Oscillation of Non-Linear Systems Close to Equilibrium Position in the Presence of Coarse-Graining in Time and Space

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Abstract. One considers the motion of nonlinear systems close to their equilibrium positions in the presence of coarse-graining in time on the one hand, and coarse-graining in time on the other hand. By considering a coarse-grained time as a time in which the increment is not $\mathrm{d}t$ but rather $(\mathrm{d}t)_c > \mathrm{d}t$, one is led to introduce a modeling in terms of fractional derivative with respect to time; and likewise for coarse-graining with respect to the space variable x. After a few prerequisites on fractional calculus via modified Riemann-Liouville derivative, one examines in a detailed way the solutions of fractional linear differential equations in this framework, and then one uses this result in the linearization of nonlinear systems close to their equilibrium positions.

Keywords: coarse-grained time, coarse-grained space, fractal time, fractional analysis, fractional Taylor's series, stability, linearization, Lyapunov function.

1 Introduction

1.1 Coarse-graining, fractals, fractional derivative

General motivation of the article

The motivation of the present article is the need to derive a mathematical framework for dealing with dynamical systems defined in coarse-grained spaces and with coarse-grained time and, to this end, to use the fact that fractional calculus appears to be intimately related to fractal and self-similar functions.

Coarse-grained space-time in physics

Coarse-grained and thus fractional spaces are basic in El Naschie's work [1–6]. Fractals are basic in Nottale's work [7–10], and transparent in Ord's paper [11]. See also [12, 13]. In these works fractals are introduced as a tool to revisit the foundation of physics as natural science.

Fractional derivative, self-similarity, fractal dimension

Fractional derivative has been introduced formally by Liouville by using an integral involving the function under consideration, and in most cases this fractional calculus is more or less a formal calculus which is very often converted into Laplace's transform. This has been true until when Kolwankar [14, 15], starting from the Hölder exponent of functions defined on Cantor's sets, arrived at the definition of fractional derivative in terms of increment. Later, we arrived at the same identification, but by using the opposite way. We started from a fractional derivative defined as the limit of fractional difference, which allowed us to obtain the generalized fractional Taylor's series, and we so found that the Hölder exponent is exactly the order of the fractional derivative of the function under consideration.

Information, coarse-graining, Hölder exponent

In an attempt to relate coarse-graining with Hölder exponent, we shall refer to information theory. Indeed, in the Shannon information theory framework, it is known that the amount of information (uncertainty should be better) involved in the random variable X which is uniform on the interval (a, b) is $h(X) = \ln(b - a)$, see for instance [16].

This being the case, assume that we have two non-random variables x and y which are related by the relation $y = f(x)$. We shall say that x and y involved the same grade of granularity when

$$\lim_{\Delta x \downarrow 0} \frac{h(|\Delta y|)}{h(|\Delta x|)} = 1,$$

and on the contrary we shall say that, there is a coarse-graining phenomenon when this limit is β, different from the unit, in other words, when one has

$$|\mathrm{d}y| = C|\mathrm{d}x|^{\beta},$$

where C is a constant, and β is such that $0 < \beta < 1$, in which case the coarse-graining effect takes place in the x-space. On the contrary when $\beta > 1$ one then has $|\,\mathrm{d}y|^{1/\beta} = \mathrm{d}x$ and it is the y-space which involves coarse-graining. This parameter β is exactly the Hölder's coefficient, and in quite a natural way we come across fractional calculus.

Non-differentiability, randomness and stochastics

As a last, but not least remark, we shall notice that non-differentiability and randomness are mutually related in their nature, in such a way that studies in fractals on the one hand and fractional Brownian motion on the other hand are often parallel in the same paper. Indeed, as pointed out by Nottale [7], a function which is continuous everywhere but is nowhere differentiable necessarily exhibits random-like or pseudo-random features, in the sense that various samplings of this functions on the same given interval will be different. This may explain the huge amount of literature which extends the theory of stochastic differential equation [17,18] to stochastic dynamics driven by fractional Brownian motion [17–24].

1.2 Purpose and organisation of the article

Organization of the paper

We shall focus on fractional derivative defined by finite difference, which, as a result, provides various extensions of the Leibnitz formula. The article is organised as follows. For the convenience of the reader, we first bear in mind the essential of the fractional modified Riemann-Liouville derivative and the fractional Taylor's series, as well as some useful formulae which one can so obtain (Section 2). Then one considers the problem of modeling velocity with coarse-grained time and coarse-grained space and by this way, via fractional analysis, we come across fractional derivative (Section 3). In the Sections 4 and 5, we consider solving different types of linear fractional linear differential equations with constant coefficients on the one hand, and time-varying coefficient on the other hand. Then we shall consider systems of linear fractional differential equations (Section 6). This material will be the basis to analyze the stability of nonlinear systems close to their equilibrium positions, and simple illustrative examples are provided in the Section 7.

Bibliographical note

Kolwankar and Gangal [14, 15] considered a function defined on a fractal set, and after introducing its derivative in terms of Hölder exponent , they arrive at fractional derivative and fractional (local) Rolle-Taylor's formula. In our approach, we worked in the opposite way. Firstly, irrespective of any fractal set, we start from the expression of the fractional derivative as the limit of a fractional difference involving an infinite number of terms, and therefore we obtain the fractional generalized Taylor's series, whereby we come across the Hölder's exponent.

In order to deal with functions which are not differentiable, Ben Adda and Cresson [12] introduced a so-called quantum derivative, different from the Nottale's scale-derivative, which also provides a (local) Rolle-Taylor's formula. Here, we shall use a different modeling based on fractional derivative.

The Section 2 below is a background on some results for the convenience of the reader, and all the remaining should be thought of as the contribution of the article.

2 Summary of some results on fractional analysis

2.1 Fractional derivative via fractional difference

This second section is a short review, but ut we think it is necessary for the convenience of the reader.

In this section, we use the term of fractional analysis instead of fractional calculus, to emphasize the fact that we do not use the formal definition of fractional derivative via integral, but rather we work on the finite increments themselves, that is to say we define fractional Brownian motion as the quotient of two increments.

Definition 1. Let $f\colon \Re \rightarrow \Re,\ x \rightarrow f(x)$ denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $FW(h)$ by the equality (the symbol := means that the left side is defined by the right side) where ρ denotes a real-valued exponent.

$$FW^{\rho}(h)f(x) := f(x + \rho h), \tag{1}$$

then the fractional difference of order α, $0 < \alpha < 1$, of $f(x)$ is defined by the expression [16, 22, 23, 25–27].

$$\Delta^{\alpha} f(x) := (FW - 1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \tag{2}$$

and its fractional derivative of order α is defined by the limit

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^{\alpha}(f(x) - f(0))}{h^{\alpha}}. \tag{3}$$

This definition is close to the standard definition of derivative (calculus for beginners), and as a direct result, the α-th derivative of a constant is zero.

2.2 Modified fractional Riemann-Liouville derivative (via integral)

An alternative to the Riemann-Liouville definition of fractional derivative

In order to circumvent some drawbacks involved in the classical Riemann-Liouville definition, we have proposed the following alternative to the Riemann-Liouville definition of F-derivative, which is moreover fully supported by the Definition 1.

Definition 2 (Riemann-Liouville definition revisited)**.** Refer to the function of the Definition 1

(i) Assume that $f(x)$ is a constant K. Then its fractional derivative of order α is

$$\mathrm{D}_x^{\alpha} K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}, \quad \alpha \le 0, \tag{4}$$

$$\mathrm{D}_x^{\alpha} K = 0, \qquad \alpha > 0. \tag{5}$$

(ii) When $f(x)$ is not a constant, then one will set

$$f(x) = f(0) + \big(f(x) - f(0)\big),$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(x) = \mathrm{D}_x^{\alpha} f(0) + \mathrm{D}_x^{\alpha}\big(f(x) - f(0)\big),$$

in which, for negative α, one has

$$\mathrm{D}_x^{\alpha}\big(f(x) - f(0)\big) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi)\, \mathrm{d}\xi, \quad \alpha < 0, \tag{6}$$

whilst for positive α, one will set

$$\begin{aligned} \mathrm{D}_x^\alpha\big(f(x)-f(0)\big) &= \mathrm{D}_x^\alpha f(0) = \big(f^{(\alpha-1)}(x)\big)' \\ &= \frac{1}{\Gamma(-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)\,\mathrm{d}\xi, \quad 0<\alpha<1. \end{aligned} \tag{7}$$

When $n<\alpha\le n+1$, one will set

$$f^{(\alpha)}(x) := \big(f^{(\alpha-n)}(x)\big)^{(n)}, \quad n<\alpha\le n+1,\ \ n\ge 1. \tag{8}$$

The main idea in this definition is that a fractional differentiable function is considered as the sum of a constant with a self-similar function which, as such, takes on the value zero at $x=0$.

We shall refer to this fractional derivative as to the *modified Riemann Liouville derivative*.

Remark that this definition is different from other definitions in the literature (see for instance [28–30] in the sense that it removes the effects of the initial value of the considered function.

2.3 Fractional Taylor's series for one-variable functions

A generalized Taylor expansion of fractional order which applies to non-differentiable functions reads as follows [31–35].

Proposition 1. *Assume that the continuous function* $f\colon \Re\to\Re,\ x\to f(x)$ *has fractional derivative of order* $k\alpha$*, for a given* α*,* $0<\alpha\le 1$*, and any positive integer* k*; then the following equality holds, which reads*

$$f(x+h) = \sum_{k=0}^{\infty}\frac{h^{\alpha k}}{\Gamma(1+\alpha k)}f^{(\alpha k)}(x), \quad 0<\alpha\le 1, \tag{9}$$

where $f^{(\alpha k)}(x)$ *holds for the fractional derivative* $\mathrm{D}^\alpha\mathrm{D}^\alpha\ldots\mathrm{D}^\alpha f(x)$*,* k *times.*

With the notation

$$\Gamma(1+\alpha k) =: (\alpha k)!,$$

one has the formula

$$f(x+h) = \sum_{k=0}^{\infty}\frac{h^{\alpha k}}{(\alpha k)!}f^{(\alpha k)}(x), \quad 0<\alpha\le 1 \tag{10}$$

which looks like the classical one.

Alternatively, in a more compact form, one can write

$$f(x+h) = E_\alpha\big(h^\alpha \mathrm{D}_x^\alpha\big)f(x),$$

where D_x is the derivative operator with respect to x and $E_\alpha(y)$ denotes the Mittag-Leffler function defined by the expression

$$E_\alpha(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(1+\alpha k)}. \tag{11}$$

Bibliographical note

Kolwankar and Gangal [14,15] had previously discovered the first term of this series (10), referred to as local fractional Taylor's series (we would prefer the terms of fractional Rolle's formula) but their approach is quite different from ours. They started from functions defined on fractal set, and then they arrive at the Hölder's exponent, whilst we began with the formal definition of fractional derivatives in terms of increment, and we came across the Mittag-Leffler function.

2.4 Some useful relations

The equation (9) provides the useful relation

$$\mathrm{d}^\alpha f \cong \Gamma(1+\alpha)\,\mathrm{d}f, \quad 0 < \alpha < 1, \tag{12}$$

or in a finite difference form $\Delta^\alpha f \cong \Gamma(1+\alpha)\Delta f$.

Corollary 1. *The following equalities hold, which are*

$$\mathrm{D}^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha}, \quad \gamma > 0, \tag{13}$$

or, what amounts to the same (we set $\alpha = n+\theta$)

$$\mathrm{D}^{n+\theta} x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta)x^{\gamma-n-\theta}, \quad 0 < \theta < 1,$$

$$\big(u(x)v(x)\big)^\alpha = u^{(\alpha)}(x)v(x) = u(x)v^{(\alpha)}(x), \tag{14}$$

$$\big(f[u(x)]\big)^\alpha = f'_u(u)u^{(\alpha)}(x), \tag{15}$$

$$\big(f[u(x)]\big)^\alpha = f_u^{(\alpha)}(u)(u'_x)^{(\alpha)}, \tag{16}$$

$$\big(f[u(x)]\big)^\alpha = (1-\alpha)!u^{\alpha-1}f_u^{(\alpha)}(u)u^{(\alpha)}(x). \tag{17}$$

$u(x)$ is non-differentiable in (14) *and* (15) *and differentiable in* (16)*, $v(x)$ is non-differentiable in* (14)*, and $f(u)$ is differentiable in* (15) *and non-differentiable in* (16)*.*

Corollary 2. *Assume that $f(x)$ and $x(t)$ are two $\Re \to \Re$ functions which both have derivatives of order α, $0 < \alpha < 1$, then one has the fractional derivative chain rule*

$$f_t^{(\alpha)}\big(x(t)\big) = \Gamma(2-\alpha)x^{\alpha-1}f_x^{(\alpha)}(x)x^{(\alpha)}(t). \tag{18}$$

For more formulae, which can be so derived with the modified Riemann-Liouville derivative, see [14].

2.5 Integration with respect to $(\mathrm{d}x)^\alpha$

The integral with respect to $(\mathrm{d}x)^\alpha$ is defined as the solution of the fractional differential equation

$$\mathrm{d}y = f(x)(\mathrm{d}x)^\alpha, \quad x \geq 0,\ \ y(0) = 0,\ \ 0 < \alpha \leq 1 \tag{19}$$

which is provided by the following result:

Lemma 1. *Let $f(x)$ denote a continuous function, then the solution $y(x)$, $y(0) = 0$, of the equation* (29) *is defined by the equality*

$$y = \int_0^x f(\xi)(\mathrm{d}\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad 0 < \alpha \leq 1. \tag{20}$$

For the proof, see for instance [35].

The *fractional* integration by part formula reads

$$\int_a^b u^{(\alpha)}(x)v(x)(\mathrm{d}x)^\alpha = [u(x)v(x)]_a^b - \int_a^b u(x)v^{(\alpha)}(x)(\mathrm{d}x)^\alpha, \tag{21}$$

as a direct consequence of (17).

For an extended bibliography on fractional calculus, one can consult for instance [29, 30, 36–52].

In the next section, we shall show that fractional difference is quite suitable to obtain a modeling of dynamics involving coarse-grained time or fractal time.

3 Application to modeling coarse-graining in time and space

3.1 Coarse-graining in time and fractional derivative

For fixing the thought, let us assume that we are dealing with a mechanical point with the mass m which is moving in a one-dimensional coarse-grained space defined by the space co-ordinate $x(t)$, where t denotes the time.

On a modeling standpoint, we shall assume that in a coarse-grained time, the instant point is not infinitely thin but rather has a thickness. So, if $\mathrm{d}t$, $\mathrm{d}t > 0$, and $(\mathrm{d}t)_c$ refer respectively to the size of the thin time increment and that one of the coarse-grained time increment, then we should have $\mathrm{d}t < (\mathrm{d}t)_c$.

Modeling coarse-graining phenomenon in time

In order to describe the coarse-graining phenomenon in time on an analytical point of view, we shall assume that the differential element, i.e. the increment in time is not $\mathrm{d}t$, but rather is $(\mathrm{d}t)^\alpha$, where α, $0 < \alpha < 1$, is a real-valued parameter which characterizes the grade of the phenomenon.

By this way, we shall have $(\mathrm{d}t)^\alpha > \mathrm{d}t$.

At first glance, we would have to select a modeling among $(\mathrm{d}t)^\alpha$ and $\alpha(\mathrm{d}t)$, but an additional underlying condition is that $(\mathrm{d}t)_c/\mathrm{d}t$ should increase indefinitely as $\mathrm{d}t$ tends to zero, therefore the model.

Coarse-grained time and fractal time

It appears that α is exactly the fractal dimension of the space in which x is running. Indeed, in the plane, let us consider a curve line of length L with a covering with disjoint balls of diameter ε. Let $N(\varepsilon)$ denote the number of balls so involved. The fractal (Hausdorf) dimension f of the curve is defined as the limit

$$f := \lim_{\varepsilon \downarrow 0} -\big(\ln N(\varepsilon)/\ln\varepsilon\big).$$

Usually one has the equality $N = L/\varepsilon$ which provides the dimension 1, but in the case of a fractal curve, one will have $N = L/\varepsilon^\alpha$, therefore

$$f := \lim_{\varepsilon \downarrow 0} -\big(\ln \varepsilon^\alpha/\ln\varepsilon\big).$$

Definition 3. With this assumption, the coarse-grained time velocity of a particle will be defined by the expression

$$u_\alpha(t) := \frac{\mathrm{d}x}{(\mathrm{d}t)^\alpha}, \quad 0 < \alpha < 1. \tag{22}$$

Application of fractional derivative to the coarse-grained velocity $u_\alpha(t)$

We have the following

Lemma 2. *The coarse-grained velocity u_α is provided by the fractional derivative*

$$u_\alpha = \frac{\mathrm{d}x}{(\mathrm{d}t)^\alpha} = (\alpha!)^{-1}\frac{\mathrm{d}^\alpha x}{(\mathrm{d}t)^\alpha} = (\alpha!)^{-1}x^{(\alpha)}(t), \quad 0 < \alpha < 1,\ \ \mathrm{d}t > 0. \tag{23}$$

Proof. According to the fractional Taylor's series one has the equality

$$\mathrm{d}^\alpha x(t) = \alpha!(\mathrm{d}t)^\alpha. \qquad \square$$

3.2 Coarse-graining in space and fractional derivative

We can now duplicate the rationale above to the space co-ordinate $x(t)$.

Modeling coarse-graining phenomenon in space

In order to describe the coarse-graining phenomenon on an analytical point of view, we shall assume that the differential in such a space is not d, $\mathrm{d} > 0$, but rather is $(\mathrm{d}x)^\alpha$, where α, $0 < \alpha < 1$, is a real-valued parameter which characterizes the grade of the phenomenon.

By this way we shall have $(\mathrm{d}x)^\alpha > \mathrm{d}x$.

Definition 4. With this assumption, the coarse-grained velocity on the right, $v_\alpha(t)$ of a particle in a coarse-grained space, will be defined by the expression

$$v_\alpha(t) := \frac{(\mathrm{d}x)^\alpha}{\mathrm{d}t}, \quad \mathrm{d}x > 0, \ \ 0 < \alpha < 1. \tag{24}$$

In the special case when $\alpha = 2k+1$, k integer, the equation (24) is meaningful for both $\mathrm{d}x > 0$ and $\mathrm{d}x < 0$, and will so define the coarse-grained velocity on the left or on the right of the particle.

Application of fractional derivative to the modeling of coarse-grained velocity $v_\alpha(t)$

Lemma 3. *Given a function* $y = f(x)$ *and its inverse* $x = g(y)$, *their fractional derivatives of order* α, $0 < \alpha < 1$ *satisfy he conditions*

$$y^{(\alpha)}(x)x^{(\alpha)}(y) = \big((1-\alpha)!\big)^{-2}(xy)^{1-\alpha}. \tag{25}$$

Proof. One has the equality

$$y^{(\alpha)}(x)x^{(\alpha)}(y) = \left(\frac{\mathrm{d}^\alpha y}{\mathrm{d}x^\alpha}\right)\left(\frac{\mathrm{d}^\alpha x}{\mathrm{d}y^\alpha}\right) = \left(\frac{\mathrm{d}^\alpha y}{\mathrm{d}y^\alpha}\right)\left(\frac{\mathrm{d}^\alpha x}{\mathrm{d}x^\alpha}\right),$$

and we take account of (13) which relates $\mathrm{d}^\alpha x$ (resp. $\mathrm{d}^\alpha y$) with $(\mathrm{d}x)^\alpha$ (resp. $(\mathrm{d}y)^\alpha$) to get the result. □

With this prerequisite, we can then state the following

Lemma 4. *The coarse-grained velocity* $v_\alpha(t)$ *is provided by the expression*

$$v_\alpha(t) = \alpha!\big((1-\alpha)!\big)^2(xt)^{\alpha-1}x^{(\alpha)}(t) := \rho(\alpha)(xt)^{\alpha-1}x^{(\alpha)}(t). \tag{26}$$

Proof. Equation (24) yields the expression

$$v_\alpha(t) = \frac{\alpha!(\mathrm{d}x)^\alpha}{\alpha!\mathrm{d}t} = \alpha!\frac{(\mathrm{d}x)^\alpha}{\mathrm{d}^\alpha t} = \alpha!\big(t^{(\alpha)}(x)\big)^{-1} \tag{27}$$

which defines $v_\alpha(t)$ in terms of the fractional derivative of time with respect to space. And substituting (23) into (24) yields the sought result

$$v_\alpha(t) = \alpha!\big((1-\alpha)!\big)^2(xt)^{\alpha-1}x^{\alpha}(t) =: \rho(\alpha)(xt)^{\alpha-1}x^{(\alpha)}(t). \tag{28}$$

4 Solution of the equation $x^{(\alpha)}(t) = a(t)x(t) + b(t)$

4.1 Linear equation with time-varying coefficient

The following results in the Section 4 and 5 are basic for our urpse, because we shall come across these types of equations in our analysis of nonlinear systems via the linearization technique.

Solution of the fractional differential equation

$$x^{(\alpha)}(t) = b(x), \quad x(0) = x_0. \tag{29}$$

One has successively

$$\int_{x_0}^{x} \mathrm{d}^\alpha \xi = x(t) - x_0 = \int_{a}^{x} b(\tau)(\mathrm{d}\tau)^\alpha,$$

therefore

$$x(t) = x_0 + \alpha \int_0^t (t-\tau)^{\alpha-1} b(\tau)\,\mathrm{d}\tau. \tag{30}$$

Solution of the fractional differential equation

$$x^{(\alpha)}(t) = a(t)x + b(t), \quad x(0) = x_0. \tag{31}$$

This solution can be obtained by duplicating the so-called Lagrange technique of constant variation as follows:

First of all, the solution of the homogeneous equation $x^{(\alpha)}(t) = a(t)x(t)$ is

$$x(t) = CE_\alpha \left\{ \int_0^t a(\tau)(\mathrm{d}\tau)^\alpha \right\}, \tag{32}$$

where C denotes a constant.

This being the case, let us look for a special solution of the complete equation in the form

$$x(t) = C(t)E_\alpha \left\{ \int_0^t a(\tau)(\mathrm{d}\tau)^\alpha \right\}.$$

Applying the rule (14) we obtain the fractional derivative

$$C^{(\alpha)}(t) = b(t)E_\alpha^{-1} \left\{ \int_0^t a(\tau)b(\tau)\mathrm{d}\tau \right\},$$

therefore the expression

$$C(t) = \int_0^t b(\tau)E_\alpha^{-1} \left\{ \int_0^t a(u)(\mathrm{d}u)^\alpha \right\} (\mathrm{d}\tau)^\alpha, \tag{33}$$

and the general solution

$$x(t) = \big(x_0 + C(t)\big)E_\alpha \left\{ \int_0^t a(\tau)(\mathrm{d}\tau)^\alpha \right\}. \tag{34}$$

4.2 Linear equation with constant coefficient

(i) Here, we consider the equation

$$x^{(\alpha)}(t) = ax(t) + b, \tag{35}$$

where a and b denote two constants. In this case, the equation (34) direct yields

$$x(t) = x_0 E_\alpha\big(at^\alpha\big) + bE_\alpha\big(at^\alpha\big) \int_0^t E_\alpha^{-1}(a\tau^\alpha)(\mathrm{d}\tau)^\alpha. \tag{36}$$

(ii) An alternative is to remark that a particular solution of the complete equation is

$$x(t) = -b/a,$$

in such a manner that the general solution could be written as well in the form

$$x(t) = \left(x_0 + \frac{b}{a}\right)E_\alpha\big(at^\alpha\big) - \frac{b}{a}. \tag{37}$$

At first glance, on the surface, it is not clear at all that (4.8) and (4.9) are exactly the same. And in effect they are. One has the following

Lemma 5. *The expressions* (36) *and* (37) *are different pictures of the same unique solution of* (35)*.*

Proof. Step 1. We refer to the basic relation (see [35] for the proof)

$$E_\alpha\big(\lambda(x+y)^\alpha\big) = E_\alpha\big(\lambda x^\alpha\big)E_\alpha\big(\lambda y^\alpha\big), \tag{38}$$

to write

$$E_\alpha\big(\lambda(x-x)^\alpha\big) = E_\alpha\big(\lambda x^\alpha\big)E_\alpha\big(\lambda(-x)^\alpha\big) = E_\alpha(0) = 1, \tag{39}$$

therefore the equality

$$E_\alpha\big(\lambda(-x)^\alpha\big) = E_\alpha\big((-1)^\alpha\lambda x^\alpha\big) = E_\alpha^{-1}\big(\lambda x^\alpha\big). \tag{40}$$

Step 2. With this data at hand, we come back to (36) to write

$$\begin{aligned} &bE_\alpha\big(at^\alpha\big)\int_0^t E_\alpha^{(-1)}\big(a\tau^\alpha\big)(\mathrm{d}\tau)^\alpha \\ &\quad = bE_\alpha\big(at^\alpha\big)\int_0^t E_\alpha\big(a(-\tau)^\alpha\big)(\mathrm{d}\tau)^\alpha = b\int_0^t E_\alpha\big(a(t-\tau)^\alpha\big)(\mathrm{d}\tau)^\alpha \\ &\quad = -\frac{b}{a}\left[E_\alpha\big(a(t-\tau)^\alpha\big)\right]_0^t = \frac{b}{a}\left[E_\alpha\big(at^\alpha\big) - 1\right]. \end{aligned}$$

Combining this result with the solution $x_0E_\alpha(at^\alpha)$ yields (37). □

5 Solution of the equation $ax^{(2\alpha)} + bx^{(\alpha)} + c = 0$

5.1 Preliminary remarks on fractional $(\sin, \cos)$ and $(\sinh, \cosh)$ functions

Following classical calculus, we shall write

$$E_\alpha(t) = \cosh_\alpha t + \sinh_\alpha t, \tag{41}$$

with

$$\cosh_\alpha t := 2^{-1}\big(E_\alpha(t) + E_\alpha(-t)\big), \tag{42}$$

and

$$\sinh_\alpha t := 2^{-1}\big(E_\alpha(t) + E_\alpha(-t)\big). \tag{43}$$

In a like manner, we shall write

$$E_\alpha(it) = \cos_\alpha t + i\sin_\alpha t, \tag{44}$$

with

$$\cos_\alpha t := 2^{-1}\big(E_\alpha(it) + E_\alpha(-it)\big), \tag{45}$$

and

$$\sin_\alpha t := (2i)^{-1}\big(E_\alpha(it) - E_\alpha(-it)\big). \tag{46}$$

5.2 Solution of the equation $x^{(2\alpha)} = ax, \;\; a = const$

It is of order to point out that the simple equation

$$x^{(2\alpha)}(t) = ax(t), \tag{47}$$

gives rise to some problems regarding its definition with respect to the fractional derivative. Indeed, on using the operator $\mathrm{d}/\,\mathrm{d}x = D$, at first glance, we could write (47) either in the form

$$\mathrm{D}^{2\alpha}x(t) = ax(t), \tag{48}$$

or

$$\mathrm{D}^{\alpha}\mathrm{D}^{\alpha}x(t) = ax(t), \tag{49}$$

but these two equations do not have the same solution. Indeed, let us assume that $0 < 2\alpha < 1$, and let us work with the Laplace's transfrom $X(s)$ of $x(t)$. Taking the Laplace's transform of the equation (47) yields

$$s^{2\alpha}X(s) - s^{2\alpha-1}x(0) = r^2X(s),$$

therefore we obtain

$$X(s) = \frac{s^{2\alpha-1}x(0)}{s^{2\alpha} - r^2}, \quad 0 < 2\alpha < 1, \tag{50}$$

whilst the equation (49) provides

$$s^{2\alpha}X(s) - s^{2\alpha-1}x(0) - s^{\alpha-1}x^{(\alpha)}(0) = r^2 X(s)$$

and

$$X(s) = \frac{s^{2\alpha-1}x(0) + s^{\alpha-1}x^{(\alpha)}(0)}{s^{2\alpha} - r^2}, \quad 0 < \alpha < 1. \tag{51}$$

As a result we have to carefully select between the equations (48) and (49) when we set define our problem. If the physical system which we are dealing with involves both $x^{(\alpha)}(t)$ and $x^{(2\alpha)}(t)$, then we shall refer to D^α and $\mathrm{D}^\alpha\mathrm{D}^\alpha$, and we shall consider the equation (49).

5.3 Solution of the differential linear equation involving both D^{α} and $\mathrm{D}^{\alpha\alpha}$

Formally, with the above remark, the equation

$$a(x)^{(2\alpha)}(t) + b(x)^{(\alpha)}(t) + c = 0, \tag{52}$$

can be re-written in the form

$$a(\mathrm{D}^\alpha - r_1)(\mathrm{D}^\alpha - r_2) = 0, \tag{53}$$

where r_1 and r_2 are the solutions of the equation

$$ar^2 + br + c = 0. \tag{54}$$

An alternative is to say that if we seek a solution in the form $x = E_\alpha(rt^\alpha)$, then on substituting into (52), we find that r is provided by the equation (54).

When $r_1 \neq r_2$, then the general solution can be written as

$$x(t) = C_1 E_\alpha\big(r_1 t^\alpha\big) + C_2 E_\alpha\big(r_2 t^\alpha\big), \tag{55}$$

where C_1 and C_2 are two constants which depend upon the initial conditions.

When the equation (54) has only one solution $\widehat{r} = r_1 = r_2$, the solution is

$$x(t) = \big(C_1 + C_2 t^\alpha\big) E_\alpha\big(\widehat{r} t^\alpha\big). \tag{56}$$

6 A class of fractional linear differential vector equations

We consider the following vector fractional linear differential equation, written in the matrix form,

$$X^{(\alpha)}(t) = a(t)AX, \tag{57}$$

with the following notations. $X(t) = (x_1(t), x_2(t))^T$ is a vector in $\Re^n$ (here $n = 2$ for convenience), $a(t)$ is a sufficiently regular scalar valued function, and A is a constant $n \times n$-matrix (here $n = 2$) with the eigen-values r_1 and r_2, with the supplementary condition $r_1 \neq r_2$. Our purpose in the following is to determine the solution of (57), and to this end, we shall proceed as follows.

Step 1. According to the basic of matrix calculus, we can write A in the form

$$A = VRV^{-1}, \tag{58}$$

where R is the diagonal matrix $\mathrm{diag}(r_1, r_2)$, and where V is the matrix the columns of which are the eigenvectors v_1 and v_2 of A.

Step 2. Using (58), we can then re-write (57) as

$$X^{(\alpha)}(t) = a(t)VRV^{-1}X(t)$$

or, what amounts to the same,

$$V^{-1}X^{(\alpha)}(t) = a(t)RV^{-1}X(t). \tag{59}$$

On making the change of variable $V^{-1}X \leftarrow Y$, (59) turns to be the vector equation

$$Y^{(\alpha)}(t) = a(t)RY(t),$$

which can be split in the two one-dimensional equations

$$y_i^{(\alpha)}(t) = a(t)r_i y_i(t), \quad i = 1, 2, \tag{60}$$

of which the solutions are respectively

$$y_i(t) = y_i(0)E_\alpha\left(r_i \int_0^t a(\tau)(\mathrm{d}\tau)^\alpha\right), \quad i = 1, 2. \tag{61}$$

In the special case when $a(t) = t^{1-\alpha}$, one has

$$\int_0^t \tau^{1-\alpha}(\mathrm{d}\tau)^\alpha = \alpha!(1-\alpha)!t$$

and

$$y_i(t) = y_i(0)E_\alpha\big(\alpha!(1-\alpha)!r_i t\big). \tag{62}$$

7 Application to dynamics close to equilibrium position

7.1 One-dimensional systems subject to fractional disturbances

The present section displays two illustrative examples which illustrate the kind f results one may so expect t obtain in this framework.

We consider the scalar-valued dynamics

$$\mathrm{d}x = f(x)\,\mathrm{d}t, \tag{63}$$

with the equilibrium position x_0, i.e. $f(x_0) = 0$ defined as the solution, when it exists, of the equation $f(x_0) = 0$. Close to this equilibrium position, one has the variation equation

$$\mathrm{d}(\delta x) = f_x(x_0)\delta x\,\mathrm{d}t. \tag{64}$$

We assume that this equation is disturbed by an external input $w(t)(\mathrm{d}t)^\alpha$, $0 < \alpha < 1$, in such a manner that it turns to be

$$\mathrm{d}(\delta x) = f_x(x_0)\delta x\,\mathrm{d}t + w(t)(\mathrm{d}t)^\alpha, \quad 0 < \alpha < 1. \tag{65}$$

$w(\mathrm{d}t)(\mathrm{d}t)^\alpha$ is an approach via Maruyama's formula (see for instance [51]) to modelling fractional white noise, *but we can drop these considerations here for the moment*, and consider the equation (65) on a formal standpoint only. For convenience we set $y := \delta x$, so that (65) now reads

$$\mathrm{d}y = f_x(x_0)y\,\mathrm{d}t + w(t)(\mathrm{d}t)^\alpha, \quad 0 < \alpha < 1. \tag{66}$$

In order to get the solution of (66) we write $y(t)$ in the form $y(t) = y_1(t)y_2(t)$, which we substitute into (66) to obtain the two equations

$$\mathrm{d}y_1 = f_x(x_0)y_1\,\mathrm{d}t, \tag{67}$$

$$\mathrm{d}y_2 = (y_1)^{-1}w(t)(\mathrm{d}t)^\alpha. \tag{68}$$

The equation (67) direct yields

$$y_1(t) = y_1(0)\exp\big(f_x(x_0)t\big). \tag{69}$$

In order to obtain the solution of (68), we multiply its both sides by $\alpha!$ and we notice that $\alpha!\,\mathrm{d}y_2 = \mathrm{d}^\alpha y_2$ to have

$$y_2^{(\alpha)}(t) = \alpha!\frac{w(t)}{y_1(t)},$$

therefore the expression

$$y_2(t) = y_2(0)\alpha!\int_0^t \frac{w(t)}{y_1(\tau)}(\mathrm{d}\tau)^\alpha \tag{70}$$

which is taken in the sense of (20).

7.2 Equilibrium position of a two-dimensional systems

We consider a theoretical two-dimensional system defined by the equations

$$\dot{x}(t) = f(x, y, u), \tag{71}$$
$$\dot{y}(t) = g(x, y, u), \tag{72}$$

with the equilibrium position (x_0, y_0, u_0) and we assume that, due to coarse-graining effect in time, the actual dynamics turns to be

$$x^{(\alpha)}(t) = f(x, y, u), \tag{73}$$
$$y^{(\alpha)}(t) = g(x, y, u). \tag{74}$$

The dynamical equations of the small deviation $(\widetilde{x}, \widetilde{y}, \widetilde{u})$ from the equilibrium position are

$$\widetilde{x}^{(\alpha)}(t) = a\widetilde{x} + b\widetilde{y} + c\widetilde{u}, \tag{75}$$
$$\widetilde{y}^{(\alpha)}(t) = a'\widetilde{x} + b'\widetilde{y} + c'\widetilde{u}, \tag{76}$$

with

$$a := f_x(x_0, y_0, u_0), \quad b := f_y \quad c := f_u, \tag{77}$$

and

$$a' := g_x, \quad b' := g_y(x_0, y_0, u_0), \quad c' := g_u. \tag{78}$$

In order to analyze the stability of the equilibrium, we shall assume that the perturbation $\widetilde{u}$ on the control parameter is zero, and according to Section 6, we refer to the equation of the eigenvalues of the system, which reads

$$r^2 - (a + b')r + (ab' - ba') = 0$$

with

$$a + b' := f_x + g_y, \tag{79}$$
$$ab' - a'b = f_x g_y - f_y g_x. \tag{80}$$

Example 1. We consider a particle with the mass m and subject to the potential function $V(x)$, driven by the dynamical equation

$$m\ddot{x}(t) = -V_x(x), \tag{81}$$

or in the vector form

$$\dot{x}(t) = y, \tag{82}$$
$$\dot{y}(t) = -m^{-1}V_x(x), \tag{83}$$

We further assume that the particle is at rest at x_0, i.e. that the equilibrium position is $(x_0, 0)$. We then have the fractional differential equations

$$x^{(\alpha)}(t) = y(t),$$
$$y^{(\alpha)}(t) = -m^{-1}V_x(x),$$

which provide the deviation equations

$$\widetilde{x}^{(\alpha)} = \widetilde{y}, \tag{84}$$
$$\widetilde{y}^{(\alpha)} = -m^{-1}V_{xx}(x_0)\widetilde{y}. \tag{85}$$

The equation (85) yields

$$\widetilde{y}(t) = \widetilde{y}(0)E_\alpha\big(-m^{-1}V_{xx}(x_0)t^\alpha\big), \tag{86}$$

whilst (84) provides

$$\widetilde{x}(t) - \widetilde{x}(0) = \widetilde{y}(0)\mathrm{D}^{-\alpha}\widetilde{y}(t) = -\frac{m\widetilde{y}(0)}{V_x(x_0)}E_\alpha\big(-m^{-1}V_{xx}(x_0)t^\alpha\big). \tag{87}$$

7.3 Coarse-graining in time and in space. Comparison

We come back to the equation (63), i.e. $\dot{x}(t) = f(x)$, to examine what happens about its equilibrium position x_0 when it is affected by coarse-graining in time (CGT), on the one hand, and by coarse-graining in space (CGS) on the other hand.

Of course, for continuously differentiable deviation, one has the equation

$$\dot{y}_1(t) = f_x(x_0)y_1(t). \tag{88}$$

This being the case, assume that the system (63) is subject to CGT, then $u_\alpha(t)$ is substituted for $\dot{x}(t)$ and inserting (23) into (63) direct yields the modeling

$$x^{(\alpha)}(t) = (\alpha!)f(x),$$

therefore the deviation equation

$$y_2^{(\alpha)}(t) = (\alpha!)f_x(x_0)y_2(t). \tag{89}$$

Assume now that the system is subject to CGS, then it is $v_\alpha(t)$ which is substituted to $\dot{x}(t)$, and on inserting (28), we obtain the dynamical equation

$$x^{(\alpha)}(t) = \rho^{-1}(\alpha)(xt)^{1-\alpha}f(x),$$

therefore the deviation equation

$$y_3^{(\alpha)}(t) = \rho^{-1}t^{1-\alpha}\big((1-\alpha)x_0^{-\alpha}f(x_0) + x_0^{1-\alpha}f_x(x_0)\big)y_3, \tag{90}$$

which suggests that $x_0 = 0$ should be considered as a special value.

These equations yield the expressions

$$y_1(t) = y_1(0)\exp\big(f_x(x_0)t\big), \tag{91}$$

$$y_2(t) = y_2(0)E_\alpha\big((\alpha!)f_x(x_0)t^\alpha\big), \tag{92}$$

$$y_3(t) = y_3(0)E_\alpha\left(\frac{(1-\alpha)x_0^{-\alpha}f(x_0) + x_0^{1-\alpha}f_x(x_0)}{(1-\alpha)!}\,t\right). \tag{93}$$

which allow us to compare the three phenomena.

8 Concluding remarks

In the present article, we have shown how fractional derivative can be used to analyze dynamical systems involving coarse-grained time, and we focused mainly on the oscillations of nonlinear systems close to their equilibrium positions. But of course, the approach could be applied to various other problems, provided that they involve a linearization technique.

For instance, let us consider the coarse-grained time dynamical system

$$x^{(\alpha)}(t) = f(x, u, t),$$

where $f(.)$ is a nonlinear function and $u(t)$ is an external control. Assume that the special control function $u_0(t)$ is selected, to ensure that the system tracks the corresponding trajectory $x_0(t),\ x_0^{(\alpha)}(t) = f(x_0, u_0, t)$.

According to the classical approach to controlling nonlinear dynamics via the linearization technique, we will consider the system

$$y_0^{(\alpha)}(t) = f_x^{(\alpha)}(x_0, u_0, t)y + f_u^{(\alpha)}(x_0, u_0, t)v,$$

in which $v(t)$ is selected in order to have that $y(t)$ tends to zero as time increases.

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