

Stokes' Problems for an Incompressible Couple Stress Fluid

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Abstract. Stokes' first and second problems for an incompressible couple stress fluid are considered under isothermal conditions. The problems are solved through the use of Laplace transform technique. Inversion of the Laplace transform of the velocity component in each case is carried out using a standard numerical approach. Velocity profiles are plotted and studied for different times and different values of couple stress Reynolds number. The results are presented through graphs in each case.

Keywords: Stokes' first problem, Stokes' second problem, couple stress fluid, Laplace transform, numerical inversion.

Nomenclature

(x, y)	space coordinates	t	time
u	velocity of the fluid along the x -direction	\bar{f}	body force per unit mass
U	velocity of the plate	\bar{c}	body couple per unit mass
l	couple stress parameter $\sqrt{\eta/\mu}$	\bar{M}	couple stress diad
R	Reynolds number $\rho Ul/\mu$	P	pressure
μ, λ	viscosity coefficients	t_{ij}	force stress tensor
η, η'	couple stress viscosity coefficients	d_{ij}	rate of deformation tensor
ρ	density	$w_{i,j}$	spin tensor
w	spin vector	m_{ij}	couple stress tensor
		m	trace of couple stress tensor
		ν	frequency of the velocity of the wall

1 Introduction

Stokes' first and second problems for the flat plate (1851) have received much attention due to their practical applications. Consider an infinitely long flat plate above which a fluid exists. Initially, both the plate and fluid are at rest. Suddenly, the plate is jerked

into motion in its own plane with a constant velocity. The resultant motion of the fluid is studied by many researchers for different types of fluids. Taking the fluid to be Newtonian, by using a simple transformation, an exact solution was obtained for this problem by Stokes in 1851 and again Rayleigh in 1911 [1]. Tanner [2] gave the exact solution to this Stokes' first problem for a Maxwell fluid in 1962. Taipel [3] studied the impulsive motion of a flat plate in a viscoelastic fluid in 1981. L. Preziosi and D. D. Joseph [4] studied the Stokes' first problem for viscoelastic fluids in 1987. N. Phan-Thien and Y. T. Chew [5] considered this problem for viscoelastic fluid in 1988. Recently many investigators have studied the Stokes' first problem for Non-Newtonian fluids with different constitutive models (see [6–18] and the references therein).

The motion of a viscous fluid caused by the sinusoidal oscillation of a flat plate is termed as Stokes' second problem by Schlichting [1]. Initially, both the plate and fluid are assumed to be at rest. At time $t = 0+$, the plate suddenly starts oscillating with the velocity $U \cos(vt)$. The study of the flow of a viscous fluid over an oscillating plate is not only of fundamental theoretical interest but it also occurs in many applied problems such as acoustic streaming around an oscillating body, an unsteady boundary layer with fluctuations etc [19]. Penton [20] has presented a closed-form to the transient component of the solution for the flow of a viscous fluid due to an oscillating plate. Puri and Kythe [21] have discussed an unsteady flow problem which deals with non-classical heat conduction effects and the structure of waves in Stokes' second problem. Erdogan [22] analyzed the unsteady flow of viscous fluid due to an oscillating plane wall by using Laplace transform technique. Vajravelu and Rivera [23] discussed the hydromagnetic flow at an oscillating plate. Much work has been published on the flow of fluid over an oscillating plate for different constitutive models (see [24–28] and the references therein).

Couple stress fluid theory developed by Vijay Kumar Stokes [29], is one among the polar fluid theories which considers couple stresses in addition to the classical Cauchy stress. It is the simplest generalization of the classical theory of fluids which allows for polar effects such as the presence of couple stresses and body couples. This fluid theory is discussed in detail by V.K. Stokes himself in his treatise "Theories of Fluids with Microstructure" [30] where in he also presented a list of problems discussed by researchers with reference to this theory. Some of the problems of recent interest can also be seen in Naduvinamani et al. [31].

In this paper, we propose to study Stokes' first and second problems for an incompressible couple stress fluid [29]. For both the problems analytical solutions are obtained in Laplace transform domain. A numerical inversion technique [32] is employed to obtain velocity component by inverting Laplace transform of the velocity in each case.

2 Basic equations and formulation of the problem

The equations of motion that characterize a couple stress fluid flow are similar to the Navier-Stokes equations and are given by [29]

$$\frac{d\rho}{dt} + \rho \operatorname{div}(\vec{q}) = 0, \quad (1)$$

$$\rho \frac{d\bar{q}}{dt} = \rho \bar{f} + \frac{1}{2} \operatorname{curl}(\rho \bar{c}) + \operatorname{div}(\tau^{(s)}) + \frac{1}{2} \operatorname{curl}(\operatorname{div}(\bar{M})), \quad (2)$$

where ρ is the density of the fluid, $\tau^{(s)}$ is the symmetric part of the force stress diad, \bar{M} is the couple stress diad and \bar{f} , \bar{c} are the body force per unit mass and body couple per unit mass respectively.

The constitutive equations concerning the force stress t_{ij} and the rate of deformation tensor d_{ij} are given by:

$$t_{ij} = -p\delta_{ij} + \lambda \operatorname{div}(\bar{q})\delta_{ij} + 2\mu d_{ij} - \frac{1}{2} \varepsilon_{ijk} [m_{,k} + 4\eta w_{k,rr} + \rho c_k]. \quad (3)$$

The couple stress tensor m_{ij} that arises in the theory has the linear constitutive relation

$$m_{ij} = \frac{1}{3} m \delta_{ij} + 4\eta w_{j,i} + 4\eta' w_{i,j}. \quad (4)$$

In the above $w = \frac{1}{2} \operatorname{curl}(\bar{q})$ is the spin vector, $w_{i,j}$ is the spin tensor, m is the trace of couple stress tensor m_{ij} , p is the fluid pressure and ρc_k is the body couple vector. Comma in the suffixes denotes covariant differentiation and $w_{k,rr}$ stands for $w_{k,11} + w_{k,22} + w_{k,33}$. The quantities λ and μ are the viscosity coefficients and η, η' are the couple stress viscosity coefficients. These material constants are constrained by the inequalities,

$$\mu \geq 0; \quad 3\lambda + 2\mu \geq 0; \quad \eta \geq 0; \quad |\eta'| \leq \eta. \quad (5)$$

There is a length parameter $l = \sqrt{\eta/\mu}$, which is a characteristic measure of the polarity of the fluid model and this parameter is identically zero in the case of non-polar fluids.

If the fluid is incompressible, in the absence of body forces and body couples the above field equations (1) and (2) reduce to

$$\operatorname{div}(\bar{q}) = 0, \quad (6)$$

$$\rho \left[\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla \bar{q}) \right] = -\operatorname{grad}(p) - \mu \operatorname{curl}(\operatorname{curl}(\bar{q})) - \eta \operatorname{curl}(\operatorname{curl}(\operatorname{curl}(\bar{q}))) \quad (7)$$

To solve the problem dealing with couple stress fluid flows, in addition to the usual assumption of no-slip condition, it is presumed that the couple stresses vanish at the boundary.

2.1 Mathematical formulation of the problem

Consider the unsteady flow of an incompressible, couple stress fluid which fills the half space $y > 0$ above a flat (solid) plate occupying xz -plane. Initially, we assume that both fluid and plate are at rest. At time $t = 0+$, whether we allow the plate to start with a constant velocity U along x -axis or oscillate with velocity $U \cos(vt)$, the flow occurs only

in x -direction. Therefore, the velocity is expected to be in the form $\bar{q} = (u(y, t), 0, 0)$ and it automatically satisfies the continuity equation (6).

The equation governing $u(y, t)$, is now seen to be

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \eta \frac{\partial^4 u}{\partial y^4}. \quad (8)$$

Introducing the non-dimensional variables

$$u^* = \frac{u}{U}, \quad y^* = \frac{y}{l}, \quad t^* = \frac{U}{l} t, \quad \text{where } l^2 = \frac{\eta}{\mu}, \quad R = \frac{\rho U l}{\mu}, \quad (9)$$

equation (8) reduces to,

$$R \frac{\partial u^*}{\partial t^*} = \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\partial^4 u^*}{\partial y^{*4}}.$$

Deleting the *'s, we get

$$R \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial y^4}. \quad (10)$$

It is imperative that we have to solve the above equation using the appropriate boundary conditions depending on whether we are dealing with Stokes' first problem or the second one.

3 Solution of the problem

3.1 Stokes' first problem

Initially, both fluid and plate are rest. At time $t = 0+$, the plate is suddenly set to move with constant velocity U . The non-dimensional conditions to be satisfied for this problem are

$$\begin{aligned} u(y, t) &\rightarrow 0 \quad \text{as } y \rightarrow \infty && \text{(regularity condition),} \\ u(0, t) &= 1 \quad \text{for all } t > 0 && \text{(usual no-slip condition),} \\ \frac{\partial^2 u}{\partial y^2} &= 0 \quad \text{at } y = 0 \quad \text{for any } t > 0 && \text{(vanishing of couple stresses} \\ &&& \text{on the boundary)} \end{aligned} \quad (11)$$

in addition to the initial condition

$$u(y, 0) \equiv 0 \quad \text{for all } y. \quad (12)$$

Taking Laplace transform to equations (10), (11) and using initial condition (12), we obtain

$$\frac{d^4 \bar{u}}{dy^4} - \frac{d^2 \bar{u}}{dy^2} + R s \bar{u} = 0 \quad (13)$$

with conditions,

$$\begin{aligned}\bar{u}(y, s) &\rightarrow 0 \quad \text{as } y \rightarrow \infty, \\ \bar{u}(0, s) &\equiv \frac{1}{s}, \\ \frac{d^2\bar{u}}{dy^2} &= 0 \quad \text{at } y = 0.\end{aligned}\tag{14}$$

The solution of equation (13) employing the boundary conditions is seen to be

$$\bar{u}(y, s) = \frac{1}{s(\beta^2 - \alpha^2)} [\beta^2 e^{-\alpha y} - \alpha^2 e^{-\beta y}],\tag{15}$$

where $\alpha^2 + \beta^2 = 1$, $\alpha^2\beta^2 = Rs$.

3.2 Stokes' second problem

Initially, both fluid and plate are rest. At time $t = 0+$, it is assumed that the plate suddenly oscillates with the velocity $U \cos(vt)$. Therefore, the non-dimensional conditions to be satisfied are

$$\begin{aligned}u(y, t) &\rightarrow 0 \quad \text{as } y \rightarrow \infty && \text{(regularity condition),} \\ u(0, t) &= \cos(vt) \quad \text{for all } t > 0 && \text{(usual no-slip condition),} \\ \frac{\partial^2 u}{\partial y^2} &= 0 \quad \text{at } y = 0 \text{ for any } t > 0 && \text{(vanishing of couple stresses} \\ &&& \text{on the boundary)}\end{aligned}\tag{16}$$

along with the initial condition

$$u(y, 0) \equiv 0 \quad \text{for all } y.\tag{17}$$

As in the case of Stokes' first problem, taking Laplace transform of equations (10), (16) and using initial condition (17), we get

$$\frac{d^4\bar{u}}{dy^4} - \frac{d^2\bar{u}}{dy^2} + Rs\bar{u} = 0\tag{18}$$

with conditions,

$$\begin{aligned}\bar{u}(y, s) &\rightarrow 0 \quad \text{as } y \rightarrow \infty, \\ \bar{u}(0, s) &\equiv \frac{s}{s^2 + v^2}, \\ \frac{d^2\bar{u}}{dy^2} &= 0 \quad \text{at } y = 0.\end{aligned}\tag{19}$$

The solution of equation (18) employing the boundary conditions (19) is seen to be

$$\bar{u}(y, s) = \frac{s}{(\beta^2 - \alpha^2)(s^2 + v^2)} [\beta^2 e^{-\alpha y} - \alpha^2 e^{-\beta y}],\tag{20}$$

where $\alpha^2 + \beta^2 = 1$, $\alpha^2\beta^2 = Rs$.

4 Numerical inversion of the Laplace transforms

In order to invert $\bar{u}(y, s)$, we adopt a numerical inversion technique due to Honig and Hirdes [32]. Using this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{e^{ct}}{t_1} \left[\frac{1}{2} \bar{f}(c) + \operatorname{Re} \left(\sum_{k=1}^N \bar{f} \left(c + \frac{ik\pi}{t_1} \right) \exp \left(\frac{ik\pi t}{t_1} \right) \right) \right], \quad 0 < t_1 < 2t,$$

where N is sufficiently large integer chosen such that,

$$e^{ct} \operatorname{Re} \left[\bar{f} \left(c + \frac{iN\pi}{t_1} \right) \exp \left(\frac{iN\pi t}{t_1} \right) \right] < \varepsilon,$$

where ε is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of c was obtained according to the criteria described in Honig and Hirdes [32].

5 Discussion of results

The velocity component $u(y, t)$ is numerically evaluated at different times t for various values of couple stress Reynolds number R in each case.

Stokes' first problem. Fig. 1 shows the variation of the velocity with distance at different times when R is fixed. As can be expected, the velocity decreases as distance from the plate y increases. It is observed that, at a fixed distance y , as time t increases, the velocity increases.

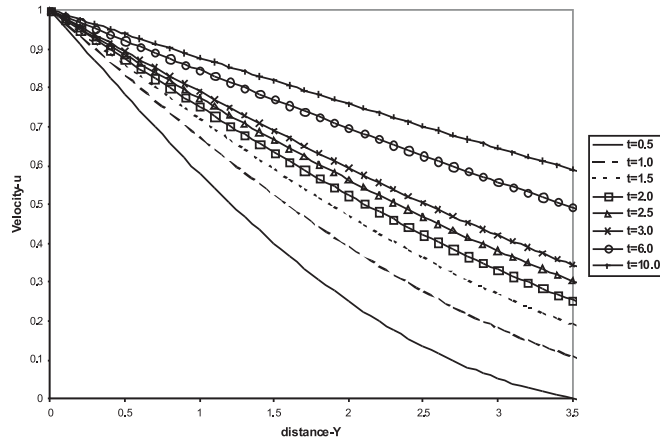


Fig. 1. Variation of velocity with distance at different times for $R = 0.5$.

From Fig. 2, it is found that, at any fixed time as R is increasing, the velocity is seen to be decreasing for a fixed distance.

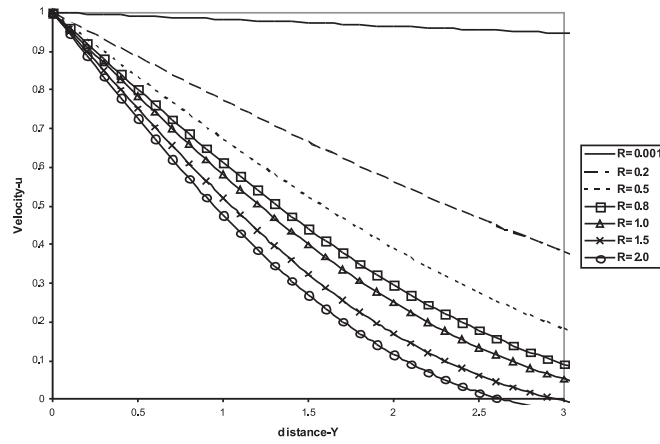


Fig. 2. Variation of velocity with distance for different values of R at $t = 1$.

Stokes' second problem. For fixed value of R the oscillatory character of the velocity is seen in the Fig. 3, as can be expected. In Fig. 4, the variation of velocity with distance is plotted for different values of couple stress Reynolds number R at a fixed time t . As R increases, it is seen that the velocity is decreasing.

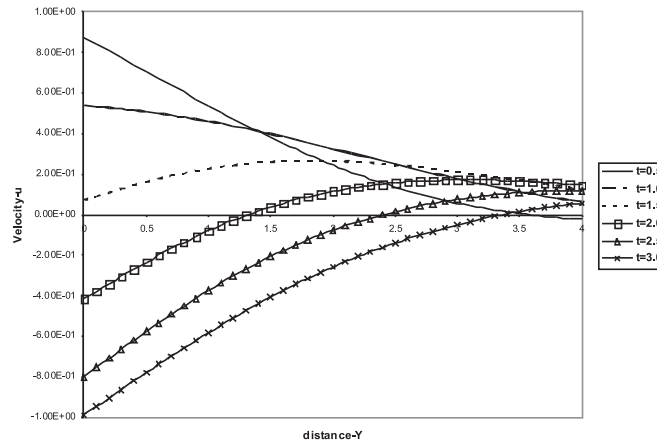


Fig. 3. Variation of velocity with distance at different times for $R = 0.5$.

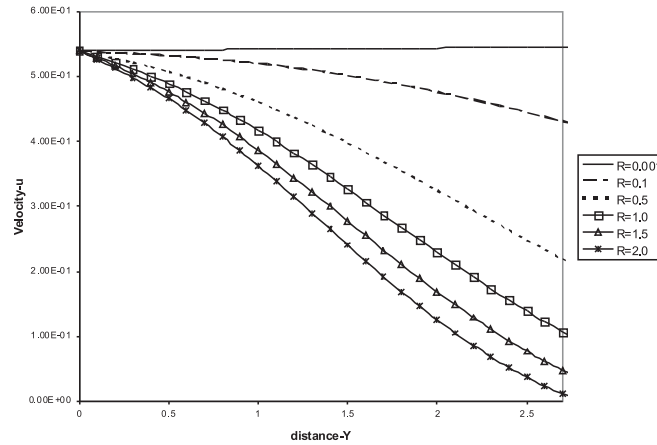


Fig. 4. Variation of velocity with distance for different R 's at $t = 1$.

6 Conclusions

Stokes' first and second problems for an incompressible couple stress fluid are studied using the condition that couple stresses vanish on the boundary. It is found that, at any fixed time t and for a fixed value of R , the velocity is decreasing as we move far away from the plate. As couple stress parameter l increases, the couple stress Reynolds number R increases which leads to a decrease in the velocity. Thus it is observed that an increase in the couple stress parameter has a decreasing influence on the velocity.

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