Vector Additive Decomposition for 2D Fractional Diffusion Equation

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Abstract. Such physical processes as the diffusion in the environments with fractal geometry and the particles' subdiffusion lead to the initial value problems for the non-local fractional order partial differential equations. These equations are the generalization of the classical integer order differential equations.

An analytical solution for fractional order differential equation with the constant coefficients is obtained in [1] by using Laplace-Fourier transform. However, nowadays many of the practical problems are described by the models with variable coefficients.

In this paper we discuss the numerical vector decomposition model which is based on a shifted version of usual Grünwald finite-difference approximation [2] for the non-local fractional order operators. We prove the unconditional stability of the method for the fractional diffusion equation with Dirichlet boundary conditions. Moreover, a numerical example using a finite difference algorithm for 2D fractional order partial differential equations is also presented and compared with the exact analytical solution.

Keywords: fractional order partial differential equations, vector decomposition methods, unconditional stability.

1 Introduction

In this paper we use the Riemann-Liouville fractional derivative

$$D_L^\alpha f(x) = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int\limits_L^x \frac{f(\xi)\,\mathrm{d}\xi}{(x-\xi)^{\alpha+1-n}}, \tag{1}$$

where n is an integer, and the αth order is in the following interval $n-1 < \alpha \le n$. Following [3] the case $L = 0$ of the formula (1) is called the Riemann form and the case $L = \infty$ is called Liouville form for the fractional derivatives. With the boundary conditions offered below the Riemann and Liouville forms become equivalent.

Grunwald-Letnikov formula for solving the one-dimensional diffusion equation is used in [2] by M.Meerschaert et al. According to the author the application of this formula leads to the unstable algorithm. This fact is the reason for the appearance of the important and interesting scientific results for the numerical methods theory to solve the fractional order differential equations [2]. Allowing for [2], in the present paper we use the right-shifted Grünwald approximation which is of the following form at $1 < \alpha \leq 2$

$$D^{\alpha}_{x^1} = \frac{1}{\Gamma(-\alpha)} \lim_{N_1 \to \infty} \sum_{k=0}^{N_1} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} u\big(x^1 - (k-1)h_1, x^2, t\big),$$

where N_1 is a non-negative integer, $\Gamma(p)$ is the gamma function and $h_1 = \frac{x^1 - x^1_H}{N_1}$.

2 Statement of the problem

On finite rectangular domain $\Omega = \{x^1_H < x^1 < x^1_K,\ x^2_H < x^2 < x^2_K\}$ we consider a two-dimensional fractional dispersion (diffusion) equation

$$\frac{\partial u}{\partial t} = c^1\big(x^1, x^2\big) D^{\alpha}_{x^1} u + c^2\big(x^1, x^2\big) D^{\beta}_{x^2} u + f\big(x^1, x^2, t\big), \tag{2}$$

where $u = u(x^1, x^2, t), \quad c^i = c^i(x^1, x^2),\ \ c^i > 0,\ \ 1 < \alpha \leq 2,\ \ 1 < \beta \leq 2.$

We assume that the differential equation (2) has a unique and sufficiently smooth solution under the following initial $u(x^1, x^2, 0) = \varphi(x^1, x^2)$ for all $x^i_H < x^i < x^i_k$ and Dirichlet boundary conditions $u(x^1, x^2, t) = Q(x^1, x^2, t)$ on the perimeter of the rectangular region Ω with the additional restriction $Q(x^1_H, x^2, t) = Q(x^1, x^2_H, t) = 0$ for $Q(x^1, x^2, t)$.

We replace the domain Ω by a discrete domain and define $t^n = n\tau,\ \ 0 \leq t^n \leq T$, $h_1 = \frac{x^1_K - x^1_H}{N_1},\ \ x^1_i = x^1_H + ih_1,\ \ i = 0, \ldots, N_1,\ \ h_2 = \frac{x^2_K - x^2_H}{N_2},\ \ x^2_j = x^2_H + jh_2$, $i = 0, \ldots, N_2$.

Function values in the discrete points are written in the following form $u(x^1, x^2, t^{n+1}) = \widehat{u}_{ij} = u^{n+1}_{ij}$. We assume that the solution function $u(x^1, x^2, t^n)$ is sufficiently smooth and vanishes on the left and lower boundary of the rectangular region. We define the finite difference operator using the right shifted Grünwald formula of these type

$$A^{\alpha}_1 {y_{ij}}^n = \frac{c^1_{ij}}{h^{\alpha}_1} \sum_{k=0}^{i+1} g_{\alpha,k} y^n_{i-k+1\ j}, \tag{3}$$

where

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \binom{\alpha}{k}. \tag{4}$$

These normalized weights (4) depend on the index k and the order α only and, for $A^{\beta}_2 y^n_{ij}$, analogously.

According to [4, Theorem 2.4] we can define

$$\frac{\partial^\alpha u_{ij}}{\partial x^{1\alpha}} - A_1^\alpha u_{ij} = O(h_1), \quad \frac{\partial^\beta u_{ij}}{\partial x^{2\beta}} - A_2^\beta u_{ij} = O(h_2). \tag{5}$$

Let a vector $y = (y_1, y_2)$ be a solution on the time interval $t^n < t < t^{n+1}$ with the operators (3). Granting this, we apply the modification version of the vector decomposition scheme [5–8]

$$\frac{y_{1,ij}^{n+1} - \widetilde{y}_{ij}}{\tau} = A_1^\alpha y_{1,ij}^{n+1} + A_2^\beta y_{2,ij}^n + f_{ij}^{n+0.5}, \quad y_{1,ij}^0 = \varphi_{ij} = \varphi\big(x_i^1, x_j^2\big), \tag{6}$$

$$\frac{y_{2,ij}^{n+1} - \widetilde{y}_{ij}}{\tau} = A_2^\beta y_{2,ij}^{n+1} + A_1^\alpha y_{1,ij}^n + f_{ij}^{n+0.5}, \quad y_{2,ij}^0 = \varphi_{ij}, \tag{7}$$

$$\widetilde{y}_{ij} = \frac{1}{2}\big(y_{1,ij}^n + y_{2,ij}^n\big), \quad i = \overline{1, N_1 - 1}, \ \ j = \overline{1, N_2 - 1}.$$

Subtracting (7) from (6), we obtain

$$y_{1,ij}^{n+1} - y_{2,ij}^{n+1} = \tau A_1^\alpha\big(y_1^{n+1} - y_1^n\big) + \tau A_2^\beta\big(y_2^n - y_2^{n+1}\big) \tag{8}$$

$$y_{1,ij} = y_{2,ij} + \psi_{ij}, \quad \psi_{ij} = O\big(\tau^2 + h_1 + h_2\big).$$

The right terms have the following form

$$\tau^2 A_1^\alpha y_{1,t} \sim O\big(\tau^2\big), \quad \tau^2 A_2^\beta y_{2,t} \sim O\big(\tau^2\big). \tag{9}$$

According to [9, 10] $(\sigma = 0.5)$, the vector decomposition method (6), (7) has the local truncation error of the form $O(\tau^2) + O(h_1) + O(h_2)$, at summing (6), (7) and using (9). It should be noted [4], that the fractional partial difference operators (3) are always $O(h)$ approximation to the αth fractional derivative.

Lets consider one of the parallel systems of the equations, for example (6). Using (3), we obtain the algebraic equations' system

$$y_{1,ij}^{n+1} - \tau A_1^\alpha y_{1,ij}^{n+1} = \widetilde{y}_{ij} + \tau A_2^\beta y_{2,ij}^n + f_{ij}^{n+0.5}, \tag{10}$$

$$y_{1,ij}^0 = \varphi_{ij}, \quad i = \overline{1, N_1 - 1}, \ \ j = \overline{1, N_2 - 1},$$

$$y_{1,0j}^{n+1} = Q\big(x_H^1, x_j^2, \widehat{t}\big) = 0, \quad y_{1,N_1 j}^{n+1} = Q\big(x_K^1, x_j^2, \widehat{t}\big), \quad \widehat{t} = t^{n+1}, \tag{11}$$

$$y_{2,i0}^{n+1} = Q\big(x_i^1, x_H^2, \widehat{t}\big) = 0, \quad y_{2,iN_2}^{n+1} = Q\big(x_i^1, x_H^2, \widehat{t}\big), \tag{12}$$

$$y_{2,ij}^0 = \varphi_{ij}, \quad i = \overline{0, N_1}, \ \ j = \overline{0, N_2}.$$

The boundary conditions provide a required approximation.

Writing (10) so that its realization is convenient

$$y_{1,ij}^{n+1} - \frac{\tau c_{ij}^1}{h_1^\alpha} \sum_{k=0}^{i+1} g_{\alpha k} y_{1,i-k+1,j}^{n+1} = \widetilde{y}_{ij} + \frac{\tau c_{ij}^2}{h_2^\beta} \sum_{m=0}^{j+1} g_{\beta m} y_{i,j-m+1}^n + f_{ij}^{n+0.5} \tag{13}$$

or

$$\left(1-\frac{\tau c_{ij}^{1}}{h_1^{\alpha}}g_{\alpha 1}\right)y_{1,ij}^{n+1}-\frac{\tau c_{ij}^{1}}{h_1^{\alpha}}g_{\alpha 0}y_{1,i+1,j}^{n+1}-\frac{\tau c_{ij}^{1}}{h_1^{\alpha}}g_{\alpha 2}y_{1,i-1,j}^{n+1}-\sum_{k=3}^{i+1}g_{\alpha k}y_{1,i-k+1,j}^{n+1}$$
$$=\widetilde{y}_{ij}+\frac{\tau c_{ij}^{2}}{h_2^{\beta}}\sum_{m=0}^{j+1}g_{\beta m}y_{2,i,j-m+1}^{n}+\tau f_{ij}^{n+0.5},\quad i=\overline{1,N_1-1},\ j=\overline{1,N_2-1}$$

with the conditions (11), (12).

On each time step algorithm (13), (11), (12) is realized by a forward sweep direction due to re-arrangement of gridpoints in the x^1-direction, thus it is economical. Analogously, in the x^2-direction.

Lets compose a matrix Q_k at the unknown $[y_{1j_0},\dots,y_{N_1-1j_0}]^T$, at each fixed value $j=j_0,\ \ (j=\overline{1,N_2-1})$, when the upper index is the line number and the lower index is the column number

$$\begin{pmatrix} Q_1^1 & Q_2^1 & 0 & 0 & \cdots & 0\\ Q_1^2 & Q_2^2 & Q_3^2 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ Q_1^{N_1-1} & Q_2^{N_1-1} & Q_3^{N_1-1} & Q_4^{N_1-1} & \cdots & Q_{N_1-1}^{N_1-1}\end{pmatrix}, \tag{14}$$

$$Q_l^i=\delta_l^i-\frac{\tau c_{il}^{1}}{h_1^{\alpha}}g_{\alpha,i-l+1},\quad l=1,\dots,N_1-1,\ \ i=1,\dots,N_1-1,$$

where δ_l^i is Croneker symbol.

In other words, the equations' system (14), (12), (13) can be presented as follows

$$\sum_{k=0}^{i+1}Q_k^i y_{1,kj_0}^{n+1}=\widetilde{y}_{ij_0}+\tau F_{j_0}^{i},\quad i=\overline{1,N_1-1},\ \ j=\overline{1,N_2}. \tag{15}$$

The F_k^i for each $x_{j_0}^2$ are defined from the last two expressions on the right-hand side of (13).

Theorem 1. *Each problem, defined by* (10)–(12), *is unconditionally stable for all* $1<\alpha,\beta\le 2$.

Proof. Let matrix $\widetilde{A_k}$ consist of the coefficients Q_l^i. For the theorem proof we will apply the useful results described below.

$$(1+z)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}z^k=\sum_{k=0}^{\infty}(-1)^k g_{\alpha k}z^k, \tag{16}$$

where the right side is the absolute convergent row at $|z|\le 1$. Placing $z=-1$ in (16), we get

$$\sum_{k=0}^{+\infty}g_{\alpha k}=0$$

and hence

$$\sum_{k=0}^{+\infty} g_{\alpha k} < 0$$

for all $N > 1$. According to Greschgorin theorem [11], we have

$$Q_i^i = 1 - C^1(i,k) g_{\alpha 1} = 1 + C^1(i,k)\alpha.$$

Moreover, the terms Q_i^i are the centers of circles with radiuses

$$r_i = \sum_{l=1, l \neq i}^{N_x - 1} \left| Q_l^i \right| \leq \sum_{l=1, l \neq i}^{i+1} C^1(i,k) g_{\alpha i - l + 1} < C^1(i,k)\alpha.$$

Thus $\operatorname{Re} \lambda(\widetilde{A_k}) > 1$ and the spectral radius of the inverse matrix is less than 1, $\rho(\widetilde{A_k^{-1}}) < 1$. We obtain the results for y_{2k}, analogously. Therefore, all problems,defined by (10)–(12), are unconditionally stable. □

Example 1. Consider fractional order partial differential equation

$$\frac{\partial u(x_1, x_2, t)}{\partial t} = c_1(x_1, x_2) \frac{\partial^{1.9} u}{\partial x_1^{1.9}} + c_2(x_1, x_2) \frac{\partial^{1.6} u}{\partial x_2^{1.6}} + f(x_1, x_2, t), \tag{17}$$

here $0 < x_1, x_2 < 1$ for $0 \leq t \leq T$ with the known exact solution

$$u = x_1^{2.9} x_2^{2.6} e^{-t}.$$

The diffusion coefficients are

$$c_1 = \frac{x_1^3 x_2^{1.4}}{\Gamma(3.9)}, \quad c_2 = \frac{x_1^{1.1} x_2^3}{\Gamma(3.6)}$$

and the forcing function is

$$f(x_1, x_2, t) = -\left(1 + 2x_1^{1.1} x_2^{1.4}\right) e^{-t} x_1^{2.9} x_2^{2.6}.$$

The algorithm was implemented using the Mathematica 5.1 compiler on a Dell Pentium PC.

Fig. 1 shows the exact solution of the equation (17) on the large grid. Fig. 2 shows the numerical solution obtained by the discussed above method (13), with $\tau = 0.1$, $h_1 = h_2 = h$. Fig. 3, 4 show the typical property of modified vector decomposition algorithms $\|y_1^{n+1} - y_2^{n+1}\| \to 0$ as $n \to \infty$. Numerical model (10)–(12) demonstrate an appropriate numerical abilities on large-size and current size grids, see Fig. 3, 4.

It should be noted that this example problem does not meet the requirement for the commutativity of the operators in (3) which was used to establish the stability of the numerical decomposition model (10)–(12).

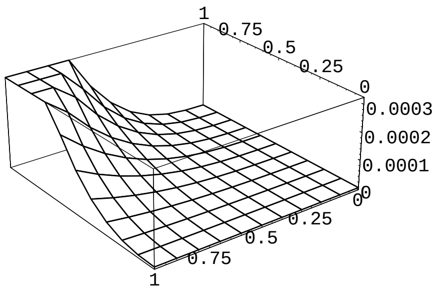

Fig. 1. $\tau = 0.1, h = 0.1$.

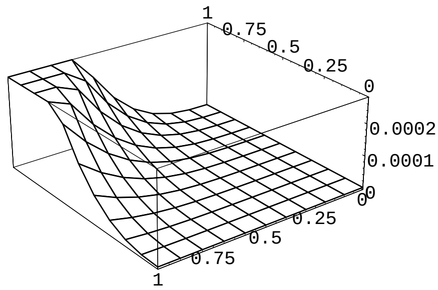

Fig. 2. $\tau = 0.1, h = 0.1$.

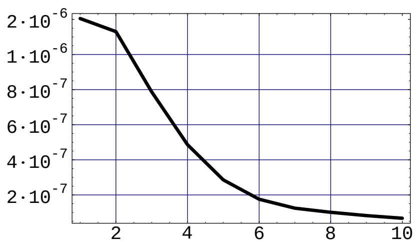

Fig. 3. $\tau = 0.1, h = 0.1$.

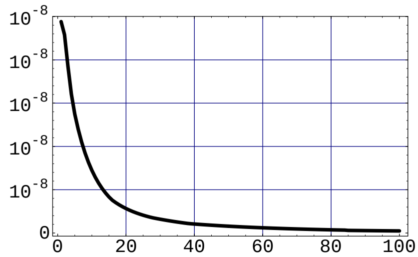

Fig. 4. $\tau = 0.01, h = 0.1$.

Indicate how one can check the exact solution. One simply substitutes back into (17) and uses the fact that

$$\frac{\partial^\alpha}{\partial x^\alpha}[x^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}$$

for the Riemann-Liouville fractional derivative (1) with $L = 0$.

3 Conclusions

The offered numerical vector decomposition model yields a numerical solution that is $O(\tau^2) + O(h_1) + O(h_2)$ accurate.

We emphasize that the vector additive methods do not meet the requirement for the commutativity of the decomposition operators A_1^α and A_2^β.

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