm{d}\theta, \tag{9}$$

where

$$g_n(\theta) = \int_0^{\infty} r e^{-\frac{r^2}{2T} - c(\theta) r} I_{\frac{n\pi}{\alpha}}\biggl(\frac{r r_0}{T}\biggr)\mathrm{d}r,$$

$$I_{\frac{n\pi}{\alpha}}\biggl(\frac{r r_0}{T}\biggr) = \sum_{m=0}^{\infty} \frac{\bigl(\frac{r r_0}{2T}\bigr)^{\frac{n\pi}{\alpha}+2m}}{m!\Gamma\bigl(\frac{n\pi}{\alpha}+m+1\bigr)}$$

is the modified Bessel function of the first kind of order $\frac{n\pi}{\alpha}$ and argument $\frac{r r_0}{T}$, $\Gamma(\cdot)$ is the

Euler's gamma function,

$$\alpha = \begin{cases} \arctan\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \textit{if } \rho < 0, \\ \pi + \arctan\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \textit{if } \rho \geq 0, \end{cases}$$

$$\theta_0 = \begin{cases} \arctan\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{-b_1\sigma_2+\rho b_2\sigma_1}\right), & \textit{if } \left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{-b_1\sigma_2+\rho b_2\sigma_1}\right) > 0 \\ \pi + \arctan\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{-b_1\sigma_2+\rho b_2\sigma_1}\right), & \textit{if } \left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{-b_1\sigma_2+\rho b_2\sigma_1}\right) \leq 0, \end{cases}$$

$$a_1 = \frac{\left(\mu_{D,1} - \mu_{V,1} - \sigma_{D,1}^2 + \rho_{1,1}^{D,V}\sigma_{D,1}\sigma_{V,1}\right)\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2} - \frac{\left(\mu_{D,2} - \mu_{V,2} - \sigma_{D,2}^2 + \rho_{2,2}^{D,V}\sigma_{D,2}\sigma_{V,2}\right)\rho\sigma_1}{\left(1-\rho^2\right)\sigma_1^2\sigma_2},$$

$$a_2 = \frac{\left(\mu_{D,2} - \mu_{V,2} - \sigma_{D,2}^2 + \rho_{2,2}^{D,V}\sigma_{D,2}\sigma_{V,2}\right)\sigma_1}{\left(1-\rho^2\right)\sigma_2^2\sigma_1} - \frac{\left(\mu_{D,1} - \mu_{V,1} - \sigma_{D,1}^2 + \rho_{1,1}^{D,V}\sigma_{D,1}\sigma_{V,1}\right)\rho\sigma_2}{\left(1-\rho^2\right)\sigma_2^2\sigma_1},$$

$$a_3 = \frac{a_1^2\sigma_1^2}{2} + \rho a_1 a_2 \sigma_1\sigma_2 + \frac{a_2^2\sigma_2^2}{2} - a_1\left(\mu_{D,1} - \mu_{V,1} - \sigma_{D,1}^2 + \rho_{1,1}^{D,V}\sigma_{D,1}\sigma_{V,1}\right) - a_2\left(\mu_{D,2} - \mu_{V,2} - \sigma_{D,2}^2 + \rho_{2,2}^{D,V}\sigma_{D,2}\sigma_{V,2}\right),$$

$$c(\theta) = \frac{b_1}{\sigma_1}\sin(\theta-\alpha) - \frac{b_2}{\sigma_2}\cos(\theta-\alpha),$$

$$d_1 = a_1\sigma_1 + \rho a_2\sigma_2, \quad d_2 = a_2\sigma_2\sqrt{1-\rho^2}, \quad r_0 = \frac{b_2}{\sigma_2\sin\theta_0}.$$

See the proof in Appendix.

Remark 3. 1. *Assume that* $V_2(0) = D_2(0)$. *Then* $P\{\tau_1 \leq T \textit{ or } \tau_2 \leq T\} = 1$.

2. *It is natural that in the case of infinite maturity T, due to increasing uncertainty, the joint probability of default approaches 1, i.e.:* $\lim_{T\to\infty} P\{\tau_1 \leq T \textit{ or } \tau_2 \leq T\} = 1$.

3. *Assume that* $\frac{\theta_0}{\alpha}$ *is an integer*[4]. *Then* $P\{\tau_1 \leq T \textit{ or } \tau_2 \leq T\} = 1$.

Corollary 1. *The probability of both companies surviving is as follows:*

$$P\{\tau_1 > T_1 \textit{ and } \tau_2 > T_2\} = 1 - P\{\tau_1 \leq T_1\} - P\{\tau_2 \leq T_2\} + P\{\tau_1 \leq T_1 \textit{ or } \tau_2 \leq T_2\}.$$

[4]Such case is possible, for instance, if the coefficient of implied correlation ρ is negative, taking α close to 0 and letting b_1 or σ_2 to be sufficiently large.

Finally, knowing the default probability of each single company and the joint default probability of both firms, it is possible to calculate the default correlation. The implementation of formula (9) is computationally complicated because presented result requires double integration of nonlinear combination of modified Bessel function of the first kind. Also, the estimation of "implied" correlation needs to computationally estimate the volatilities and the correlations of Brownian motions $\{V_i(t),\ t \geq 0\}$ and $\{D_i(t),\ t \geq 0\}$, $i = 1, 2$. On the other hand, it is clear that the given solution is not symmetric with respect to different defaults. So, the problem of selection of defaults arise, and it is not clear in what order select different defaults[5].

Remark 4. *Assume that $T_1 \leq T_2$. Then*

$$P\{\tau_1 \leq T_1 \text{ or } \tau_2 \leq T_2\} = P\{\tau_1 \leq T_1 \text{ or } \tau_2 \leq T_1\} + P\{\tau_1 \leq T_1 \text{ and } T_1 \leq \tau_2 \leq T_2\}.$$

Remark 5. *All the propositions in this paper are made for calculations at initial time $t = 0$. The same formulas hold in the case of any $t > 0$. In such case, the constant T must be changed by the variable $T - t$ for any $t \geq 0$ and the default time of i-th company, $i = 1, 2$, $\tau_i = \inf\{t \geq 0 :\ V_i(t) = D_i(t)\}$ by $\tau_i = \inf\{s \geq t \colon V_i(s) = D_i(s)\}$.*

6 Conclusion

The main result of this paper is the generalization of Zhou's approach. The contribution of this paper is to derive analytical expression for the joint probabiloity of correlated defaults in the first passage time approach of credit risk which is necessary to quickly assess the credit risk of corporate bonds portfolio. The main assumptions in this paper are that both firm's asset value and the default threshold follow geometric Brownian motions, i.e. by assuming stochastic behavior of default boundary, we also consider other exogenously defined type of financial risk, notably, liquidity and market changes risk and in such way expand credit risk modelling. On the other hand, one of the largest disadvantages of presented formulas of these formulas is the limited data for defining implied correlations, i.e., that its application requires a lot of aggregated information concerning secondary market of corporate bonds and the internal information about firm's asset. In addition, the provided expression of joint probability of correlated defaults is not symmetric with respect to different company's default.

The further step of this research could be the generalization of given formula of joint probability of $n > 2$ correlated defaults. On the other hand, the generalization of probability of corporate bonds portfolio default in jump-diffusion case remains.

Appendix

Proof of Proposition 1. Using results of Zhou [1], where the default boundary follows exponential function, i.e. $D_i(t) = D_i(0)e^{\mu_{D,i}t}$ the probability of single company default

[5] In practice, it is possible to follow so called principle of conservatism.

is given by

$$P\{\tau_i \le T\} = \Phi\biggl(- \frac{b_i}{\sigma_{V,i}\sqrt{T}} - \frac{\mu_{V,i} - \mu_{D,i}}{\sigma_{V,i}}\sqrt{T}\biggr) + e^{-2\frac{(\mu_{D,i}-\mu_{V,i})b_i}{\sigma_{V,i}^2}}\Phi\biggl(- \frac{b_i}{\sigma_{V,i}\sqrt{T}} + \frac{\mu_{V,i} - \mu_{D,i}}{\sigma_{V,i}}\sqrt{T}\biggr).$$

Assume that default boundary, (in special case, the value of i-th bond) follows geometric Brownian motion. Then after some simple calculations we have the expression of stopping time, i.e. the time when default of the i-th firm, $i = 1, 2$ occurs:

$$\begin{aligned} &\inf\{t \ge 0\colon V_i(t) = D_i(t)\} \\ &\quad = \inf\Bigl\{t \ge 0\colon V_i(0)e^{\left(\mu_{V,i}-\mu_{D,i}+\sigma_{D,i}^2-\rho_{i,i}^{D,V}\sigma_{D,i}\sigma_{V,i}-\frac{\sigma_i^2}{2}\right)t+\sigma_i W_i(t)} = D_i(0)\Bigr\}. \end{aligned} \tag{10}$$

Let us define implied stochastic processes for any $t \ge 0$ by formula $\widetilde{Y}_i(t) = D_i(0)\frac{V_i(t)}{D_i(t)}$, $\widetilde{Y}_i(0) = V_i(0),\ i = 1, 2$. Then we obtain implied geometric Brownian motions, for any $t \ge 0$ described by equation

$$\mathrm{d}\widetilde{Y}_i(t) = \widetilde{Y}_i(t)\bigl((\mu_i - \mu_{D,i})\mathrm{d}t + \sigma_i \mathrm{d}W_i(t)\bigr), \quad i = 1, 2$$

with coefficients, defined in formula (6), and respective default thresholds $D_i(0)$, and unchanged initial conditions: $\tilde{Y}_i(0) = V_i(0) > D_i(0),\ i = 1, 2.$

Let us rewrite the term in right-hand side in expression (10):

$$V_i(0)e^{\left(\mu_{V,i}-\mu_{D,i}+\sigma_{D,i}^2-\rho_{i,i}^{D,V}\sigma_{D,i}\sigma_{V,i}-\frac{\sigma_i^2}{2}\right)t+\sigma_i W_i(t)} = V_i(0)\mathrm{e}^{\sigma_i Z_i(t)}, \quad i = 1, 2.$$

where $Z_i(t) = \nu_i t + W_i(t),\ t \ge 0$, and $\nu_i = \frac{\mu_i - \mu_{D,i}}{\sigma_i} - \frac{\sigma_i}{2},\ i = 1, 2$. Hence, using the reflection principle for geometric Brownian motions (for more details, see [13]) the probability of i-th firm default is given by

$$\begin{aligned} P\{\tau_i \le T\} &= P\bigl\{\inf\{t \ge 0\colon \widetilde{Y}_i(t) = D_i(0)\} \le T\bigr\} \\ &= P\biggl\{\inf\Bigl\{t \ge 0\colon Z_i(t) = -\frac{b_i}{\sigma_i}\Bigr\} \le T\biggr\} = 1 - P\Bigl\{m_T^{Z_i} \ge -\frac{b_i}{\sigma_i}\Bigr\} \\ &= \Phi\biggl(- \frac{b_i}{\sigma_i\sqrt{T}} - \frac{\nu_i}{\sigma_i}\sqrt{T}\biggr) + e^{-2\frac{b_i\nu_i}{\sigma_i}}\Phi\biggl(- \frac{b_i}{\sigma_i\sqrt{T}} + \frac{\nu_i}{\sigma_i}\sqrt{T}\biggr). \end{aligned}$$

where $m_T^{Z_i} = \inf_{0\le t\le T} Z_i(t),\ i = 1, 2.$ □

Proof of Proposition 2. Using the definition of default time of the i-th firm we have $P\{\tau_1 \le T$ or $\tau_2 \le T\}$. Technically, we calculate the probability

$$P\{\tau_1 > T \text{ and } \tau_2 > T\} = 1 - P\{\tau_1 \le T \text{ or } \tau_2 \le T\}.$$

After some simple calculations we have the expression of stopping time, i.e. the time when default of the i-th firm occurs:

$$\inf\{t \geq 0\colon V_i(t) = D_i(t)\} = \inf \Big\{ t \geq 0\colon V_i(0)e^{\left(\mu_{V,i}+\sigma^2_{D,i}-\rho^{D,V}_{i,i}\sigma_{D,i}\sigma_{V,i}-\frac{\sigma_i^2}{2}\right)t} + \sigma_i W_i(t) = D_i(0)\mathrm{e}^{\mu_{D,i}t} \Big\}, \quad i = 1, 2.$$

Let us define implied stochastic processes for any $t \geq 0$ by formula

$$Y_i(t) = D_i(0)\frac{V_i(t)}{D_i(t)}\mathrm{e}^{\mu_{D,i}t}, \quad Y_i(0) = V_i(0), \quad i = 1, 2.$$

Then we obtain implied geometric Brownian motions, described in formulas (5) and (6) for any $t \geq 0$ with correlation

$$\rho = \mathrm{Corr}[\log Y_1(t), \log Y_2(t)] = \mathrm{Corr}[W_1(t), W_2(t)].$$

The respective default thresholds are $D_i(0)\mathrm{e}^{\mu_{D,i}t}$ and the initial conditions left unchanged: $Y_i(0) = V_i(0) > D_i(0),\ i = 1, 2$. Let us define

$$X_i(t) = -\log\left(\mathrm{e}^{-\mu_{D,i}t}\frac{Y_i(t)}{Y_i(0)}\right), \quad i = 1, 2$$

that follows two-dimensional correlated arithmetic Brownian motion,

$$\begin{pmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mathrm{d}t - \Theta \begin{pmatrix} \mathrm{d}W_1(t) \\ \mathrm{d}W_2(t) \end{pmatrix},$$

where $\lambda_i = \mu_{D,i} - \mu_i + \frac{\sigma_i^2}{2},\ i = 1, 2$ and Θ is a 2×2 covariance matrix such that

$$\Theta \cdot \Theta' = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

with the transformed initial conditions $X_i(0) = 0$ and $b_i = \log\frac{V_i(0)}{D_i(0)},\ i = 1, 2$. Let $f(b_1, b_2, x_1, x_2, \rho, T)$ be the transition probability density of the particle in the region $\{(x_1, x_2):\ x_1 < b_1 \text{ and } x_2 < b_2\}$ before the maturity T. The finding of the probability $P\{\tau_1 \leq T \text{ or } \tau_2 \leq T\}$ is equivalent to finding the first passage time of $X_i(t)$ to a boundary $b_i,\ i = 1, 2$. Let us define the probability that the particle does not reach the fixed barrier $\partial(b_1, b_2)$ in the time period $[0, T]$, i.e.,

$$\begin{aligned} &F(b_1, b_2, T) \\ &\quad\equiv P\big\{X_1(T) < b_1 \text{ and } X_2(T) < b_2 \mid\ X_1(s) < b_1 \text{ and } X_2(s) < b_2,\ 0 < s < T\big\} \\ &\quad= P\{\tau_1 > T \text{ and } \tau_2 > T\} = \int\limits_{-\infty}^{b_1}\int\limits_{-\infty}^{b_2} f(b_1, b_2, x_1, x_2, \rho, T)\mathrm{d}x_1\mathrm{d}x_2. \end{aligned}$$

The problem of computing probability density is classical. It can be usually tackled using the separation of variables technique or more sophisticated methods such as contour integration and Laplace transform for the closely related problem of the solution of the heat equation on a wedge. Zhou [1], He, Keirstead and Rebholz [5] and Patras [7] followed the first approach. Patras analyzed the correlated default problem using planar Brownian motion constructions and its local isometry. Patras used this method in more general case, i.e. when induced by local (where local means with respect to time and space simultaneously) isometry from planar Brownian motion stochastic process behaves therefore locally as the usual planar Brownian motion and the subspace is only locally Euclidean. The transition probability density is the solution of Kolmogorov forward equation

$$\frac{\partial f}{\partial T} = \frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial x_2^2}, \quad x_1 < b_1, \;\; x_2 < b_2, \tag{11}$$

subject to the boundary conditions:

$$\begin{aligned}
&f(b_1, b_2, -\infty, x_2, \rho, T) = f(b_1, b_2, x_1, -\infty, \rho, T) = 0,\\
&f(b_1, b_2, x_1, x_2, \rho, 0) = \delta(x_1)\delta(x_2),\\
&\int_{-\infty}^{b_1}\int_{-\infty}^{b_2} f(b_1, b_2, x_1, x_2, \rho, T)\mathrm{d}x_1\mathrm{d}x_2 \le 1, \quad T > 0,\\
&f(b_1, b_2, b_1, x_2, \rho, T) = f(b_1, b_2, x_1, b_2, \rho, T) = 0,
\end{aligned}$$

where $\delta(x)$ is a Dirac's Delta function and the equation $f(b_1, b_2, x_1, x_2, 0) = \delta(x_1)\delta(x_2)$ means the initial condition $X_i(0) = 0, \; i = 1, 2$. The solution of Kolmogorov forward equation subject to boundary conditions (for more details, see [5, 14] and [1]) for any fixed $t > 0$ is given by

$$\begin{aligned}
&f(b_1, b_2, x_1, x_2, \rho, T)\\
&\quad = \frac{2}{\sigma_1\sigma_2\sqrt{1-\rho^2}\alpha T} e^{-\frac{r^2+r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{T}\right),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
x_1 &= b_1 - \sigma_1\left(\sqrt{1-\rho^2}r\cos\theta + \rho r \sin\theta\right),\\
x_2 &= b_2 - \sigma_2 r \sin\theta.
\end{aligned}$$

The probability that the particle does not reach the barrier $\partial(b_1, b_2)$ during the period $[0, T]$ is given by

$$F(b_1, b_2, T) = \int_{-\infty}^{b_1}\int_{-\infty}^{b_2} f(b_1, b_2, x_1, x_2, \rho, T)\mathrm{d}x_1\mathrm{d}x_2$$

$$= \int_0^{\alpha}\int_0^{\infty} r\sigma_1\sigma_2\sqrt{1-\rho^2} f(\alpha,\infty,r,\theta,\rho,T)\mathrm{d}\theta\mathrm{d}r$$

$$= \frac{2}{\alpha T} e^{-\frac{r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right)\mathrm{d}\theta \int_0^{\infty} r e^{-\frac{r^2}{2T}} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{T}\right)\mathrm{d}r.$$

Hence, the probability of either company defaulting is as follows:

$$\begin{aligned} P\{\tau_1 \le T \text{ or } \tau_2 \le T\} &= 1 - F(b_1, b_2, T) \\ &= 1 - \frac{2}{\alpha T} e^{-\frac{r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right)\mathrm{d}\theta \int_0^{\infty} r e^{-\frac{r^2}{2T}} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{T}\right)\mathrm{d}r \\ &= 1 - \frac{2}{\alpha T} e^{-\frac{r_0^2}{2T}} e^{a_1 d_1 + a_2 d_2 + a_3 T} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta)\mathrm{d}\theta. \end{aligned} \tag{13}$$

□

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