Estimation of Convergence Rate in the Transfer Theorem for Maxima

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Abstract. Let Z_N be a maximum of independent identically distributed random variables. In this paper, a nonuniform estimate of convergence rate in the transfer theorem max-scheme is obtained. Presented results make the estimates, given in [1] and [2], more precise.

Keywords: max-scheme, transfer theorem, limit theorem, rate of convergence.

1 Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables (r.v.'s) with the distribution function $P(X_j \leq x) = F(x),\ j \geq 1$; let $N_1, N_2, \ldots, N_n, \ldots$ be a sequence of positive integer-valued r.v.'s independent of $X_j,\ j \geq 1$ and $P(N_n \leq x) = A_n(x)$.

Let us define two random variables

$$Z_n = \max(X_1, \ldots, X_n), \quad Z_{N_n} = \max(X_1, \ldots, X_{N_n}),$$

and consider linear normalized maxima

$$\overline{Z}_n = b_n^{-1}(Z_n - a_n), \quad \overline{Z}_{N_n} = b_n^{-1}(Z_{N_n} - a_n),$$

$-\infty < a_n < +\infty,\ \ b_n > 0.$

Now, let us denote:

$$u_n(x) = n\big(1 - F(xb_n + a_n)\big).$$

The necessary and sufficient conditions for weak convergence $P(\overline{Z}_n \leq x) \Rightarrow H(x)$ to nonsingular distribution function H, $n \to \infty$ are presented in the fundamental work of

Gnedenko [3]. In terms of the function $u_n(x)$ the necessary and sufficient condition is defined to be

$$\lim_{n\to\infty} u_n(x) = u(x).$$

Then $H(x) = e^{-u(x)}$ is a limit distribution function.

Nonuniform estimate of convergence rate

$$\left|P\big(\overline{Z}_n \le x\big) - H(x)\right| \le \overline{\Delta}_n(x)$$

is presented in the monograph of Galambos [4]. More general estimates of convergence rate are given in the paper of Aksomaitis [1].

We are interested in nonuniform estimates of convergence rate in the transfer theorem of Gnedenko [5], i.e. in estimates of type

$$\left|P\big(\overline{Z}_{N_n} \le x\big) - \Psi(x)\right| \le \Delta_n(x),$$

where

$$\Psi(x) = \int_0^\infty H^z(x)dA(z)$$

and

$$A(z) = \lim_{n\to\infty} P\left(\frac{N_n}{n} \le z\right) = \lim_{n\to\infty} A_n(nz).$$

Let $\rho_n(x) = u_n(x) - u(x)$. When supposing $\rho_n(x) \ge 0$ an estimate of the rate $\Delta_n(x)$ has already been proposed by the author [2, 6] and is given by Aksomaitis.

2 Main results

A nonuniform estimate of the convergence rate in transfer theorem is given by the following theorem.

Theorem. *Let H be the limit distribution of the r.v. $\overline{Z}_n$ and $\lim_{n\to\infty} P(\frac{N_n}{n} \le x) = A(x)$, $A(+0) = 0$. Then, for any x satisfying condition $\frac{u_n(x)}{n} \le \frac{1}{2}$, the following estimate holds true:*

$$\Delta_n(x) \le \left(\frac{u_n^2(x)}{n} + |\rho_n(x)|\right)\int_0^\infty z\delta_n^z(x)dA_n(nz) + u(x)\int_0^\infty |A_n(nz) - A(z)|H^z(x)dz,$$

where $\delta_n(x) = \max(F^n(xb_n + a_n), H(x))$, $\rho_n(x) = u_n(x) - u(x)$.

This theorem simplifies and makes more precise results obtained in the author's previous publication [2].

Proposition 1. $\delta_n(x) \le H(x)\max(e^{-\rho_n(x)}, 1)$.

Proposition 2. *If* $EN_n < \infty$, *then*

$$\int_0^\infty z\delta_n^z(x)dA_n(nz) \le \frac{EN_n}{n}.$$

Proposition 3. *If* $P(N_n = n) = 1$, *then*

$$\int_0^\infty z\delta_n^z(x)dA_n(nz) = \delta_n(x).$$

Proposition 4. *If* $\rho_n(x) \ge 0$, *then*

$$\int_0^\infty z\delta_n^z(x)dA_n(nz) \le \int_0^\infty zH^z(x)dA_n(nz).$$

3 Proof of the theorem

We have

$$\begin{aligned} &\big|P\big(\overline{Z}_{N_n} \le x\big) - \Psi(x)\big| \\ &\quad\le \big|P\big(\overline{Z}_{N_n} \le x\big) - EH^{\frac{N_n}{n}}(x)\big| + \big|EH^{\frac{N_n}{n}}(x) - \Psi(x)\big| \\ &\quad= I_n^{(1)}(x) + I_n^{(2)}(x). \end{aligned} \tag{1}$$

The total probability formula implies:

$$\begin{aligned} I_n^{(1)}(x) &= \Big|\sum_{j\ge1} F^j(xb_n + a_n)P(N_n = j) - \sum_{j\ge1} H^{\frac{j}{n}}(x)P(N_n = j)\Big| \\ &= \Big|\int_0^\infty F^{nz}(xb_n + a_n)dA_n(nz) - \int_0^\infty H^z(x)dA_n(nz)\Big| \\ &\le \int_0^\infty \big|F^{nz}(xb_n + a_n) - H^z(x)\big|dA_n(nz). \end{aligned} \tag{2}$$

Since

$$\begin{aligned} &\big|F^{nz}(xb_n + a_n) - H^z(x)\big| = z\Big|\int_{H(x)}^{F^n(xb_n+a_n)} t^{z-1}dt\Big| \\ &\quad\le z\Big(\max\big(F^n(xb_n + a_n), H(x)\big)\Big)^z \big|\ln F^n(xb_n + a_n) - \ln H(x)\big| \\ &\quad\le z\delta_n^z(x)\left(\left|n\ln\left(1 - \frac{u_n(x)}{n}\right) + u_n(x)\right| + |u_n(x) - u(x)|\right), \end{aligned}$$

then, using the inequality $|\ln(1-t)+t| \le t^2,\ 0 \le t \le \frac{1}{2}$, we obtain

$$\big|F^{nz}(xb_n+a_n)-H^z(x)\big| \le z\delta_n^z(x)\left(\frac{u_n^2(x)}{n}+|\rho_n(x)|\right), \tag{3}$$

provided $\frac{u_n(x)}{n} \le \frac{1}{2}$.

From (2) and (3) it follows that

$$I_n^{(1)}(x) \le \left(\frac{u_n^2(x)}{n}+|\rho_n(x)|\right)\int_0^\infty z\delta_n^z(x)dA_n(nz).$$

Also,

$$I_n^{(2)}(x) = \left|\int_0^\infty H^z(x)dA_n(nz) - \int_0^\infty H^z(x)dA(z)\right| = \left|\int_0^\infty H^z(x)d\big(A_n(nz)-A(z)\big)\right|.$$

Integrating by parts, we get

$$I_n^{(2)}(x) = \left|\int_0^\infty \big(A_n(nz)-A(z)\big)dH^z(x)\right| \le u(x)\int_0^\infty \big|A_n(nz)-A(z)\big|H^z(x)dz.$$

The proof of the theorem follows from (1), (2) and (3).

4 Example

Let $P(N_n=k)=\frac{1}{n},\ 1 \le k \le n$. We get:

$$\lim_{n\to\infty} A_n(nx) = A(x) = x, \quad 0 \le x \le 1;$$

$$|A_n(nx)-A(x)| \le \frac{1}{n}, \quad \frac{EN_n}{n} = \frac{n}{2}+\frac{1}{2};$$

$$I_n^{(2)}(x) \le \frac{u(x)}{n}\int_0^1 H^z(x)dz = \frac{1-H(x)}{n} \le \frac{u(x)}{n};$$

$$\Delta_n(x) \le \left(\frac{u_n^2(x)}{n}+|\rho_n(x)|\right)\left(\frac{1}{2}+\frac{1}{2n}\right)+\frac{u(x)}{n}.$$

If, for instance,

$$F(x) = 1-\frac{c}{x^\alpha}, \quad x \ge c^{\frac{1}{\alpha}}, \quad \alpha > 0,$$

then

$$u_n(x) = n\Big(1-F\big(x(cn)^{\frac{1}{\alpha}}\big)\Big) = x^{-\alpha}$$

and

$$\Delta_n(x) \leq \frac{x^{-\alpha}}{n}\left(1 + \frac{x^{-\alpha}}{2}\left(1 + \frac{1}{n}\right)\right),$$

provided $nx^\alpha \geq 2$.

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