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Numerical Approximation of Some Infinite Gaussian Series and Integrals*

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Abstract. The paper deals with numerical computation of the asymptotic variance of the so-called increment ratio (IR) statistic and its modifications. The IR statistic is useful for estimation and hypothesis testing on fractional parameter $H \in (0,1)$ of random process (time series), see Surgailis et al. [1], Bardet and Surgailis [2]. The asymptotic variance of the IR statistic is given by an infinite integral (or infinite series) of 4-dimensional Gaussian integrals which depend on parameter H. Our method can be useful for numerical computation of other similar slowly convergent Gaussian integrals/series. Graphs and tables of approximate values of the variances $\sigma_p^2(H)$ and $\hat{\sigma}_p^2(H)$, $p = 1, 2$ are included.

Keywords: increment ratio statistic, Gaussian integrals, Gaussian process, numerical approximation.

1 Introduction

Let $X_1, \dots, X_n$ be n observations of the discrete time process $(X(t),\ t \in \mathbf{Z})$ and let

$$S_m(\tau) := \sum_{t=1}^{[m\tau]} X_t, \quad \tau \in [0,1],\ \ m \in \mathbf{N}$$

be the partial sums process. The increment ratio statistics are defined as

$$IR^{p,n} = \frac{1}{\frac{n+1}{m} - p - 2} \int_0^{\frac{n+1}{m}-p-2} \frac{|\Delta_\tau^p S_m(\tau) + \Delta_\tau^p S_m(\tau+1)|}{|\Delta_\tau^p S_m(\tau)| + |\Delta_\tau^p S_m(\tau+1)|}\, \mathrm{d}\tau, \quad p = 1, 2.$$

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The statistic $IR^{2,n}$ was introduced in Surgailis et al. [1], while $IR^{1,n}$ is its modification. The increment ratio statistics were applied to continuous time processes $(X(t),\ t \in \mathbf{R})$ instead of discrete time sequences in Bardet and Surgailis [2]:

$$R^{p,n} = \frac{1}{1-\frac{p}{n}} \int\limits_0^{1-\frac{p}{n}} \frac{\left|\Delta^p_{[n\tau]} X_n([n\tau]) + \Delta^p_{[n\tau]} X_n([n\tau]+1)\right|}{\left|\Delta^p_{[n\tau]} X_n([n\tau])\right| + \left|\Delta^p_{[n\tau]} X_n([n\tau]+1)\right|}\, \mathrm{d}\tau, \quad p = 1, 2,$$

where $X_n(\tau) := X(\tau/n)$. Results in [1] state that if X is stationary/stationary increment Gaussian sequence and satisfies some semi-parametric assumptions, then

$$\sqrt{\frac{n}{m}}\bigl(IR^{p,n} - \mathrm{E}IR^{p,n}\bigr) \Rightarrow \mathcal{N}\bigl(0, \sigma_p^2(H)\bigr), \qquad \begin{cases} p = 1, & 0 < H < 7/4, \\ p = 2, & 0 < H < 3/4, \end{cases}$$

where $\Rightarrow$ denotes weak convergence of random variables and $\mathcal{N}(0, \sigma_p^2(H))$ is Gaussian random variable with zero mean and variance

$$\sigma_p^2(H) := \int\limits_{-\infty}^{\infty} \mathrm{Cov}\bigl(\eta_0^{(p)}, \eta_\tau^{(p)}\bigr)\, \mathrm{d}\tau. \tag{1}$$

Here

$$\eta_\tau^{(p)} := \psi\bigl(Z_H^{(p)}(\tau), Z_H^{(p)}(\tau+1)\bigr), \quad \tau \in \mathbf{R} \tag{2}$$

is a stationary process,

$$\psi(x_1, x_2) = \frac{|x_1 + x_2|}{|x_1| + |x_2|}, \quad (x_1, x_2) \in \mathbf{R}^2 \tag{3}$$

is a homogeneous function taking values in the interval $[0, 1]$, and $Z_H^{(p)}(\tau)$, $\tau \in \mathbf{R}$ is a stationary Gaussian process with zero mean and autocovariance function

$$\rho_H(\tau) := \mathrm{Cov}\bigl(Z_H^{(1)}(0), Z_H^{(1)}(\tau)\bigr) = -\frac{1}{2}\Delta_s \Delta_t |t-s|^{2H}\Big|_{t-s=\tau}, \tag{4}$$

$$\mathrm{Cov}\bigl(Z_H^{(2)}(0), Z_H^{(2)}(\tau)\bigr) = \frac{1}{2(4^H - 4)} \Delta_s^2 \Delta_t^2 |t-s|^{2H}\Big|_{t-s=\tau},$$

where Δ is the difference operator, defined by

$$\Delta_x g(x,y) = g(x+1,y) - g(x,y), \quad \Delta_x^2 g(x,y) = \Delta_x\bigl(\Delta_x g(x,y)\bigr),$$
$$\Delta_y g(x,y) = g(x,y+1) - g(x,y), \quad \Delta_y^2 g(x,y) = \Delta_y\bigl(\Delta_y g(x,y)\bigr).$$

Bardet and Surgailis [2] derived results on asymptotic behavior of $R^{p,n}$, $p = 1, 2$ statistics. Corollary 4.3 in [2] says that if X continuous time Gaussian process with stationary increments and its variogram $V(t) = \mathrm{E}X^2(t)$ satisfy some additional assumptions, then

$$\sqrt{n}\bigl(R^{p,n} - \mathrm{E}R^{p,n}\bigr) \Rightarrow \mathcal{N}\bigl(0, \tilde{\sigma}_p^2(H)\bigr), \qquad \begin{cases} p = 1, & 0 < H < 3/4, \\ p = 2, & 0 < H < 1, \end{cases}$$

where

$$\tilde{\sigma}_p^2(H) := \sum_{\tau=-\infty}^{\infty} \mathrm{Cov}\big(\eta_0^{(p)}, \eta_\tau^{(p)}\big). \tag{5}$$

The $R^{p,n}$ ($IR^{p,n}$) statistics are useful for testing nonparametric hypotheses about the "fractional parameter" (the long-memory parameter $d = H - 1/2$) of observations (see $[1, 2]$). Precise values of the asymptotic variances in (1) are needed for construction of confidence intervals of the above mentioned tests.

The aim of this paper is to discuss numerical approximation of the infinite Gaussian integrals (1) and series (5). We provide graphs and tables of numerical values of functions $\sigma_p^2(H),\ \tilde{\sigma}_p^2(H),\ p = 1, 2$. For $p = 1$, the tabulated interval is $(0, 3/4)$, since for $H \geq 3/4$ these quantities become infinite. On the other hand, for $p = 2$ the tabulated interval is $(0, 7/4)$, due to the fact that for $H \geq 7/4$ (and $p = 2$) the series and the integral in (1) again diverge (see [1]). For the "discrete" quantities $\tilde{\sigma}_p^2(H), p = 1, 2$, we provide a theoretical justification of the approximation with the absolute error $\varepsilon > 0$. For the "continuous" quantities $\sigma_p^2(H), p = 1, 2$, the error of our approximation is only heuristically assessed, because of the lack of available methods for numerical integration of Gaussian integrals as in (1) on a finite interval $[-T, T] \subset (-\infty, \infty)$. Because of similarities between the cases $p = 1$ and $p = 2$, the subsequent discussion focuses on the approximation of $\sigma_1^2(H)$ and $\tilde{\sigma}_1^2(H)$ only.

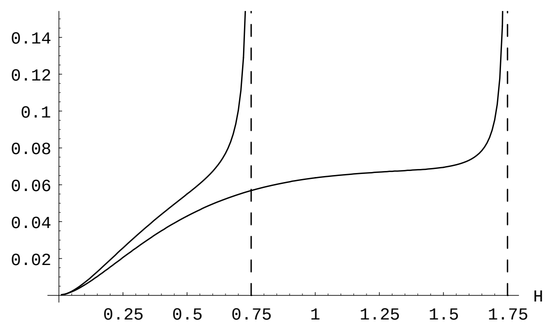

Fig. 1. The graphs of $\sigma_p^2(H),\ p = 1, 2$.

The above mentioned stationarity of the random process $\eta_\tau^{(1)}$, $\tau \in \mathbf{R}$ allows simplifying integration/summation:

$$\sigma_1^2(H) = 2\int_0^\infty \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big)\,\mathrm{d}\tau,$$

$$\tilde{\sigma}_1^2(H) := \mathrm{Cov}\big(\eta_0^{(1)}, \eta_0^{(1)}\big) + 2\sum_{\tau=1}^{\infty} \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big).$$

The hyperbolic decay of the autocovariance $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)})$ is characteristic for the processes in (2).

Proposition 1. *Let $0 < H < 3/4$, $H \neq 1/2$ and random process $\eta_\tau^{(1)}$, $\tau \in \mathbf{R}$ is defined in* (2)*. There exists $\tau_0 = \tau_0(H) < \infty$, such that*

$$\left|\mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big) - C(H)\tau^{4H-4}\right| \leq g(\tau, \tau_0), \quad \tau > \tau_0, \tag{6}$$

where the asymptotic constant $C(H)$ has the form

$$C(H) = \frac{(2H-2)^2(2H-1)^2(2H)^2(\ln 2)^2}{\pi^2(2-2^{2H-1})2^{2H}} \tag{7}$$

and the function $g(\tau, \tau_0)$ is defined in (32)*.*

In the case $H \neq 1/2$, the method of the approximation of $\sigma_1^2(H)$ and $\tilde{\sigma}_1^2(H)$ is based on (6). Write $\sigma_1^2(H) = 2(I^-(\tau_0) + I^+(\tau_0))$ and $\tilde{\sigma}_1^2(H) = 2(S^-(\tau_0) + S^+(\tau_0))$, where

$$I^-(\tau_0) = \int_0^{\tau_0} \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big)\,\mathrm{d}\tau, \tag{8}$$

$$I^+(\tau_0) = \int_{\tau_0}^{\infty} \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big)\,\mathrm{d}\tau,$$

$$S^-(\tau_0) = \frac{1}{2}\mathrm{Cov}\big(\eta_0^{(1)}, \eta_0^{(1)}\big) + \sum_{\tau=1}^{[\tau_0]} \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big), \tag{9}$$

$$S^+(\tau_0) = \sum_{\tau=[\tau_0]+1}^{\infty} \mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big). \tag{10}$$

Here and below $[x]$ denotes the integer part of a real x. Using (6), one has approximate values

$$\sigma_1^2(H) \approx 2\bar{I}^-(\tau_0) + 2C(H)\int_{\tau_0}^{+\infty} \tau^{4H-4}\,\mathrm{d}\tau, \tag{11}$$

$$\tilde{\sigma}_1^2(H) \approx 2\bar{S}^-(\tau_0) + 2C(H)\zeta(4-4H, [\tau_0]+1), \tag{12}$$

where $\zeta(\alpha, k_0) = \sum_{k=k_0}^{\infty} k^{-\alpha}$ is Hurwitz zeta function, $\bar{I}^-(\tau_0)$ and $\bar{S}^-(\tau_0)$ denote approximate values of $I^-(\tau_0)$ and $S^-(\tau_0)$ with respective truncation error ε_{I^-} and ε_{S^-}. The corresponding truncation errors in (11) and (12) do not exceed

$$\varepsilon_I = 2\varepsilon_{I^-} + 2\int_{\tau_0}^{+\infty} g_H(\tau_0, \tau)\,\mathrm{d}\tau,$$

$$\varepsilon_S = 2\varepsilon_{s^-} + 2C(H)\varepsilon_\zeta + 2\sum_{k=[\tau_0]+1}^{\infty} g_H(\tau_0, \tau)$$

where ε_ζ is a truncation error for $\zeta(b-aH,[\tau_0]+1)$. See eg. [3] for numerical evaluation of Hurwitz zeta function. If $H=1/2$, approximation of $\sigma_1^2(H)$ and $\tilde\sigma_1^2(H)$ is based on Lemma 3(i) (see below):

$$\sigma_1^2(1/2)\approx 2\bar I^-(2),\quad \tilde\sigma_1^2(1/2)\approx 2\bar S^-(1). \tag{13}$$

As the substantiation of the suggested method of numerical approximation of the functions $\sigma_1^2(H)$, $\tilde\sigma_1^2(H)$ serves noting that a simple truncation of the integral/series in (1) and (5) at $\tau=T(H,\varepsilon)$ such that

$$\left|\int_T^\infty \operatorname{Cov}\big(\eta_0^{(1)},\eta_\tau^{(1)}\big)\,\mathrm d\tau\right|\le\varepsilon,\quad \left|\sum_{\tau=T}^\infty \operatorname{Cov}\big(\eta_0^{(1)},\eta_\tau^{(1)}\big)\right|\le\varepsilon$$

works for small values of H only, since for large values of H (in particular, for $7/10<H<3/4$), the covariances in (6) decay very slowly and the required truncation point T becomes extremely large (the decay rate of the covariances can be accurately estimated from Arcones [4] inequality).

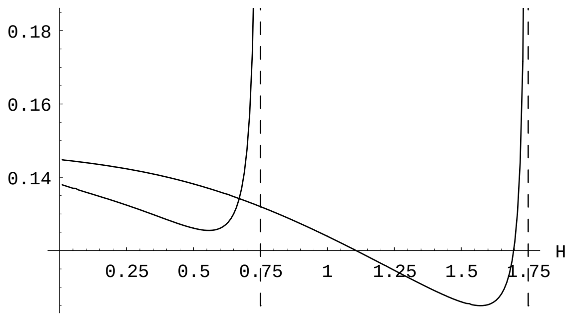

Fig. 2. The graphs of $\tilde\sigma_p^2(H),\ p=1,2$.

2 Numerical approximation of $S^-(\tau_0)$ and $I^-(\tau_0)$

In this Section, assuming that the splitting point τ_0 is given, we will obtain numerical approximations of $I^-(\tau_0)$ and $S^-(\tau_0)$ in (8) and (9).

In fact, to obtain numerical approximation of $S^-(\tau_0)$ with a given error estimation ε_{S^-}, it is enough to find numerical approximations $\bar f_H(\tau)$ of covariances $\operatorname{Cov}(\eta_0^{(1)},\eta_\tau^{(1)})$ evaluated at the nodes $\tau=0,1,\dots,[\tau_0]$ with truncation error $\varepsilon_{S^-}/([\tau_0]+1)$.

Let us shortly describe semi-heuristic method, used for numerical approximation of $I^-(\tau_0)$, see [5] for details. Recall that rectangle (midpoint) and trapezium formulas can be presented as

$$I^-(\tau_0) \approx P_m\big(\bar f_H\big) = h\sum_{i=0}^{m-1} \bar f_H(\tau)\big(ih + (h/2)\big),$$
$$I^-(\tau_0) \approx Q_m\big(\bar f_H\big) = \frac{h\big(\bar f_H(0) + \bar f_H(\tau_0)\big)}{2} + h\sum_{i=1}^{m-1} \bar f_H\big((i-1)h\big),$$

where $m \in \mathbf{N}$, $h = \tau_0/(m-1)$. The resulting approximation has the form

$$I^-(\tau_0) \approx Q_m\big(\bar f_H\big) + \theta\big\{P_m\big(\bar f_H\big) - Q_m\big(\bar f_H\big)\big\} \tag{14}$$

where m is enough big that inequality $|P_m(\bar f_H) - Q_m(\bar f_H)| < \varepsilon_{I^-}$ is satisfied. The most popular choice is $\theta = 0$.

Except the cases $\tau = 0$ and $\tau = 1$, the covariances $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)})$ has presentation of 4-tulpe integral. Approximate values of 2-tuple integral of (integrable) function over finite integration interval are easy to be obtained using *NIntegrate* function in Mathematica 5.2 (software for technical and scientific computing). *NIntegrate* uses adaptive Genz-Malik [6] method to get estimates of the countable integral (options *AccuracyGoal*, *PrecisionGoal* and *WorkingPrecision* allow controlling the absolute error). The rest of this section will discuss the simplification of $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)})$ for $\tau \in [0, \tau_0]$.

Lets consider the case $\tau = 0$. A random vector $(Z_H^{(1)}(0), Z_H^{(1)}(1))$ has a probability density function

$$p(x_1, x_2) = \frac{1}{2\pi(1-A^2)^{1/2}} \exp\left\{-\frac{1}{2}\big(q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{11}x_2^2\big)\right\},$$

where $q_{11} = q_{22} = (1-A^2)^{-1}$, $q_{12} = q_{21} = -A/(1-A^2)$ are elements of the inverse matrix of the covariance matrix of $(Z_H^{(1)}(0), Z_H^{(1)}(1))$ and $A := \rho_H(1) = 2^{2H-1} - 1$, see (4). Put $a_{0,0} := \mathrm{E}(\eta_0^{(1)})$. In [1] it is proved that

$$a_{0,0} = \frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{1+A}{1-A}}\right) + \frac{1}{\pi}\sqrt{\frac{1+A}{1-A}}\,\ln\left(\frac{2}{1+A}\right). \tag{15}$$

Lemma 1. *Let* $0 < H < 3/4$. *Then* $\mathrm{Cov}(\eta_0^{(1)}, \eta_0^{(1)}) = 1 + 8J_1(H) - a_{0,0}^2$, *where*

$$J_1(H) = \frac{1}{4\pi(1-A)}\left(\sqrt{1-A^2} - \frac{\pi}{2} + \tan^{-1}\left(\frac{A}{\sqrt{1-A^2}}\right)\right).$$

Proof. By covariance definition, $\mathrm{Cov}(\eta_0^{(1)}, \eta_0^{(1)}) = \mathrm{E}(\eta_0^{(1)})^2 - (\mathrm{E}\eta_0^{(1)})^2$. In view of (15), it is enough to find $\mathrm{E}(\eta_0^{(1)})^2$. Using the identity

$$(x+y)^2/(x-y)^2 = 1 + 4xy/(x-y)^2,$$

where $x - y \neq 0$, we have

$$\begin{aligned}
\mathrm{E}(\eta_0^{(1)})^2 &= \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \left(\frac{|x+y|}{|x|+|y|}\right)^2 p(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} p(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 8\int\limits_{0}^{+\infty}\int\limits_{-\infty}^{0} \frac{xy}{(x-y)^2} p(x,y)\,\mathrm{d}x\,\mathrm{d}y =: 1 + 8J_1(H).
\end{aligned}$$

After substituting $x = r\cos\phi$, $y = r\sin\phi$ we get

$$\begin{aligned}
J_1(H) = \frac{1}{2\pi\sqrt{1-A}} &\int\limits_{3\pi/2}^{2\pi} \frac{\cos\phi\sin\phi}{(\cos\phi - \sin\phi)^2}\,\mathrm{d}\phi \\
&\times \int_0^\infty r\exp\left\{-\frac{1-2A\cos\phi\sin\phi}{2(1-A)}\,r^2\right\}\mathrm{d}r.
\end{aligned}$$

Integrate with respect to r first, giving $(1-A)/(1-2A\cos\phi\sin\phi)$. In order to obtain the value of the integral with respect to ϕ use the substitution $\tan\phi = v$. □

We now turn to the case $\tau = 1$. Introduce the functions

$$\begin{aligned}
f_1(\phi,\theta) &:= \frac{|\sin\phi\sin\theta + \cos\phi|}{\cos\phi - \sin\phi\sin\theta}, \\
f_2(\phi,\theta) &:= a_{11}(\sin\phi\cos\theta)^2 + a_{22}(\sin\phi\sin\theta)^2 + a_{11}(\cos\phi)^2 \\
&\quad + 2a_{12}\sin\phi\cos\theta\sin\phi\sin\theta + 2a_{13}\sin\phi\cos\theta\cos\phi \\
&\quad + 2a_{12}\sin\phi\sin\theta\cos\phi,
\end{aligned}$$

where $a_{ij}, i,j = 1,2,3$ are elements of the inverse matrix of the covariance matrix

$$\begin{pmatrix}
\rho_H(0) & \rho_H(1) & \rho_H(2) \\
\rho_H(1) & \rho_H(0) & \rho_H(1) \\
\rho_H(2) & \rho_H(1) & \rho_H(0)
\end{pmatrix} \tag{16}$$

of the random vector $(Z_H^{(1)}(0), Z_H^{(1)}(1), Z_H^{(1)}(2))$.

Lemma 2. *Let* $0 < H < 3/4$. *Then* $\mathrm{Cov}(\eta_0\eta_1) = 2J_2(H) + 4J_3(H) + 2J_4(H) - a_{0,0}^2$, *where*

$$J_2(H) = \frac{1}{4\pi\sqrt{D_3(H)}} \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \frac{1}{(f_2(\phi,\theta))^{3/2}} \sin\phi \ \mathrm{d}\phi,$$

$$J_3(H) = -\frac{1}{4\pi\sqrt{D_3(H)}} \int_0^{\pi/2} \mathrm{d}\theta \int_{\pi/2}^{\pi} \frac{f_1(\phi,\theta)}{(f_2(\phi,\theta))^{3/2}} \sin\phi \ \mathrm{d}\phi,$$

$$J_4(H) = \frac{1}{4\pi\sqrt{D_3(H)}} \int_{3\pi/2}^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \frac{|\cos\theta+\sin\theta|}{\cos\theta-\sin\theta} \frac{f_1(\phi,\theta)}{(f_2(\phi,\theta))^{3/2}} \sin\phi \ \mathrm{d}\phi,$$

and $D_3(H)$ *denotes the determinant of the matrix in* (16)*.*

Proof is similar to the following lemma and, thus, it is skipped.

Now we assume that $\tau > 0$ and $\tau \neq 1$. As above, we denote by $(a_{ij}(\tau), i,j = 1,2,3,4)$ the inverse matrix of the covariance matrix

$$\begin{pmatrix} \rho_H(0) & \rho_H(1) & \rho_H(\tau) & \rho_H(\tau+1) \\ \rho_H(1) & \rho_H(0) & \rho_H(\tau-1) & \rho_H(\tau) \\ \rho_H(\tau) & \rho_H(\tau-1) & \rho_H(0) & \rho_H(1) \\ \rho_H(\tau+1) & \rho_H(\tau) & \rho_H(1) & \rho_H(0) \end{pmatrix} \tag{17}$$

of the random vector

$$\big(Z_H^{(1)}(0),\ Z_H^{(1)}(1),\ Z_H^{(1)}(\tau),\ Z_H^{(1)}(\tau+1)\big). \tag{18}$$

Let $\phi \in [0,2\pi)$, $\theta \in [0,2\pi)$. Introduce the functions

$$\begin{aligned} c_{11}(\tau,\phi) &= a_{11}(\tau)\cos^2\phi + a_{22}(\tau)\sin^2\phi + 2a_{12}(\tau)\cos\phi\sin\phi, \\ c_{22}(\tau,\theta) &= a_{22}(\tau)\cos^2\theta + a_{11}(\tau)\sin^2\theta + 2a_{12}(\tau)\cos\theta\sin\theta, \\ c_{12}(\tau,\phi,\theta) &= a_{13}(\tau)\cos\phi\cos\theta + a_{13}(\tau)\sin\phi\sin\theta \\ &\quad + a_{14}(\tau)\cos\phi\sin\theta + a_{23}(\tau)\sin\phi\cos\theta, \\ c(\tau,\phi,\theta) &= c_{11}(\tau,\phi)c_{22}(\tau,\theta) - c_{12}^2(\tau,\phi,\theta), \\ f_3(\tau,\phi,\theta) &= \frac{1}{c(\tau,\phi,\theta)} + \frac{c_{12}(\tau,\phi,\theta)}{(c(\tau,\phi,\theta))^{3/2}} \arctan\Big(\frac{c_{12}(\tau,\phi,\theta)}{(c(\tau,\phi,\theta))^{1/2}}\Big), \\ f_4(\theta) &= |\cos\theta+\sin\theta|/(\cos\theta-\sin\theta). \end{aligned}$$

Lemma 3. *Let* $0 < H < 3/4$ *and* $\tau > 0$, $\tau \neq 1$.

(i) *If* $H = 1/2$ *and* $\tau \geq 2$, *then* $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)}) = 0$;

(ii) *If at least one of the assumptions in (i) is not satisfied, then*

$$\mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big) = 2J_5(H) + 4J_6(H) + 2J_7(H) - a_{0,0}^2,$$

where

$$J_5(H) := \frac{1}{(2\pi)^2\sqrt{D_4(\tau)}} \int_0^{\pi/2} \mathrm{d}\phi \int_0^{\pi/2} f_3(\tau,\phi,\theta)\,\mathrm{d}\theta,$$

$$J_6(H) := -\frac{1}{(2\pi)^2\sqrt{D_4(\tau)}} \int_0^{\pi/2} \mathrm{d}\phi \int_{\pi/2}^{\pi} f_3(\tau,\phi,\theta) f_4(\theta)\,\mathrm{d}\theta,$$

$$J_7(H) := -\frac{1}{(2\pi)^2\sqrt{D_4(\tau)}} \int_{3\pi/2}^{2\pi} f_4(\phi)\,\mathrm{d}\phi \int_{\pi/2}^{\pi} f_3(\tau,\phi,\theta) f_4(\theta)\,\mathrm{d}\theta,$$

and $D_4(H)$ denotes determinant of the matric in (17)*.*

Proof. (i) Note that $\rho_{1/2}(\tau) = 0$ for $\tau \geq 1$ (see (4)). Thus

$$\begin{aligned}
&\mathrm{E}\big(v_1 Z_{1/2}^{(1)}(0) + v_2 Z_{1/2}^{(1)}(1)\big)\big(v_3 Z_{1/2}^{(1)}(\tau) + v_4 Z_{1/2}^{(1)}(\tau+1)\big) \\
&\qquad = (v_1v_3 + v_2v_4)\rho_{1/2}(\tau) + v_1v_4\rho_{1/2}(\tau+1) + v_2v_3\rho_{1/2}(\tau-1) = 0
\end{aligned}$$

for any $(v_1, v_2, v_3, v_4) \in \mathbf{R}^4$ and $\tau \geq 2$. The last inequality means that Gaussian random vectors $(Z_{1/2}^{(1)}(0), Z_{1/2}^{(1)}(1))$ and $(Z_{1/2}^{(1)}(\tau),\ Z_{1/2}^{(1)}(\tau+1))$ are independent for $\tau \geq 2$. To obtain independence of $\eta_0^{(1)}$ and $\eta_\tau^{(1)}$ $(\tau \geq 2)$, it rests to use (2) with $p = 1$.

(ii) We have $\mathrm{E}(\eta_0^{(1)}\eta_\tau^{(1)}) = E_1 + E_2 + E_3$, where

$$\begin{aligned}
E_1 &= \mathrm{E}\big(\eta_0^{(1)}\eta_\tau^{(1)} I\big\{Z_H^{(1)}(0)Z_H^{(1)}(1) > 0,\ Z_H^{(1)}(\tau)Z_H^{(1)}(\tau+1) > 0\big\}\big),\\
E_2 &= \mathrm{E}\big(\eta_0^{(1)}\eta_\tau^{(1)} I\big\{Z_H^{(1)}(0)Z_H^{(1)}(1)Z_H^{(1)}(\tau)Z_H^{(1)}(\tau+1) < 0\big\}\big),\\
E_3 &= \mathrm{E}\big(\eta_0^{(1)}\eta_\tau^{(1)} I\big\{Z_H^{(1)}(0)Z_H^{(1)}(1) < 0,\ Z_H^{(1)}(\tau)Z_H^{(1)}(\tau+1) < 0\big\}\big),
\end{aligned}$$

and $I\{\cdot\}$ denotes the indicator function. The random vector in (18) has the probability density function

$$p_\tau(x_1, x_2, x_3, x_4) = \frac{1}{(2\pi)^2\sqrt{D_4(\tau)}} \exp\Big\{-\frac{1}{2}\sum_{i,j=1}^{4} a_{ij}(\tau)x_ix_j\Big\}.$$

The identity

$$p_\tau(x_1, x_2, x_3, x_4) = p_\tau(-x_1, -x_2, -x_3, -x_4) \tag{19}$$

follows from noting that the matrix $(a_{ij}(\tau), i, j = 1, 2, 3, 4)$ is symmetric with respect to both its diagonals. In view of $\psi(x, y)I\{xy > 0\} = 1$ (see (3)) and (19), the quantity

$(1/2)E_1$ equals to

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \big(p_\tau(x_1,x_2,x_3,x_4)+p_\tau(x_1,x_2,-x_3,-x_4)\big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4.$$

Using substitutions $x_1 = y_1\cos\phi$, $x_2 = y_1\sin\phi$, $x_3 = y_2\cos\theta$, $x_4 = y_2\sin\theta$ we get $E_1 = 2\int_0^{\pi/2}\int_0^{\pi/2} K(\tau,\phi,\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta$, where

$$\begin{aligned}K(\tau,\phi,\theta) = \int_0^\infty\int_0^\infty &\big(p_\tau(y_1\cos\phi, y_1\sin\phi, y_2\cos\theta, y_2\sin\theta)\\ &+ p_\tau(y_1\cos\phi, y_1\sin\phi, -y_2\cos\theta, -y_2\sin\theta)\big)y_1y_2\,\mathrm{d}y_1\,\mathrm{d}y_2.\end{aligned}$$

Use direct integration to prove $K(\tau,\phi,\theta) = f_3(\tau,\phi,\theta)$. The quantities E_2 and E_3 can be similarly considered. □

3 Asymptotic behavior of covariance $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)})$

In this Section, we prove Proposition 1 and discuss the numerical approximation strategy of the functions $\sigma_1^2(H)$ and $\tilde{\sigma}_1^2(H)$ for a given $0 < H < 3/4$ and a specified tolerance $\varepsilon > 0$.

We start with the definition of the (univariate) Hermite polynomials (see also [7, 8]) $H_n(x),\ n = 0, 1, 2, \ldots$ They can be defined by reccurence relations

$$\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} = nH_{n-1}(x), \quad n = 1, 2, \ldots$$

with $H_0(x) = 1$ and $\int_{-\infty}^{+\infty} H_n(x)\mathrm{e}^{-x^2/2}\mathrm{d}x = 0$ providing the constants of integration.

Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent random variables. If $E[\Psi^2(X,Y)] < \infty$, then $f(X,Y)$ can be expanded as

$$\Psi(X,Y) = \sum_{i,j=0}^{\infty} \frac{a_{i,j}}{i!j!} H_i(X)H_j(Y), \tag{20}$$

where $a_{i,j} = E[\Psi(X,Y)H_i(X)H_j(Y)]$, see eg. [1]. The right-hand side of (20) converges in square mean.

The Hermite rank of Ψ with respect to Gaussian vector (X,Y) is defined by

$$\mathrm{rank}(f) = \inf\{r\colon\ \exists\ i \ \text{ and }\ j\ \text{ with }\ i+j=r\ \text{ and }\ a_{i,j}\neq 0\},$$

where the infimum of the empty set is infinity, see [4].

Suppose $|A| < 1$. Let us split the function

$$\Psi(x,y) := \psi\big(x, Ax + \sqrt{1-A^2}y\big), \quad (x,y)\in\mathbb{R}^2 \tag{21}$$

into $\Psi(x,y) = \Psi_1(x,y) + \Psi_2(x,y)$, where

$$\Psi_1(x,y) := \sum_{0\le i+j\le 2} \frac{a_{i,j}}{i!j!} H_i(x)H_j(y), \tag{22}$$

$$\Psi_2(x,y) := \Psi(x,y) - \Psi_1(x,y). \tag{23}$$

Lemma 4. (i) *The function $\Psi_1(x,y)$ has the form*

$$\Psi_1(x,y) = a_{0,0} + \frac{a_{0,2}}{2}H_2(y) + a_{1,1}H_1(x)H_1(y) + \frac{a_{2,0}}{2}H_2(x),$$

where $a_{0,0}$ is given in (15), $a_{2,0} = -a_{0,2} = q_A A, \quad a_{1,1} = q_A\sqrt{1-A^2}$ *and*

$$q_A := \pi^{-1}\sqrt{(1+A)/(1-A)}\cdot \ln(2/(1+A)), \quad A := 2^{2H-1} - 1.$$

(ii) *The Hermite rank of the function $\Psi_2(x,y)$ in* (23) *with respect to a standard Gaussian vector (X,Y) is not less than 4.*

Proof. (i) Let n and m denote any non-negative integers. If $m+n$ is odd then the function

$$u_{n,m}(x,y) := \frac{1}{\sqrt{2\pi}} x^n y^m \Psi(x,y) \exp\left\{\left(x^2+y^2\right)/2\right\}$$

satisfies identity $u_{n,m}(x,y) = -u_{n,m}(-x,-y)$ for any $(x,y) \in \mathbf{R}^2$. Whence follows $a_{0,1} = \int_{\mathbf{R}^2} u_{0,1}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$ and $a_{1,0} = \int_{\mathbf{R}^2} u_{1,0}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$. Use direct integration to find $a_{0,2}$ and $a_{2,0}$.

(ii) It suffices to show that $a_{0,3} = a_{1,2} = a_{2,1} = a_{3,0} = 0$. With $H_1(x) = x$, $H_2(x) = x^2 - 1$ and $H_3(x) = x^3 - 3x$ in mind, we have

$$a_{0,3} = \int_{\mathbf{R}^2} \left(u_{0,3}(x,y) - 3u_{0,1}(x,y)\right)\mathrm{d}x\,\mathrm{d}y = 0,$$

$$a_{1,2} = \int_{\mathbf{R}^2} \left(u_{1,2}(x,y) - u_{1,0}(x,y)\right)\mathrm{d}x\,\mathrm{d}y = 0.$$

The computations of $a_{2,1}$ and $a_{3,0}$ are similar. □

Lemma 5. *Let $X_i \sim N(0,1),\ i = 1,2,3,4$. Assume that X_{2k-1} and X_{2k} are independent random variables for $k = 1,2$. Then*

$$\begin{aligned}\mathrm{Cov}\big(\eta_0^{(1)}, \eta_\tau^{(1)}\big) &= \mathrm{Cov}\big(\Psi_1(X_1,X_2), \Psi_1(X_3,X_4)\big)\\ &\quad + \mathrm{Cov}\big(\Psi_2(X_1,X_2), \Psi_2(X_3,X_4)\big),\end{aligned} \tag{24}$$

where $\eta_\tau^{(1)}$ is defined in (2) *and the functions $\Psi_k,\ k = 1,2$ are defined in* (22) *and* (23).

Proof. Let us prove firstly

$$\mathrm{Cov}\big(\eta_0^{(1)},\eta_\tau^{(1)}\big)=\mathrm{Cov}\big(\Psi(X_1,X_2),\Psi(X_3,X_4)\big), \tag{25}$$

where the function Ψ is defined in (21). Using the fact that probability distribution of the Gaussian random vector is determined by its mean vector and covariance matrix, one can easily prove

$$\begin{aligned}\big(Z_H^{(1)}(0),Z_H^{(1)}(1)\big)&\stackrel{d}{=}\big(X_1,AX_1+\sqrt{1-A^2}X_2\big),\\ \big(Z_H^{(1)}(\tau),Z_H^{(1)}(\tau+1)\big)&\stackrel{d}{=}\big(X_3,AX_3+\sqrt{1-A^2}X_4\big),\end{aligned}$$

where $\stackrel{d}{=}$ denotes equality of the probability distributions of the random elements. Then $\eta_0^{(1)}\stackrel{d}{=}\Psi(X_1,X_2)$ and $\eta_\tau^{(1)}\stackrel{d}{=}\Psi(X_3,X_4)$, thereby proving the claim (25).

We have

$$\begin{aligned}\mathrm{Cov}\big(\Psi(X_1,X_2),\Psi(X_3,X_4)\big)&=\mathrm{Cov}\big(\Psi_1(X_1,X_2),\Psi_1(X_3,X_4)\big)\\&\quad+\mathrm{Cov}\big(\Psi_1(X_1,X_2),\Psi_2(X_3,X_4)\big)\\&\quad+\mathrm{Cov}\big(\Psi_2(X_1,X_2),\Psi_1(X_3,X_4)\big)\\&\quad+\mathrm{Cov}\big(\Psi_2(X_1,X_2),\Psi_2(X_3,X_4)\big).\end{aligned}$$

Lets prove $\mathrm{Cov}(\Psi_1(X_1,X_2),\Psi_2(X_3,X_4))=0$. By the Lemma 4,

$$\begin{aligned}&\mathrm{Cov}\big(\Psi_1(X_1,X_2),\Psi_2(X_3,X_4)\big)\\&\quad=\sum_{(i_1,i_2)}{}'\frac{a_{i_1,i_2}}{i_1!i_2!}\sum_{(i_3,i_4)}{}''\frac{a_{i_3,i_4}}{i_3!i_4!}\mathrm{Cov}\big(H_{i_1}(X_1)H_{i_2}(X_2),H_{i_3}(X_3)H_{i_4}(X_4)\big),\end{aligned}$$

where the sum $\sum'$ is taken over pairs $(i_1,i_2)=(0,2),(1,1),(2,0)$ and the sum $\sum''$ is taken over pairs of non-negative integers (i_3,i_4), such that sum $i_3+i_4\geq 4$ is even (similarly as in the Lemma 4 proof one can check that if $i+j$ is odd, then $a_{i,j}=0$ in the Hermite expansion of the function Ψ (21)).

The covariance $\mathrm{Cov}(H_{i_1}(X_1)H_{i_2}(X_2),H_{i_3}(X_3)H_{i_4}(X_4))$ is equal to expectation $\mathrm{E}(H_{i_1}(X_1)H_{i_2}(X_2)H_{i_3}(X_3)H_{i_4}(X_4))$. Use independence of random variables X_{2k-1} and X_{2k} ($k=1,2$) to prove it. To prove

$$\mathrm{E}(H_{i_1}(X_1)H_{i_2}(X_2)H_{i_3}(X_3)H_{i_4}(X_4))=0$$

we use the diagram formula for expectations of products of Hermite polynomials over Gaussian random vector (see eg. [4, 9]). A diagram G of order (i_1,i_2,i_3,i_4) is a set of points

$$\begin{aligned}&(1,1),\ (1,2),\dots,\ (1,i_1)\\&(2,1),\ (2,2),\dots,\ (2,i_2)\\&(3,1),\ (3,2),\dots,\ (3,i_3)\\&(4,1),\ (4,2),\dots,\ (4,i_4)\end{aligned}$$

called vertices and a set of pairs of these points called edges, such that each vertex is met by at most one edge and such that if vertices (j_1, k_1) and (j_2, k_2) are joined by an edge, it follows that $j_1 \neq j_2$. Let $\Gamma(i_1, i_2, i_3, i_4)$ denote the set of all diagrams of order (i_1, i_2, i_3, i_4) having $(i_1 + i_2 + i_3 + i_4)/2$ edges. In Fox and Taqqu [10] such diagrams are called complete. If in a given diagram rows i and j are joined by v_{ij} edges, then the diagram formula is:

$$\begin{aligned}
&\mathrm{E}\big(H_i(X_1)H_j(X_2)H_s(X_3)H_t(X_4)\big) \\
&\quad = \sum_{G\in\Gamma(i_1,i_2,i_3,i_4)} \big(\mathrm{E}(X_1X_2)\big)^{v_{12}}\big(\mathrm{E}(X_1X_3)\big)^{v_{13}}\big(\mathrm{E}(X_1X_4)\big)^{v_{14}} \\
&\qquad\qquad \times \big(\mathrm{E}(X_2X_3)\big)^{v_{23}}\big(\mathrm{E}(X_2X_4)\big)^{v_{24}}\big(\mathrm{E}(X_3X_4)\big)^{v_{34}},
\end{aligned}$$

with convention $0^0 = 1$. Let the diagram G is such that $v_{1,2} \neq 0$. Then, by independence of X_1 and X_2, $\mathrm{E}(X_1X_2) = 0$. Whence follows that $(\mathrm{E}(X_1X_2))^{v_{12}} = 0$. If conversely, $v_{1,2} \neq 0$, then the i_1 vertices in the first row and i_2 vertices in the second row have not interconnections. Recall that $i_1 + i_2 = 2$ and $i_3 + i_4 \geq 4$, so it rests $(i_3 + i_4 - i_1 - i_2)/2 \geq 1$ edges, which connect the vertices in the third and in the fourth rows. Thus, $(\mathrm{E}(X_3X_4))^{v_{34}} = 0$ and a sum over diagrams $G \in \Gamma(i_1, i_2, i_3, i_4)$, for which $v_{1,2} \neq 0$, also is equal to zero. The equality $\mathrm{Cov}(\Psi_2(X_1, X_2), \Psi_1(X_3, X_4)) = 0$ follows similarly. ☐

Let c_k denotes the coefficients in Maclaurin series for $(1 + \tau^{-1})^{2H}$. By applying Maclaurin expansion to (4), we have

$$\rho_H(\tau \pm 1) = c_2\tau^{2H-2} \pm 3c_3\tau^{2H-3} + 7c_4\Lambda_1(\tau)\tau^{2H-4} \pm 15c_5\Lambda_2(\tau)\tau^{2H-5}, \quad (26)$$

$$\rho_H(\tau) = c_2\tau^{2H-2} + c_4L(\tau)\tau^{2H-4}, \quad (27)$$

where

$$\begin{aligned}
L(\tau) &:= (c_4)^{-1}\sum_{n=0}^{\infty} c_{2n+4}\tau^{-2n}, \\
\Lambda_1(\tau) &:= \frac{1}{7c_4}\sum_{n=0}^{\infty}\big(2^{2n+3} - 1\big)c_{2n+4}\tau^{-2n}, \\
\Lambda_2(\tau) &:= \frac{1}{15c_5}\sum_{n=0}^{\infty}\big(2^{2n+4} - 1\big)c_{2n+5}\tau^{-2n}.
\end{aligned}$$

In the next lemma, we show that functions $L(\tau)$, $\Lambda_k(\tau),\ k = 1, 2$ can be estimated by a function having an easily computable form.

Lemma 6. *Let $\tau_0 > \sqrt{5}$. For any $0 < H < 3/4,\ H \neq 1/2$ hold the inequalities*

$$1 < L(\tau) < \Lambda_1(\tau) < \Lambda_2(\tau) < \ell\big(\tau_0/\sqrt{5}\big), \quad \tau \in [\tau_0, \infty), \quad (28)$$

where $\ell(\tau) := (1 - \tau^{-2})^{-c_7/c_5}$.

Proof. The inequality $1 < L(\tau)$ is obvious, to prove $L(\tau) < \Lambda_1(\tau)$, use the inequality

$$4 < \left(2^{n+2} - 1\right)/\left(2^n - 1\right), \quad n \geq 3$$

and the monotonicity of the function $L(\tau)$. The first terms of a series $\Lambda_1(\tau)$ and $\Lambda_2(\tau)$ are equal, thus, the inequality $\Lambda_1(\tau) < \Lambda_2(\tau)$ follows from

$$\frac{2^{2n+3} - 1}{7} \frac{c_{2n+4}}{c_4} < \frac{2^{2n+4} - 1}{15} \frac{c_{2n+5}}{c_5}, \quad n \geq 1.$$

Use c_n definition to get the equivalent inequality

$$H > -\frac{2^{2n+4}(n-1) + 5n + 16}{2^{2n+3}(6n+1) - 2(3n+4)}, \quad n \geq 1,$$

i.e. the inequality $\Lambda_1(\tau) < \Lambda_2(\tau)$ holds for any $H > 0$. A function $\ell(\tau), \tau > 1$ is monotonically decreasing, thus, to prove the last inequality in (28), it is enough to prove

$$\Lambda_2(\tau) < \tilde{L}\left(\tau/\sqrt{5}\right), \quad \tilde{L}(\tau) \leq \ell(\tau), \tag{29}$$

where $\tilde{L}(\tau) := c_5^{-1} \sum_{n=0}^{\infty} c_{2n+5} \tau^{-2n}$. Use the inequality

$$\left(2^{n+2} - 1\right)/\left(2^n - 1\right) < 5, n \geq 3$$

to prove the first inequality in (29). Let $\delta = \delta(H) > 0$. The inequality $\tilde{L}(\tau) \leq (1 - \tau^{-2})^{-\delta}$ is satisfied if the corresponding coefficients of the function $\tilde{L}(\tau)$ do not exceed the coefficients in Maclaurin series for $(1 - \tau^{-2})^{-\delta}$, i.e.

$$\frac{c_7}{c_5} \leq \delta, \; \frac{c_9}{c_5} \leq \frac{\delta(\delta+1)}{2!}, \; \frac{c_{11}}{c_5} \leq \frac{\delta(\delta+1)(\delta+2)}{3!}, \ldots$$

The last inequalities are equivalent to the system of inequalities $d_n \leq \delta, \; n = 2, 3, \ldots,$ where

$$d_n := \frac{(n-1)c_{2n+3}}{c_{2n+1}} - (n-2).$$

The sequence $\{d_n, n \geq 2\}$ is strictly decreasing with the upper bound $\delta = d_2 = c_7/c_5$. Indeed, the inequalities $d_n \geq d_{n+1}, \; n = 2, 3, \ldots$ follow from

$$2\left(n^2 - n - 5\right)H^2 + \left(11n^2 + 29n + 15\right)H + 5(n+1)(n+2) \geq 0, \quad n = 2, 3, \ldots \tag{30}$$

For $n = 2$, the inequality (30) holds with $-1/2 < H < 20$, and for $n \geq 3$, the inequality (30) holds for $H > -1/2$. □

Proof of Proposition 1. In view of (24), it is enough to consider covariances $\mathrm{Cov}(\Psi_k(X_1, X_2), \Psi_k(X_3, X_4)), \; k = 1, 2,$ where $X_i, \; i = 1, 2, 3, 4$ satisfy assumptions of the Lemma 5.

By the diagram formula, the covariance $\mathrm{Cov}(\Psi_1(X_1,X_2),\Psi_1(X_3,X_4))$ is equal to

$$\frac{q_A^2}{2(1-A^2)^2}\big\{A^2\{\rho_H(\tau-1)+\rho_H(\tau+1)\}^2+2\big(1+A^2\big)\rho_H^2(\tau) \\ +2\big(1-A^2\big)\rho_H(\tau-1)\rho_H(\tau+1)-4A\rho_H(\tau)[\rho_H(\tau-1)+\rho_H(\tau+1)]\big\}.$$

Combining obtained equality with (26), (27) and Lemma 6, we get that for $\tau \ge \tau_0 > \sqrt{5}$,

$$\big|\mathrm{Cov}\big(\Psi_1(X_1,X_2),\Psi_1(X_3,X_4)\big)-C(H)\tau^{4H-4}\big| \le g_1(\tau,\tau_0)\tau^{4H-6}$$

where $C(H)$ is defined in (7) and

$$\begin{aligned} g_1(\tau,\tau_0) &=: \frac{q_A^2}{2(1-A^2)^2}\big\{\delta_1(H)\ell(\tau_0)+\delta_2(H)\ell^2(\tau_0)\tau^{-2}+\delta_3(H)\ell^2(\tau_0)\tau^{-4}\big\},\\ \delta_1(H) &:= 32c_2c_4(|A|+1)^2+18c_3^2\big(1+A^2\big),\\ \delta_2(H) &:= c_4^2\big(100A^2+56|A|+100\big)+180c_3c_5\big(A^2+1\big),\\ \delta_3(H) &:= 450\big(1-A^2\big)c_5^2. \end{aligned}$$

Let us estimate $\mathrm{Cov}(\Psi_2(X_1,X_2),\Psi_2(X_3,X_4))$. We will use the elegant bound due to Arcones. The maximal correlation coefficient between (X_1,X_2) and (X_3,X_4) is $\alpha(\tau,H) := \{|\mathrm{E}[X_1X_3]|\vee|\mathrm{E}[X_2X_4]|\}+\{|\mathrm{E}[X_1X_4]|\vee|\mathrm{E}[X_2X_3]|\}$. Under the condition

$$\alpha(\tau,H) \le 1 \tag{31}$$

the Lemma 1 in [4] yields

$$\big|\mathrm{Cov}\big(\Psi_2(X_1,X_2),\Psi_2(X_3,X_4)\big)\big| \le \big(\alpha(\tau,H)\big)^r\mathrm{E}\big[\Psi_2(X_1,X_2)\big]^2,$$

where $r=4$ is Hermite rank of the function $\Psi_2(x,y)$ with respect to (X_1,X_2), see Lemma 4. By the definition (21), the function $\Psi(x,y)$ is bounded by 1 a.s., thus $\mathrm{E}[\Psi(X_1,X_2)]^2 \le 1$. The bound $\mathrm{E}[\Psi_2(X_1,X_2)]^2 \le 1$ follows from $\mathrm{E}[\Psi_2(X_1,X_2)]^2 \le \mathrm{E}[\Psi(X_1,X_2)]^2$.

Let us estimate $\alpha(\tau,H)$. Introduce function

$$\begin{aligned} g_2(\tau,\tau_0) =: |c_2|&\left\{\left(1\vee\frac{1-A}{1+A}\right)+\sqrt{\frac{1-A}{1+A}}\right\}\\ &+\frac{|c_4|}{\tau^2}\left\{\frac{A^2+14|A|+1}{1-A^2}+\frac{7+|A|}{\sqrt{1-A^2}}\right\}\ell\left(\frac{\tau_0}{\sqrt{5}}\right)\\ &+\frac{3|c_3|}{\tau}+\frac{15|c_5|}{\tau^3}\ell\left(\frac{\tau_0}{\sqrt{5}}\right). \end{aligned}$$

A bound $\alpha(H,\tau) \le \tau^{2H-2}g_2(\tau,\tau_0), \quad \tau \ge \tau_0 > \sqrt{5}$ is immediate by (4) and (26), (27). Now it is possible to define the function $g(\tau,\tau_0)$ in (6):

$$g(\tau,\tau_0) := g_1(\tau,\tau_0)\tau^{4H-6}+\big(g_2(\tau,\tau_0)\big)^4\tau^{8H-8}. \tag{32}$$

Recall that $\ell(\tau)$ is monotonically decreasing for $\tau > 1$ (see proof of Lemma 6). This, together with definitions of the functions $g_1(\tau, \tau_0)$ and $g_2(\tau, \tau_0)$ gives that a function $g(\tau, \tau_0)$ is monotonically decreasing in both arguments for $\sqrt{5} < \tau_0 \le \tau$. The integrability/summiability of the function $g(\tau, \tau_0)$ in $\tau \in [\tau_0, \infty)$ follows from the bound

$$g(\tau, \tau_0) \le g_1(\tau_0, \tau_0)\tau^{4H-6} + \big(g_2(\tau_0, \tau_0)\big)^4 \tau^{8H-8}.$$

It rests to prove that for any $0 < H < 3/4$, $H \ne 1/2$ exists a splitting point $\sqrt{5} < \tau_0(H) < \infty$, such that (31) is satisfied. Define

$$\tau_0 := \inf_{\tau \ge 3} \big\{\tau \colon \tau^{2H-2} g_2(\tau, \tau) < 1\big\}, \tag{33}$$

where the infimum of the empty set is infinity. It is easy to check that $\tau_0 \le (g_2(3,3))^{1/(2-2H)} < \infty$. □

Suppose that we want to approximate $\sigma_1^2(H)$, $0 < H < 3/4$ to within a specified tolerance $\varepsilon > 0$. For solving this problem in the case $H \ne 1/2$, find a splitting point τ_0 in (33) such that the inequality $\int_{\tau_0}^{\infty} g(\tau, \tau_0)\,\mathrm{d}\tau < \varepsilon/4$ is satisfied. Then use (14) for finding an approximate value of $I^-(\tau_0)$ with truncation error $\varepsilon_{I^-} = \varepsilon/4$. The resulting approximation is given in (11). In the case $H = 1/2$, use (13), where approximate value of $I^-(2)$ must be found with truncation error $\varepsilon_{I^-} = \varepsilon/2$.

The approximation of $\sigma_1^2(H)$, $0 < H < 3/4$ to within a specified tolerance $\varepsilon > 0$ is similar. In the case $H \ne 1/2$, the splitting point τ_0 in (33) must satisfy the condition $\sum_{\tau=[\tau_0]+1}^{\infty} g(\tau, \tau_0) < \epsilon/6$. The last sum can be expressed as a linear combination on powers of Hurwitz zeta functions. Use composition of the functions *N[Zeta[α, t],k]* in Mathematica 5.2 to find an approximate value of Hurwitz zeta function $\zeta(\alpha, t)$ with k digits precision. Next, find approximate values of summands in (9) with truncation error $\varepsilon/(6([\tau_0]+1))$ and approximate value of $\zeta(4-4H, [\tau_0]+1)$ with the truncation error $\varepsilon/(6C(H))$. Then it rests to use (12). In the case $H = 1/2$, it is enough to find an approximate value of $\mathrm{Cov}(\eta_0^{(1)}, \eta_\tau^{(1)})$, $\tau = 0, 1$ with the truncation error $\epsilon/3$ and use (13).

The approximate values $\sigma_p^2(H)$, $\tilde{\sigma}_p^2(H)$, $p = 1, 2$ with a specified tolerance $\varepsilon = 10^{-4}$ are presented in Tables 1, 2.

Table 1. Approximate values of $\sigma_1^2(H)$ and $\tilde{\sigma}_1^2(H)$

H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$	H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$	H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$
0.01	0.0002	0.1379	0.17	0.0154	0.1343	0.33	0.0357	0.1304
0.02	0.0005	0.1377	0.18	0.0167	0.1341	0.34	0.0369	0.1301
0.03	0.0009	0.1375	0.19	0.018	0.1339	0.35	0.0381	0.1298
0.04	0.0015	0.1372	0.2	0.0193	0.1336	0.36	0.0393	0.1296
0.05	0.0022	0.137	0.21	0.0205	0.1334	0.37	0.0405	0.1293
0.06	0.003	0.137	0.22	0.0218	0.1332	0.38	0.0417	0.129
0.07	0.0039	0.1366	0.23	0.0232	0.1329	0.39	0.0428	0.1288
0.08	0.0048	0.1363	0.24	0.0245	0.1327	0.4	0.0439	0.1285
0.09	0.0059	0.1361	0.25	0.0257	0.1324	0.41	0.0451	0.1282

H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$	H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$	H	$\sigma_1^2(H)$	$\tilde{\sigma}_1^2(H)$
0.1	0.0069	0.1359	0.26	0.027	0.1322	0.42	0.0462	0.128
0.11	0.008	0.1357	0.27	0.0283	0.1319	0.43	0.0473	0.1277
0.12	0.0091	0.1355	0.28	0.0296	0.1317	0.44	0.0484	0.1274
0.13	0.0104	0.1352	0.29	0.0308	0.1314	0.45	0.0495	0.1272
0.14	0.0116	0.135	0.3	0.032	0.1312	0.46	0.0506	0.127
0.15	0.0128	0.1348	0.31	0.0333	0.1309	0.47	0.0517	0.1267
0.16	0.0141	0.1345	0.32	0.0345	0.1306	0.48	0.0528	0.1265
0.49	0.0539	0.1263	0.58	0.0644	0.1257	0.67	0.0833	0.1339
0.5	0.055	0.1261	0.59	0.0658	0.1258	0.68	0.0875	0.137
0.51	0.0561	0.126	0.6	0.0673	0.1261	0.69	0.093	0.1413
0.52	0.0572	0.1258	0.61	0.0689	0.1265	0.7	0.1004	0.1476
0.53	0.0583	0.1257	0.62	0.0706	0.1271	0.71	0.1112	0.1572
0.54	0.0595	0.1256	0.63	0.0725	0.1278	0.72	0.129	0.1739
0.55	0.0606	0.1255	0.64	0.0747	0.1287	0.73	0.1639	0.2076
0.56	0.0619	0.1255	0.65	0.0771	0.13	0.74	0.2679	0.3105
0.57	0.0631	0.1256	0.66	0.0799	0.1317	0.75	∞	∞

Table 2. Approximate values of $\sigma_2^2(H)$ and $\tilde{\sigma}_2^2(H)$

H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$	H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$	H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$
0.01	0.0004	0.1448	0.24	0.0195	0.1424	0.47	0.0408	0.1388
0.02	0.0006	0.1447	0.25	0.0206	0.1423	0.48	0.0415	0.1386
0.03	0.0009	0.1446	0.26	0.0216	0.1422	0.49	0.0423	0.1384
0.04	0.0014	0.1445	0.27	0.0226	0.1421	0.5	0.043	0.1382
0.05	0.0019	0.1444	0.28	0.0236	0.1419	0.51	0.0437	0.138
0.06	0.0025	0.1443	0.29	0.0247	0.1418	0.52	0.0444	0.1378
0.07	0.0032	0.1443	0.3	0.0256	0.1416	0.53	0.0451	0.1376
0.08	0.004	0.1442	0.31	0.0266	0.1415	0.54	0.0458	0.1373
0.09	0.0048	0.1441	0.32	0.0276	0.1414	0.55	0.0464	0.1371
0.1	0.0056	0.144	0.33	0.0286	0.1412	0.56	0.0472	0.1369
0.11	0.0065	0.1439	0.34	0.0295	0.1411	0.57	0.0478	0.1367
0.12	0.0074	0.1438	0.35	0.0305	0.1409	0.58	0.0484	0.1364
0.13	0.0083	0.1437	0.36	0.0314	0.1407	0.59	0.049	0.1362
0.14	0.0093	0.1436	0.37	0.0324	0.1406	0.6	0.0496	0.136
0.15	0.0102	0.1435	0.38	0.0333	0.1404	0.61	0.0502	0.1357
0.16	0.0113	0.1434	0.39	0.0342	0.1402	0.62	0.0507	0.1355
0.17	0.0122	0.1433	0.4	0.035	0.1401	0.63	0.0513	0.1353
0.18	0.0133	0.1432	0.41	0.0358	0.1399	0.64	0.0518	0.135
0.19	0.0143	0.143	0.42	0.0368	0.1397	0.65	0.0523	0.1347
0.2	0.0154	0.1429	0.43	0.0376	0.1395	0.66	0.0528	0.1345
0.21	0.0164	0.1428	0.44	0.0384	0.1394	0.67	0.0533	0.1342
0.22	0.0175	0.1427	0.45	0.0392	0.1392	0.68	0.0538	0.134
0.23	0.0184	0.1426	0.46	0.0400	0.139	0.69	0.0543	0.1337
0.7	0.0547	0.1334	1.06	0.0647	0.1218	1.42	0.0683	0.1084
0.71	0.0552	0.1331	1.07	0.0649	0.1214	1.43	0.0684	0.1081
0.72	0.0556	0.1329	1.08	0.065	0.121	1.44	0.0686	0.1078

H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$	H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$	H	$\sigma_2^2(H)$	$\tilde{\sigma}_2^2(H)$
0.73	0.0561	0.1326	1.09	0.0651	0.1207	1.45	0.0687	0.1074
0.74	0.0565	0.1323	1.1	0.0653	0.1203	1.46	0.0688	0.1071
0.75	0.0569	0.132	1.11	0.0654	0.1199	1.47	0.069	0.1068
0.76	0.0572	0.1317	1.12	0.0655	0.1195	1.48	0.0691	0.1066
0.77	0.0576	0.1314	1.13	0.0656	0.1192	1.49	0.0693	0.1063
0.78	0.058	0.1311	1.14	0.0657	0.1188	1.5	0.0694	0.106
0.79	0.0583	0.1308	1.15	0.0659	0.1184	1.51	0.0697	0.1058
0.8	0.0587	0.1305	1.16	0.066	0.118	1.52	0.07	0.1056
0.81	0.059	0.1302	1.17	0.0661	0.1177	1.53	0.0702	0.1055
0.82	0.0593	0.1299	1.18	0.0662	0.1173	1.54	0.0705	0.1052
0.83	0.0597	0.1296	1.19	0.0663	0.1169	1.55	0.0709	0.1051
0.84	0.06	0.1293	1.2	0.0664	0.1165	1.56	0.0712	0.105
0.85	0.0603	0.129	1.21	0.0665	0.1162	1.57	0.0717	0.105
0.86	0.0606	0.1287	1.22	0.0666	0.1158	1.58	0.0721	0.105
0.87	0.0608	0.1283	1.23	0.0667	0.1154	1.59	0.0727	0.1051
0.88	0.0611	0.128	1.24	0.0668	0.115	1.6	0.0733	0.1053
0.89	0.0614	0.1277	1.25	0.0669	0.11463	1.61	0.074	0.1055
0.9	0.0616	0.1273	1.26	0.067	0.1143	1.62	0.0749	0.1059
0.91	0.062	0.127	1.27	0.0671	0.1139	1.63	0.0759	0.1064
0.92	0.0622	0.1267	1.28	0.0671	0.1135	1.64	0.0771	0.1072
0.93	0.0624	0.1263	1.29	0.0673	0.1131	1.65	0.0786	0.1082
0.94	0.0626	0.126	1.3	0.0673	0.1127	1.66	0.0804	0.1095
0.95	0.0628	0.1257	1.31	0.0675	0.1124	1.67	0.0827	0.1113
0.96	0.063	0.1253	1.32	0.0674	0.112	1.68	0.0856	0.1138
0.97	0.0632	0.125	1.33	0.0675	0.1116	1.69	0.0896	0.1173
0.98	0.0634	0.1246	1.34	0.0676	0.1112	1.7	0.0952	0.1224
0.99	0.0636	0.1243	1.35	0.0677	0.1109	1.71	0.1037	0.1303
1	0.0638	0.1239	1.36	0.0678	0.1105	1.72	0.1178	0.1439
1.01	0.064	0.1236	1.37	0.0679	0.1102	1.73	0.1461	0.1717
1.02	0.0641	0.1232	1.38	0.068	0.1098	1.74	0.231	0.256
1.03	0.0643	0.1228	1.39	0.0681	0.1094	1.75	∞	∞
1.04	0.0644	0.1225	1.4	0.0682	0.1091			
1.05	0.0646	0.1221	1.41	0.0682	0.1088			

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References

1. D. Surgailis, G. Teyssière, M. Vaičiulis, The increment ratio statistic, *J. Multivariate Anal.*, **99**, pp. 510–541, 2008.
2. J. M. Bardet, D. Surgailis, *Measuring Roughness of Random Paths by Increment Ratios*, Preprint, 2007.

3. D. Cvijovic, J. Klinowski, Values of the Legendre chi and Hurwitz zeta functions at rational arguments, *Math. Comput.*, **68**, pp. 1623–1630, 1999.
4. M. A. Arcones, Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors, *Ann. Probab.*, **22**, pp. 2242–2274, 1994.
5. L. D. Kudriavcev, *Mathematical Analysis Course*, 2nd edition, Vysshaya shkola, Moscow, 1988 (in Russian).
6. A. C. Genz, A. A. Malik, An adaptive algorithm for numerical integration over an N-dimensional rectangular region, *J. Comput. Appl. Math.*, **6**, pp. 295–302, 1980.
7. M. Avarucci, D. Marinucci, Polynomial cointegration among stationary processes with long memory, *J. Time Ser. Anal.*, **28**(6), pp. 923–942, 2007.
8. L. Giraitis, M. S. Taqqu, Whittle estimator for finite-variance non-Gaussian time series with long memory, *Ann. Stat.*, **27**(1), pp. 178–203, 1999.
9. L. Giraitis, D. Surgailis, ARCH-type bilinear models with double long memory, *Stoch. Proc. Appl.*, **100**, pp. 275–300, 2002.
10. R. Fox, M. S. Taqqu, Noncentral limit theorems for quadratic forms in random variables having long-range dependence, *Ann. Probab.*, **13**(2), pp. 428–446, 1985.