A Two-Dimentional Discrete Limit Theorem in the Space of Analytic Functions for Mellin Transforms of the Riemann Zeta-Function*

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Abstract. In the paper, a two-dimentional discrete limit theorem in the sense of weak convergence of probability measures in the space of analytic functions for Mellin transforms of the Riemann zeta-function on the critical line is obtained.

Keywords: limit theorem, Mellin transform, Riemann zeta-function, weak convergence.

1 Introduction

Let $s = \sigma + it$ be a complex variable, and let, as usual, $\zeta(s)$ denote the Riemann zeta-function. The modified Mellin transform $\mathcal{Z}_k(s)$ of $|\zeta(\frac{1}{2} + it)|^{2k}$, $k \geq 0$, is defined, for $\sigma \geq \sigma_0(k) > 1$, by

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s}\,\mathrm{d}x.$$

In [1] and [2], discrete limit theorems on the complex plane for the functions $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$ were proved. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and put, for $N \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$,

$$\mu_N(\ldots) = \frac{1}{N+1} \sum_{\substack{0 \leq m \leq N \\ \ldots}} 1,$$

where in place of dots a condition satisfied by m is to be written. Let $h > 0$ be a fixed number.

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Theorem 1 ([1]). *Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N\big(\mathcal{Z}_1(\sigma + imh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \to \infty$.

Theorem 2 ([2]). *Let $\sigma > \frac{5}{6}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N\big(\mathcal{Z}_2(\sigma + imh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \to \infty$.

In [3], a two-dimentional generalization of Theorems 1 and 2 was obtained.

Theorem 3 ([3]). *Suppose that $\sigma_1 > \frac{1}{2}$ and $\sigma_2 > \frac{5}{6}$. Then on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ there exists a probability measure P_{σ_1,σ_2} such that the probability measure*

$$\mu_N\big(\big(\mathcal{Z}_1(\sigma_1 + imh), \mathcal{Z}_2(\sigma_2 + imh)\big) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^2),$$

converges weakly to P_{σ_1,σ_2} as $N \to \infty$.

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $D_1 = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$ and $D_2 = \{s \in \mathbb{C}\colon \frac{5}{6} < \sigma < 1\}$. Then in [1] and [2] discrete limit theorems for $\mathcal{Z}_1(s)$ in $H(D_1)$ and for $\mathcal{Z}_2(s)$ in $H(D_2)$ were obtained, respectively.

Theorem 4 ([1]). *On $(H(D_1), \mathcal{B}(H(D_1)))$, there exists a probability measure P_1 such that the probability measure*

$$\mu_N\big(\mathcal{Z}_1(s + imh) \in A\big), \quad A \in \mathcal{B}\big(H(D_1)\big),$$

converges weakly to P_1 as $N \to \infty$.

Theorem 5 ([2]). *On $(H(D_2), \mathcal{B}(H(D_2)))$, there exists a probability measure P_2 such that the probability measure*

$$\mu_N\big(\mathcal{Z}_2(s + imh) \in A\big), \quad A \in \mathcal{B}\big(H(D_2)\big),$$

converges weakly to P_2 as $N \to \infty$.

The aim of this paper is to prove a two-dimentional discrete limit theorem in the space of analytic functions for the pair $(\mathcal{Z}_1(s), \mathcal{Z}_2(s))$.
Let $H = H(D_1, D_2) = H(D_1) \times H(D_2)$, and

$$P_N(A) = \mu_N\big(\big(\mathcal{Z}_1(s_1 + imh), \mathcal{Z}_2(s_2 + imh)\big) \in A\big), \quad A \in \mathcal{B}(H).$$

Theorem 6. *On $(H, \mathcal{B}(H))$, there exists a probability measure P such that the probability measure P_N converges weakly to P as $N \to \infty$.*

2 A limit theorem for the integrals over finite interval

Let $a > 1$ and $\sigma_0 > \frac{1}{2}$ be fixed numbers, for $x \geq y$, $y \geq 1$, define $v(x,y) = \exp\{-(\frac{x}{y})^{\sigma_0}\}$, and let

$$\mathcal{Z}_{k,a,y}(s) = \int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s} v(x,y)\,\mathrm{d}x, \quad k=1,2.$$

For brevity, we put

$$\underline{\mathcal{Z}}_{a,y}(s_1,s_2,\tau) = \big(\mathcal{Z}_{1,a,y}(s_1+i\tau), \mathcal{Z}_{2,a,y}(s_2+i\tau)\big),$$

and consider the probability measure

$$P_{N,a,y}(A) = \mu_N\big(\underline{\mathcal{Z}}_{a,y}(s_1,s_2,mh) \in A\big), \quad A \in \mathcal{B}(H).$$

Theorem 7. *On $(H,\mathcal{B}(H))$, there exists a probability measure $P_{a,y}$ such that the probability measure $P_{N,a,y}$ converges weakly to $P_{a,y}$ as $N \to \infty$.*

For the proof of Theorem 7, we will apply a limit theorem on the torus

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \{s \in \mathbb{C} : |s| = 1\} \stackrel{\text{def}}{=} \gamma$ for all $u \in [1,a]$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω_a is a compact topological Abelian group. On $(\Omega_a, \mathcal{B}(\Omega_a))$ define the probability measure

$$Q_{N,a}(A) = \mu_N\big(\big(u^{-imh}:\ u \in [1,a]\big) \in A\big).$$

Lemma 1. *On $(\Omega_a,\mathcal{B}(\Omega_a))$ there exists a probability measure Q_a such that the probability measure $Q_{N,a}$ converges weakly to Q_a as $N \to \infty$.*

Proof. The lemma is Theorem 5 from [2]. □

Proof of Theorem 7. Define a function $h_{a,y} : \Omega_a \to H$ by the formula

$$\begin{aligned} &h_{a,y}(\{y_x:\ x \in [1,a]\}) \\ &\quad = \left(\int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s_1} v(x,y)\widehat{y}_x\,\mathrm{d}x, \int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^4 x^{-s_2} v(x,y)\widehat{y}_x\,\mathrm{d}x\right), \end{aligned}$$

where

$$\widehat{y}_x = \begin{cases} y_x \text{ if } y_x \text{ is integrable over [1,}a\text{]}, \\ \text{an arbitrary circle integrable over } [1,a] \text{ function otherwise.} \end{cases}$$

The function $h_{a,y}$ is continuous, and

$$\begin{aligned} & h_{a,y}\big(\{x^{-imh}:\ x \in [1,a]\}\big) \\ & \quad = \left(\int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-(s_1+imh)} v(x,y)\,\mathrm{d}x, \int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^4 x^{-(s_2+imh)} v(x,y)\,\mathrm{d}x \right) \\ & \quad = \underline{\mathcal{Z}}_{a,y}(s_1,s_2,mh). \end{aligned}$$

Therefore, $P_{N,a,y}(A) = Q_{N,a}h_{a,y}^{-1}(A) = Q_{N,a}(h_{a,y}^{-1}A)$, and the theorem is a consequence of Theorem 5.1 from [4], Lemma 1 and continuity of $h_{a,y}$. ☐

3 A limit theorem for absolutely convergent integrals

In [1] and [2], it was observed that the integrals

$$\mathcal{Z}_{k,y}(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s} v(x,y)\,\mathrm{d}x, \quad k = 1,2,$$

are absolutely convergent for $\sigma > \frac{1}{2}$ and $\sigma > \frac{5}{6}$, respectively. We put

$$\underline{\mathcal{Z}}_y(s_1,s_2,\tau) = \big(\mathcal{Z}_{1,y}(s_1+i\tau), \mathcal{Z}_{2,y}(s_2+i\tau)\big)$$

and define the probability measure

$$P_{N,y}(A) = \mu_N\big(\underline{\mathcal{Z}}_y(s_1,s_2,mh) \in A\big), \quad A \in \mathcal{B}(H).$$

Theorem 8. *On $(H,\mathcal{B}(H))$, there exists a probability measure P_y such that the probability measure $P_{N,y}$ converges weakly to P_y as $N \to \infty$.*

Proof. By Theorem 7, the probability measure $P_{N,a,y}$ converges weakly to the measure $P_{a,y}$ as $N \to \infty$. First we will prove the tightness of the family of probability measures $\{P_{a,y}\}$ for fixed y.
On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, define a random variable θ_N by

$$\mathbb{P}(\theta_N = hm) = \frac{1}{N+1}, \quad m = 0,1,\dots,N,$$

and put

$$\begin{aligned} \underline{X}_{N,a,y}(s_1,s_2) &= \big(X_{N,a,y,1}(s_1), X_{N,a,y,2}(s_2)\big) \\ &= \big(\mathcal{Z}_{1,a,y}(s_1+i\theta_N), \mathcal{Z}_{2,a,y}(s_2+i\theta_N)\big). \end{aligned}$$

Then, by Theorem 7,

$$\underline{X}_{N,a,y}(s_1,s_2) \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_{a,y}(s_1,s_2), \tag{1}$$

where $\underline{X}_{a,y} = \underline{X}_{a,y}(s_1, s_2)$ is a H-valued random element with the distribution $P_{a,y}$, and $\xrightarrow[N\to\infty]{\mathcal{D}}$ denotes the convergence in distribution.
Now we introduce a metric in the space H which induces its topology. For $j = 1, 2$, let $\{K_{jl}\}$ be a sequence of compact subsets of the strip D_j such that

$$D_j = \bigcup_{l=1}^{\infty} K_{jl},$$

$K_{jl} \subset K_{j,l+1}, l \in \mathbb{N}$, and if $K_j \subset D_j$ is a compact subset, then $K_j \subseteq K_{jl}$ for some $l \in \mathbb{N}$. Then, for $j = 1, 2$,

$$\rho_j(g_{j1}, g_{j2}) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_{jl}} |g_{j1}(s) - g_{j2}(s)|}{1 + \sup_{s\in K_{jl}} |g_{j1}(s) - g_{j2}(s)|},$$

where $g_{j1}, g_{j2} \in H(D_j)$, is a metric on $H(D_j)$ which induces its topology of uniform convergence on compacta. Putting

$$\rho(\underline{g}_1, \underline{g}_2) = \max_{1\le j\le 2} \rho_j(g_{j1}, g_{j2}),$$

where $\underline{g}_1 = (g_{11}, g_{21})$, $\underline{g}_2 = (g_{12}, g_{22}) \in H$, we obtain the desired metric in H.
We take $M_{jl} > 0$, $l \in \mathbb{N}$, $j = 1, 2$. Then

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\Big(\sup_{s_j\in K_{jl}} |X_{N,a,y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2\Big) \\
&\quad\le \limsup_{N\to\infty} \sum_{j=1}^{2} \mathbb{P}\Big(\sup_{s_j\in K_{jl}} |X_{N,a,y,j}(s_j)| > M_{jl}\Big) \\
&\quad= \limsup_{N\to\infty} \sum_{j=1}^{2} \mu_N\Big(\sup_{s_j\in K_{jl}} |\mathcal{Z}_{j,a,y}(s_j + imh)| > M_{jl}\Big) \\
&\quad\le \limsup_{N\to\infty} \sum_{j=1}^{2} \frac{1}{M_{jl}} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j\in K_{jl}} |\mathcal{Z}_{j,a,y}(s_j + imh)|.
\end{aligned} \tag{2}$$

The integrals for $\mathcal{Z}_{j,y}(s)$ converge absolutely on D_j, thus uniformly on compact subsets of D_j, $j = 1, 2$. Hence,

$$\sup_{a\ge1} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j\in K_{jl}} |\mathcal{Z}_{j,a,y}(s_j + imh)| \le R_{jl} < \infty. \tag{3}$$

Now we choose $M_{jl} = R_{jl} 2^{l+1} \varepsilon^{-1}$, $j = 1, 2$, $l \in \mathbb{N}$, where ε is an arbitrary positive number. Then from (2) and (3) we deduce

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\Big(\sup_{s_j\in K_{jl}} |X_{N,a,y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2\Big) \\
&\quad\le \sum_{j=1}^{2} \frac{R_{jl}}{M_{jl}} = \frac{\varepsilon}{2^l}, \quad l \in \mathbb{N}.
\end{aligned}$$

This and (1) imply

$$\mathbb{P}\Big(\sup_{s_j\in K_{jl}}|X_{N,a,y,j}(s_j)|>M_{jl}\text{ for at least one } j=1,2\Big)\leq\frac{\varepsilon}{2^l},\quad l\in\mathbb{N}. \tag{4}$$

Now we set

$$K_\varepsilon=\Big\{(g_1,g_2)\in H\colon \sup_{s_j\in K_{jl}}|g_j(s_j)|\leq M_{jl},\quad j=1,2,\ l\in\mathbb{N}\Big\}.$$

Then the set K_ε is compact on H, and in view of (4)

$$\mathbb{P}(\underline{X}_{a,y}\in K_\varepsilon)\geq 1-\varepsilon,$$

or, by the definition of $\underline{X}_{a,y}$,

$$P_{a,y}(K_\varepsilon)\geq 1-\varepsilon$$

for all $a\geq 1$. Thus, we proved that the family $\{P_{a,y}\}$ is tight. Therefore, by the Prokhorov theorem, see, for example, [4], it is relatively compact. Hence, we have that there exists a sequence $\{P_{a_k,y}\}\subset\{P_{a,y}\}\}$ such that $P_{a_k,y}$ converges weakly to a certain probability measure P_y on $(H,\mathcal{B}(H))$ as $k\to\infty$. In other words,

$$\underline{X}_{a_k,y}\xrightarrow[k\to\infty]{\mathcal{D}}P_y. \tag{5}$$

The definition of $\mathcal{Z}_{j,a,y}(s)$, and the absolute convergence of the integral for $\mathcal{Z}_{j,y}(s)$ show that

$$\lim_{a\to\infty}\mathcal{Z}_{j,a,y}(s)=\mathcal{Z}_{j,y}(s)$$

uniformly on compact subsets of $D_j,\ j=1,2$. Therefore, putting

$$\begin{aligned}\underline{X}_{N,y}&=\underline{X}_{N,y}(s_1,s_2)=\big(X_{N,y,1}(s_1),X_{N,y,2}(s_2)\big)\\&=\big(\mathcal{Z}_{1,y}(s_1+i\theta_N),\mathcal{Z}_{2,y}(s_2+i\theta_N)\big),\end{aligned}$$

we find that, for every $\varepsilon>0$,

$$\begin{aligned}&\lim_{a\to\infty}\limsup_{N\to\infty}\mathbb{P}\big(\rho\big(\underline{X}_{N,y}(s_1,s_2),\underline{X}_{N,a,y}(s_1,s_2)\big)\geq\varepsilon\big)\\&\quad=\lim_{a\to\infty}\limsup_{N\to\infty}\mu_N\big(\rho\big(\underline{\mathcal{Z}}_y(s_1,s_2,mh),\underline{\mathcal{Z}}_{a,y}(s_1,s_2,mh)\big)\geq\varepsilon\big)\\&\quad\leq\lim_{a\to\infty}\limsup_{N\to\infty}\frac{1}{\varepsilon(N+1)}\sum_{m=0}^{N}\big(\rho\big(\underline{\mathcal{Z}}_y(s_1,s_2,mh),\underline{\mathcal{Z}}_{a,y}(s_1,s_2,mh)\big)\geq\varepsilon\big)=0.\end{aligned}$$

This, (1), (5) and Theorem 4.2 of [4] imply that

$$\underline{X}_{N,y}\xrightarrow[N\to\infty]{\mathcal{D}}P_y,$$

or that $P_{N,y}$ converges weakly to P_y as $N\to\infty$. The theorem is proved.

4 Approximation in the mean

Let, for brevity,

$$\underline{\mathcal{Z}}(s_1, s_2, \tau) = \big(\mathcal{Z}_1(s_1 + i\tau), \mathcal{Z}_2(s_2 + i\tau)\big).$$

In this section, we approximate $\underline{\mathcal{Z}}(s_1, s_2, mh)$ by $\underline{\mathcal{Z}}_y(s_1, s_2, mh)$ in the mean.

Theorem 9. *We have*

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \rho\big(\underline{\mathcal{Z}}(s_1, s_2, mh), \underline{\mathcal{Z}}_y(s_1, s_2, mh)\big) = 0.$$

Proof. In [1] it was obtained that

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \rho_1\big(\mathcal{Z}_1(s + imh), \mathcal{Z}_{1,y}(s + imh)\big) = 0. \tag{6}$$

Similarly, in [5] it was proved that

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \rho_2\big(\mathcal{Z}_2(s + imh), \mathcal{Z}_{2,y}(s + imh)\big) = 0. \tag{7}$$

Therefore, the assertion of the theorem follows from (6) and (7), and the metric ρ as defined in the proof of Theorem 8.

5 Proof of Theorem 6

We use a method similar to that of the proof of Theorem 9. By this theorem,

$$\underline{X}_{N,y}(s_1, s_2) \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_y(s_1, s_2), \tag{8}$$

where $\underline{X}_y = \underline{X}_y(s_1, s_2)$ is a H-valued random element with the distribution P_y.
Let $M_{jl} > 0$, $l \in \mathbb{N}$, $j = 1, 2$. Then

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\Big(\sup_{s_j\in K_{jl}} |X_{N,y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2\Big) \\
&\quad\le \limsup_{N\to\infty} \sum_{j=1}^{2} \mathbb{P}\Big(\sup_{s_j\in K_{jl}} |X_{N,y,j}(s_j)| > M_{jl}\Big) \\
&\quad= \limsup_{N\to\infty} \sum_{j=1}^{2} \mu_N\Big(\sup_{s_j\in K_{jl}} |\mathcal{Z}_{j,y}(s_j + imh)| > M_{jl}\Big) \\
&\quad\le \limsup_{N\to\infty} \sum_{j=1}^{2} \frac{1}{M_{jl}} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j\in K_{jl}} |\mathcal{Z}_{j,y}(s_j + imh)|.
\end{aligned} \tag{9}$$

We have that, for $j = 1, 2$,

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j \in K_{jl}} |\mathcal{Z}_{j,y}(s_j + imh)| \\ &\quad \le \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j \in K_{jl}} |\mathcal{Z}_{j,y}(s_j + imh) - \mathcal{Z}_j(s_j + imh)| \\ &\qquad + \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_j \in K_{jl}} |\mathcal{Z}_j(s_j + imh)|. \end{aligned} \tag{10}$$

Since by [1]

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_1 \in K_{1l}} |\mathcal{Z}_1(s_1 + imh)| \ll_{j,l} 1,$$

and by [5]

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s_2 \in K_{2l}} |\mathcal{Z}_2(s_2 + imh)| \ll_{j,l} 1,$$

(9), (10) and (6), (7) show that

$$\limsup_{N\to\infty} \mathbb{P}\Big(\sup_{s_j \in K_{jl}} |X_{N,y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2 \Big) \le \sum_{j=1}^{2} \frac{R_{jl}}{M_{jl}}.$$

Now taking $M_{jl} = R_{jl} 2^{l+1} \varepsilon^{-1}$, $\varepsilon > 0$, from this we have that

$$\limsup_{N\to\infty} \mathbb{P}\Big(\sup_{s_j \in K_{jl}} |X_{N,y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2 \Big) \le \frac{\varepsilon}{2^l}, \quad l \in \mathbb{N}.$$

This together with (8) shows that

$$\mathbb{P}\Big(\sup_{s_j \in K_{jl}} |X_{y,j}(s_j)| > M_{jl} \text{ for at least one } j = 1, 2 \Big) \le \frac{\varepsilon}{2^l}, \quad l \in \mathbb{N}.$$

Thus, preserving the notation of Section 3, we obtain that

$$P_y(K_\varepsilon) \ge 1 - \varepsilon$$

for all $y \ge 1$, that is, the family of probability measures $\{P_y \colon y \ge 1\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact, and there exists a sequence $\{P_{y_k}\} \subset \{P_y\}$ such that P_{y_k} converge to some probability measure P on $(H, \mathcal{B}(H))$ as $k \to \infty$, or

$$\underline{X}_{y_k}(s_1, s_2) \xrightarrow[k\to\infty]{\mathcal{D}} P.$$

Now this, (8), Theorem 9 and Theorem 4.2 of [4] prove Theorem 6. ◻

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