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Pseudo Almost Periodic Sequence Solutions of Discrete Time Cellular Neural Networks

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Abstract. In this paper we discuss the existence and uniqueness of a k-pseudo almost periodic sequence solutions of a discrete time neural network. We give several sufficient conditions for the exponential and global attractivity of the solution.

Keywords: pseudo almost periodic function, almost periodic sequence, cellular neural network, discrete time model.

1 Introduction

The theory of almost periodic functions was introduced by Harald Bohr during 1924–1926. In his study of Dirichlet series he developed the notion of a uniformly almost periodic functions. Later, Bochner extended Bohr's theory to general abstract spaces. Almost periodic functions have been widely treated by Favard, Levitan [1] and Besicovich [2] in their monographs. Amerio [3] extended certain results of Favard and Bochner to differential equations in abstract spaces. The concept of pseudo almost periodicity is a natural genelization of almost periodicity. The theory of pseudo almost periodicity was first treated by Zhang [4] around 1990. The existence and uniqueness of pseudo almost periodic solutions of differential equations have been of great interest to mathematicians in the past few decades.

A cellular neural network is a nonlinear dynamic circuit consisting of many processing units called cells arranged in two or three dimensional array. This is very useful in the areas of signal processing, image processing, pattern classification and associative memories. Hence the application of cellular networks is of great interest to many researchers ([5–8] have dealt with the global exponential stability and the existence of a periodic solution of a cellular neural network with delays using the general Lyapunov functional). Many authors have established the almost periodic solutions of cellular neural networks ([9, 10] and references cited therein). The discrete analogues of continuous time cellular network models are very important for theoretical analysis as well as for implementation.

Thus it is essential to formulate a discrete time analogue of continuous time network. A most acceptable method is to discretize the continuous time network. For detailed analysis on the discretization method the reader may consult Mohamad [8], Stewart [11], Broomhead and Iserles [12]. Huang et al. [13] have considered the following model of neural network with piecewise constant arguments

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i([t])x_i(t) + \sum_{j=1}^{m} b_{ij}([t])f_j\big(x_j([t])\big) + I_i([t]).$$

The authors have proved the existence of an almost periodic sequence solution for the following discrete time analogue

$$x_i(n+1) = x_i(n)e^{-a_i(n)} + \frac{1-e^{-a_i(n)}}{a_i(n)}\left\{\sum_{j=1}^{m} b_{ij}(n)f_j\big(x_j(n)\big) + I_i(n)\right\}.$$

Huang, Xia and Wang in [14] have considered the following network model with the piecewise constant arguments in the following equation,

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j\left(x_j\left(\left[\frac{t}{k}\right]k\right)\right) + I_i(t), \tag{1}$$

where $i = 1, 2, \ldots, m$. The authors in [14] have proved the existence and uniqueness of a k-almost periodic sequence solution of the discrete analogue of (1), and also shown the exponential attractivity of the solution.

In this paper we study the problem of existence, uniqueness and exponential attractivity of a k-pseudo almost periodic solution of the following differential equation,

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^{m} c_{ij}(t)f_j\big(x_j(t-\tau_{ij})\big) + I_i(t), \tag{2}$$

where $x_i(t)$ the potential of the cell i at time t, f_i is the nonlinear output function, b_{ij} and c_{ij} denote the strengths of connectivity between the cells i and j at the instants t and $t - \tau_{ij}$, respectively. We have τ_{ij} the time delay required in processing and transmitting a signal from j-th cell to the i-th cell. We denote the i-th component of an external input source from outside the network to the cell i by I_i.

2 Preliminaries

We consider a continuous time neural network consisting of m interconnected cells described by the following system of delay differential equations

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) &+ \sum_{j=1}^{m} b_{ij}(t)f_j\big(x_j(t)\big) \\ &+ \sum_{j=1}^{m} c_{ij}(t)f_j\big(x_j(t-\tau_{ij})\big) + I_i(t)\end{aligned} \tag{3}$$

for $i \in \{1, 2, \dots, m\}$, and $t > 0$. A discrete analogue of (3) can be written as,

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\left(x_j\left(\left[\frac{t}{k}\right]k\right)\right) + \sum_{j=1}^{m} c_{ij}(t) f_j\left(x_j\left(\left[\frac{t}{k}\right]k - \left[\frac{\tau_{ij}}{k}\right]k\right)\right) + I_i(t) \tag{4}$$

for $i = 1, 2, \dots, m$, and $[\cdot]$ denotes the greatest integer function and $k > 0$ the transmission step size. Throughout the paper we impose the following conditions,

Assumptions:

(A1) $a_i(t) > 0$, $b_{ij}(t)$, $c_{ij}(t)$ and $I_i(t)$ are pseudo almost periodic functions for $i, j = 1, 2, \dots, m$.

(A2) There exist positive constants M_j and L_i such that $|f_j(x)| \leq M_j$ and $|f_i(x) - f_i(y)| \leq L_i|x - y|$ for each $x, y \in \mathbb{R}$ and $j = 1, 2, \dots, m; i = 1, 2, \dots, m$.

Definition 1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if for each $\epsilon > 0$,

$$T(f, \epsilon) = \{\tau \in \mathbb{R} : |f(t + \tau) - f(t)| < \epsilon, \text{ for all } t \in \mathbb{R}\} \tag{5}$$

is relatively dense in $\mathbb{R}$. That is there exists a positive number l_ϵ such that any interval of the length l_ϵ contains at least one point of $T(f, \epsilon)$.

The set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$ are denoted by AP.

Denote

$$AP_0 = \left\{ f \in BC(\mathbb{R}, \mathbb{R}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |f(t)| \, \mathrm{d}t = 0 \right\}.$$

Definition 2. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be pseudo almost periodic if it can be written as $f = f_1 + f_2$, where $f_1 \in AP$ and $f_2 \in AP_0$.

The set of all such functions is denoted by PAP. Now we have a similar definition for pseudo almost periodic sequence.

Definition 3. A real sequence $x : \mathbb{Z} \to \mathbb{R}$ is called almost periodic sequence if for each $\epsilon > 0$,

$$\hat{T}(x, \epsilon) = \{\tau \in \mathbb{Z} : |x(n + \tau) - x(n)| < \epsilon \text{ for all } n \in \mathbb{Z}\}$$

is relatively dense set in $\mathbb{Z}$. That is there exists a positive integer l_ϵ such that any interval with the length l_ϵ contains at least one point of $\hat{T}(x, \epsilon)$.

The class of all almost periodic sequences is denoted by APS.

Definition 4. A real sequence $x\colon \mathbb{Z} \to \mathbb{R}$ is said to be in AP_0S if

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} |x(i)| = 0.$$

Definition 5. A real sequence $x\colon \mathbb{Z} \to \mathbb{R}$ is said to be pseudo almost periodic sequence if it can be written as $x = x_1 + x_2$, where $x_1 \in APS$ and $x_2 \in AP_0S$.

The set of all such sequences is denoted by $PAPS$.

Definition 6. A real sequence $x : k\mathbb{Z} \to \mathbb{R}$ is called k-almost periodic sequence if for each $\epsilon > 0$,

$$T(x,\epsilon) = \{\tau \in k\mathbb{Z}\colon |x(\nu+\tau) - x(\nu)| < \epsilon \text{ for all } \nu \in k\mathbb{Z}\}$$

is relatively dense set in $k\mathbb{Z}$. That is there exists a positive integer l_ϵ such that any integer interval with length l_ϵ contains at least one point of $T(x,\epsilon)$.

The set of all such sequences is denoted by APS_k.

Definition 7. A real sequence $x\colon k\mathbb{Z} \to \mathbb{R}$ is said to be in AP_0S_k if

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} |x(i)| = 0.$$

Definition 8. A real sequence $x\colon k\mathbb{Z} \to \mathbb{R}$ is said to be k-pseudo almost periodic sequence if it can be written as $x = x_1 + x_2$, where $x_1 \in APS_k$ and $x_2 \in AP_0S_k$.

The set of all such sequences is denoted by $PAPS_k$.

Denote $PAPS_k^m$ the set of all $x = (x_1, x_2, \dots, x_m)$ in which every component is k-pseudo almost periodic sequence, that is $x_i \in PAPS_k$ for $i = 1, 2, \dots, m$.

Using the discretization scheme one can have the following difference equations for equation (4),

$$\begin{aligned} x_i\big((n+1)k\big) &= x_i(nk) e^{-\int_{nk}^{(n+1)k} a_i(u)\,\mathrm{d}u} + \int_{nk}^{(n+1)k} \Bigg(\sum_{j=1}^{m} b_{ij}(s) f_j\big(x_j(nk)\big) \\ &\quad + \sum_{j=1}^{m} c_{ij}(s) f_j\big(x_j(n-k_{ij})k\big) + I_i(s) \Bigg) e^{-\int_{s}^{(n+1)k} a_i(u)\,\mathrm{d}u} \,\mathrm{d}s, \end{aligned} \tag{6}$$

where $i = 1, 2, \dots, m$ and $n \in \mathbb{Z}$. Define the followings,

$$\begin{aligned} C_i(n) &= e^{-\int\limits_{nk}^{(n+1)k} a_i(u)\,\mathrm{d}u}, \\ D_{ij}(n) &= \int\limits_{nk}^{(n+1)k} b_{ij}(s) e^{-\int\limits_{s}^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s, \\ E_{ij}(n) &= \int\limits_{nk}^{(n+1)k} c_{ij}(s) e^{-\int\limits_{s}^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s, \\ F_i(n) &= \int\limits_{nk}^{(n+1)k} I_i(s) e^{-\int\limits_{s}^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s. \end{aligned} \tag{7}$$

Fixing the k_{ij} as k^* and using above notations, equation (6) can be written as,

$$\begin{aligned} x_i^k(n+1) = C_i(n) x_i^k(n) &+ \sum_{j=1}^{m} D_{ij}(n) f_j\big(x_i^k(n)\big) \\ &+ \sum_{j=1}^{m} E_{ij}(n) f_j\big(x_i^k(n-k^*)\big) + F_i(n), \end{aligned} \tag{8}$$

for $i = 1, 2, \dots, m$, where $x_i^k(n) = x_i(nk)$.

Denote:

$$\begin{aligned} C_i^* &= \sup_{n\in\mathbb{Z}} |C_i(n)|, \quad & I_i^* &= \sup_{t\in\mathbb{R}} |I_i(t)|, \\ D_{ij}^* &= \sup_{n\in\mathbb{Z}} |D_{ij}(n)|, \quad & E_{ij}^* &= \sup_{n\in\mathbb{Z}} |E_{ij}(n)|, \\ F_i^* &= \sup_{n\in\mathbb{Z}} |F_i(n)|, \quad & a_i^* &= \inf_{t\in\mathbb{R}} a_i(t), \\ b_{ij}^* &= \sup_{t\in\mathbb{R}} |b_{ij}(t)|, \quad & c_{ij}^* &= \sup_{t\in\mathbb{R}} |c_{ij}(t)|, \\ P_i &= \sum_{j=1}^{m} \big(D_{ij}^* + E_{ij}^*\big) M_j + F_i^*. \end{aligned}$$

Definition 9. A solution $x(\nu) = (x_1(\nu), \dots, x_m(\nu))^T$ of (8) is said to be globally attractive if for any other solution $y(\nu) = (y_1(\nu), \dots, y_m(\nu))^T$ of (8), we have

$$\lim_{\nu\to\infty} |x_i(\nu) - y_i(\nu)| = 0.$$

The following Lemmas are easy to verify:

Lemma 1. *Suppose that $g_i \in AP$ for $i = 1, 2$, then $T_k(g_1, \epsilon) \cap T_k(g_2, \epsilon), T_{k\mathbb{Z}}(g_1, \epsilon)$ and $T_{k\mathbb{Z}}(g_1, \epsilon) \cap T_{k\mathbb{Z}}(g_2, \epsilon)$ are relatively dense, where*

$$T_k(g_i, \epsilon) = \{\tau \in \mathbb{R};\ |g_i(t + k\tau) - g_i(t)| < \epsilon \ \text{for all} \ t \in \mathbb{R}\}$$

and

$$T_{k\mathbb{Z}}(g_i, \epsilon) = \{\tau \in \mathbb{Z};\ |g_i(t + k\tau) - g_i(t)| < \epsilon \ \text{for all} \ t \in \mathbb{R}\}.$$

If $g_i \in PAPS$ for $i = 1, 2$, then we can decompose g_i in two components g_{i1} and g_{i2}, first one is almost periodic sequence and other is in AP_0S.

Lemma 2. *Suppose that $g_i \in PAPS$ for $i = 1, 2$, then $g_i|k\mathbb{Z} \in PAPS_k$, that is for all $\epsilon > 0$,*

$$T(g_{i1}, \epsilon) = \{\tau \in k\mathbb{Z}\colon |g_{i1}(\nu + \tau) - g_{i1}(\nu)| < \epsilon \ \text{for all} \ \nu \in k\mathbb{Z}\}$$

is relatively dense set in $k\mathbb{Z}$ and

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{\nu=-n}^{n} |g_{i2}(\nu)| = 0.$$

Also $T(g_{11}, \epsilon) \cap T(g_{21}, \epsilon)$ is relatively dense set in $k\mathbb{Z}$.

For $x \in PAPS_k$, denoting $x^k(n) = x(nk)$ for all $n \in \mathbb{Z}$, the following properties are true for sequence x^k:

- $x \in PAPS_k$ if and only if $x^k \in PAPS$.
- $f \in PAP$, then $f|\Gamma_k \in PAPS_k$.
- Any $x \in PAPS_k$ is bounded.

3 Pseudo almost periodic solutions

Lemma 3. *Suppose assumption* (A1) *holds, then $C_i, D_{ij}, E_{ij}, F_i \in PAPS$ for $i, j = 1, 2, \ldots, m$.*

Proof. Because a_i is pseudo almost periodic, one have $a_i = a_{i1} + a_{i2}$ where $a_{i1} \in AP$ and $a_{i2} \in AP_0$. For any $\tau \in Z$, we have

$$\int_{(n+\tau)k}^{(n+\tau+1)k} a_{i1}(s)\,\mathrm{d}s - \int_{nk}^{(n+1)k} a_{i1}(s)\,\mathrm{d}s = \int_{nk}^{(n+1)k} \big(a_{i1}(s + k\tau) - a_{i1}(s)\big)\,\mathrm{d}s. \tag{9}$$

a_{i1} is almost periodic so given $\epsilon > 0, i = 1, 2, \ldots, m$, we have the set

$$T_{kZ}(a_{i1}, \epsilon) = \{\tau \in Z;\ |a_{i1}(t + k\tau) - a_{i1}(t)| < \epsilon \ \text{for each} \ t \in \mathbb{R}\}$$

is relatively dense. Next, one can easily observe that

$$\sum_{l=-n}^{n}\left|\int_{lk}^{(l+1)k} a_{i2}(s)\,\mathrm{d}s\right| \le \sum_{l=-n}^{n}\int_{lk}^{(l+1)k} |a_{i2}(s)|\,\mathrm{d}s \le \int_{-nk}^{nk}|a_{i2}(s)|\,\mathrm{d}s+\int_{nk}^{(n+1)k}|a_{i2}(s)|\,\mathrm{d}s$$
$$\le I_1 + I_2. \tag{10}$$

As we know that $a_{i2} \in AP_0$, we have

$$\lim_{n\to\infty}\frac{k}{2nk}\int_{-nk}^{nk}|a_{i2}(s)|\,\mathrm{d}s = 0.$$

Also because a_{i2} is bounded, we get

$$\lim_{n\to\infty}\frac{1}{2n}\int_{nk}^{(n+1)k}|a_{i2}(s)|\,\mathrm{d}s = 0.$$

Combining these two one have

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{l=-n}^{n}\left|\int_{lk}^{(l+1)k} a_{i2}(s)\,\mathrm{d}s\right| = 0.$$

Thus we have $\int_{nk}^{(n+1)k} a_i(s)\,\mathrm{d}s \in PAPS_k$.

Because $I_i \in PAPS$, thus we have $I_i = I_{i1} + I_{i2}$, where $I_{i1} \in APS$ and $I_{i2} \in AP_0S$. For $I_{i1} \in APS$, we have for any $t \in Z$,

$$\begin{aligned}
&|F_{i1}(n+\tau) - F_{i1}(n)| \\
&\quad= \left|\int_{(n+\tau)k}^{(n+\tau+1)k} I_{i1}(s)e^{-\int_s^{(n+\tau+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s - \int_{nk}^{(n+1)k} I_{i1}(s)e^{-\int_s^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s\right| \\
&\quad\le \int_{nk}^{(n+1)k}|I_{i1}(s+k\tau) - I_{i1}(s)|\,\mathrm{d}s \\
&\qquad+ \int_{nk}^{(n+1)k}|I_{i1}(s)|\left(\int_s^{(n+1)k}|a_{i1}(u+k\tau) - a_{i1}(u)|\,\mathrm{d}u\right)\mathrm{d}s \le \epsilon,
\end{aligned} \tag{11}$$

because of almost periodicity of $I_{i1}(t)$ and $a_{i1}(t)$. Now we show that $F_{i2} \in AP_0S$. Consider

$$\sum_{l=-n}^{n} |F_{i2}(l)| = \sum_{l=-n}^{n} \left| \int_{lk}^{(l+1)k} I_{i2}(s) e^{-\int_{s}^{(l+1)k} a_i(u)\,\mathrm{d}u} \,\mathrm{d}s \right|$$
$$\leq \sum_{l=-n}^{n} v \int_{lk}^{(l+1)k} |I_{i2}(s)| \left| e^{-\int_{s}^{(l+1)k} a_i(u)\,\mathrm{d}u} \right| \mathrm{d}s \leq \sum_{l=-n}^{n} \int_{lk}^{(l+1)k} |I_{i2}(s)| \,\mathrm{d}s$$
$$= \int_{-nk}^{nk} |I_{i2}(s)| \,\mathrm{d}s + \int_{nk}^{(n+1)k} |I_{i2}(s)| \,\mathrm{d}s \tag{12}$$

Thus we get the following

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{l=-n}^{n} |F_{i2}(l)| = 0.$$

Hence, we conclude that $F_i \in PAPS$. By similar argument one can show that D_{ij} and E_{ij} are pseudo almost periodic. □

Lemma 4. *Under the assumptions* (A1), (A2)*, every solution of* (8) *is bounded.*

Proof. This Lemma is a direct consequence of Theorem 3.1 of [14].

A brief summary is as follows: One can easily observe that the relation

$$C_i(n)x_i^k(n) - R_i \leq x_i^k(n+1) \leq C_i(n)x_i^n(n) + R_i,$$

where $R_i = \sum_{j=1}^{m}(D_{ij}^* + E_{ij}^*) + F_i^*$ holds. Considering the following difference equations

$$\bar{x}_i^k(n+1) = C_i(n)\bar{x}_i^k(n) + R_i,$$

where $\bar{x}_i^k(0) = x_i^k(0)$ one get the following estimate

$$-\left|x_i^k(0)\right| - \frac{R_i}{1-e^{-a_{ik}^*}} \leq x_i^k(n) \leq \left|x_i^k(0)\right| + \frac{R_i}{1-e^{-a_{ik}^*}}.$$

□

Consider the following difference equations

$$x_i^k(n+1) = C_i(n)x_i^k(n) + F_i(n). \tag{13}$$

Lemma 5. *Under assumption* (A1)*, there exists a* k*-pseudo almost periodic sequence solution of* (13)*.*

Proof. Using the induction argument, one obtain

$$
\begin{aligned}
x_i^k(n+1) &= \prod_{k=0}^{n} C_i(n-k)x_i^k(0) + \sum_{l=0}^{n}\prod_{k=n-l+1}^{n} C_i(k)F_i(n-l) \\
&= e^{-\int\limits_0^{(n+1)k} a_i(u)\,\mathrm{d}u} x_i^k(0) + \sum_{m=0}^{\infty}\int\limits_{(n-m)k}^{(n-m+1)k} I_i(s)e^{-\int\limits_s^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s.
\end{aligned}
$$

Consider the sequence

$$
\hat{x}_i^k(n) = \sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k} I_i(s)e^{-\int\limits_s^{nk} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s.
$$

Since

$$
|\hat{x}_i^k(n)| \le \sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k} I_i^* e^{-\int\limits_s^{nk} a_i^*\,\mathrm{d}u}\,\mathrm{d}s = \sum_{m=0}^{\infty}\frac{1-e^{a_i^*k}}{a_i^*}e^{-mka_i^*} \le \frac{I_i^*}{a_i^*}.
$$

Thus the sequence $\hat{x}_i^k(n)$ is well defined. It is easy to verify that

$$
\hat{x}_i^k(n+1) = C_i(n)\hat{x}_i^k(n) + F_i(n).
$$

Hence the sequence $\hat{x}_i^k = \{\hat{x}_i^k(n)\}$ is bounded. Next consider

$$
\begin{aligned}
&\left|\hat{x}_i^k(n+\tau) - \hat{x}_i^k(n)\right| \\
&= \left|\sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k}\left[I_i(s+k\tau)e^{-\int\limits_s^{nk} a_i(u+k\tau)\,\mathrm{d}u} - I_i(s)e^{-\int\limits_s^{nk} a_i(u)\,\mathrm{d}u}\right]\mathrm{d}s\right| \\
&\le \sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k}\Bigg(|I_i(s+k\tau) - I_i(s)|e^{-\int\limits_s^{nk} a_i^*\,\mathrm{d}u} \\
&\qquad\qquad + I_i^*\left|e^{-\int\limits_s^{nk} a_i(u+k\tau)\,\mathrm{d}u} - e^{-\int\limits_s^{nk} a_i(u)\,\mathrm{d}u}\right|\Bigg)\mathrm{d}s \\
&\le \epsilon\sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k} e^{-a_i^*(nk-s)}\,\mathrm{d}s + I_i^*\sum_{m=0}^{\infty}\int\limits_{(n-m-1)k}^{(n-m)k} e^{-\theta\int\limits_s^{nk} a_i(u+\tau^*)\,\mathrm{d}u-(1-\theta)\int\limits_s^{nk} a_i(u)\,\mathrm{d}u} \\
&\qquad\qquad \times \int\limits_s^{nk} |a_i(u+\tau^*) - a_i(u)|\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\left|\hat{x}_i^k(n+\tau) - \hat{x}_i^k(n)\right| \\
&\quad \leq \epsilon \sum_{m=0}^{\infty} \frac{1 - e^{-a_i^* k}}{a_i^*} e^{-a_i^* mk} + I_i^* \epsilon \sum_{m=0}^{\infty} \int\limits_{(n-m-1)k}^{(n-m)k} (nk - s) e^{a_i^*(s-nk)}\,\mathrm{d}s \\
&\quad \leq \epsilon \frac{1}{a_i^*} + \epsilon I_i^* \sum_{m=0}^{\infty} \int\limits_{mk}^{(m+1)k} s e^{-a_i^* s}\,\mathrm{d}s \leq \frac{1}{a_i^*}\epsilon + \frac{I_i^*}{{a_i^*}^2}\epsilon, \quad 0 < \theta < 1. \qquad (14)
\end{aligned}$$

Thus $\hat{x}_i^k$ is a k almost periodic sequence if I_i and a_i are k almost periodic sequences. That is for I_{i1} and a_{i1} this is an almost periodic sequence, denote it by $\hat{x}_{i1}^k$.

Consider

$$\begin{aligned}
\sum_{l=-n}^{n} \hat{x}_i^k(l) &= \sum_{l=-n}^{n} \sum_{m=0}^{\infty} \int\limits_{(l-m-1)k}^{(l-m)k} I_i(s) e^{-\int_s^{lk} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s \\
&\leq \sum_{l=-n}^{n} \sum_{m=0}^{\infty} \int\limits_{(l-m-1)k}^{(l-m)k} I_i(s) e^{-a_i^*(lk-s)}\,\mathrm{d}s. \qquad (15)
\end{aligned}$$

Now we can easily observe that

$$lk - s \geq (l - l + m)k = mk \geq 0, \quad s \in [(l-m-1)k, (l-m)k].$$

Thus above inequality becomes

$$\begin{aligned}
\sum_{l=-n}^{n} \hat{x}_i^k(l) &\leq \sum_{l=-n}^{n} \sum_{m=0}^{\infty} \int\limits_{(l-m-1)k}^{(l-m)k} I_i(s)\,\mathrm{d}s \\
&\leq \int\limits_{(-n-1)k}^{nk} I_i(s)\,\mathrm{d}s + \int\limits_{(-n-2)k}^{(n-1)k} I_i(s)\,\mathrm{d}s + \dots \\
&\leq \int\limits_{-nk}^{nk} I_i(s)\,\mathrm{d}s + \int\limits_{(-n-1)k}^{-nk} I_i(s)\,\mathrm{d}s + \dots. \qquad (16)
\end{aligned}$$

For I_{i2} we have

$$\sum_{l=-n}^{n} |\hat{x}_{i2}^k(l)| \leq \int\limits_{-nk}^{nk} |I_{i2}(s)|\,\mathrm{d}s + \int\limits_{(-n-1)k}^{-nk} |I_{i2}(s)|\,\mathrm{d}s + \dots.$$

As we know that $I_{i2} \in AP_0S$ and bounded, we get

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-nk}^{nk} |I_{i2}(s)| \,\mathrm{d}s = 0$$

and

$$\lim_{n\to\infty} \frac{1}{2n} \int_{(-n-1)k}^{-nk} |I_{i2}(s)| \,\mathrm{d}s \le \lim_{n\to\infty} \frac{k}{2n} \|I_{i2}\| = 0.$$

One can easily observe that every finite sum of these integral when divided by $2n$ and passing limit as $n \to \infty$ is zero. Thus we have

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{l=-n}^{n} \left|\hat{x}_{i2}^{k}(l)\right| = 0.$$

Therefore we get that $\hat{x}_i = \hat{x}_{i1} + \hat{x}_{i2}$ is a k-pseudo almost periodic sequence solution of equation (13). □

Theorem 1. *Suppose assumptions* (A1), (A2) *holds. There exists a unique k-pseudo almost periodic sequence solution of* (8) *which is globally attractive, if*

$$\max_{1\le i\le m} \left\{ C_i^* + \sum_{j=1}^{m} (D_{ij}^* + E_{ij}^*) L_j \right\} < 1.$$

Proof. Denote a metric $d\colon PAPS_k^m \times PAPS_k^m \to \mathbb{R}^+$, by

$$d(x,y) = \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \left| x_i^k(n) - y_i^k(n) \right|.$$

Now define a mapping $F\colon PAPS_k^m \to PAPS_k^m$ by $Fx = y$, where

$$Fx = (F_1x, F_2x, \ldots, F_mx)^T$$

such that $F_ix = y_i$ and $y_i = \{y_i^k(n)\}$. Define

$$y_i^k(n+1) = C_i(n)\hat{x}_i^k(n) + \sum_{j=1}^{m} \left[D_{ij}(n) f_j\big(x_j^k(n)\big) + E_{ij}(n) f_j\big(x_j^k(n-k^*)\big) \right] + F_i(n),$$

where $\hat{x}_i^k$ is k-pseudo almost periodic sequence solution of (13). Using Lemma 3 and assumption (A2), one can observe that F maps k-pseudo almost periodic sequences into k-pseudo almost periodic sequences. Now denote

$$\max_{1\le i\le m} \left\{ C_i^* + \sum_{j=1}^{m} \big(D_{ij}^* + E_{ij}^*\big) L_j \right\} = r, \quad \max_{1\le i\le m} \frac{I_i^*}{a_i^*} = \gamma$$

and

$$\|x - \hat{x}\| = \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \left|x_i^k(n) - \hat{x}_i^k(n)\right|.$$

Define the set

$$BA_k^* = \left\{x \in PAPS_k^m \colon \|x - \hat{x}\| \le \frac{\gamma(1+r)}{1-r}\right\}.$$

Note that for $x \in BA_k^*$,

$$\|x\| \le \|x - \hat{x}\| + \|\hat{x}\| \le \gamma + \frac{\gamma(1+r)}{1-r} = \frac{2\gamma}{1-r}.$$

Also we have,

$$\begin{aligned}
\|Fx - \hat{x}\| &\le \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \left|y_i^k(n+1) - \hat{x}_i^k(n+1)\right| \\
&\le \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \Bigg| C_i(n)\big(x_i^k(n) - \hat{x}_i^k(n)\big) \\
&\qquad + \sum_{j=1}^m \left[D_{ij}(n) f_j\big(x_j^k(n)\big) + E_{ij}(n) f_j\big(x_j^k(n-k^*)\big)\right] \Bigg| \\
&\le \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \left|C_i(n)\hat{x}_i^k(n)\right| + \max_{1\le i\le m} \left(C_i^* + \sum_{j=1}^m \big(D_{ij}^* + E_{ij}^*\big) L_j\right) \|x\| \\
&\le \gamma + r\frac{2\gamma}{1-r} = \frac{\gamma(1+r)}{1-r}.
\end{aligned} \tag{17}$$

Thus we conclude that $Fx \in BA_k^*$. For $x, y \in BA_k^*$, we have

$$\begin{aligned}
\|Fx - Fy\| &= \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \sum_{j=1}^m \Big| \Big[D_{ij}(n)\big(f_j\big(x_j^k(n)\big) - f_j\big(y_j^k(n)\big)\big) \\
&\qquad + E_{ij}(n)\big(f_j\big(x_j^k(n-k^*)\big) - f_j\big(y_j^k(n-k^*)\big)\big)\Big] \Big| \\
&\le \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \sum_{j=1}^m D_{ij}^* L_j \left|x_j^k(n) - y_j^k(n)\right| \\
&\qquad + E_{ij}^* L_j \left|x_j^k(n-k^*) - y_j^k(n-k^*)\right| \\
&\le r\|x - y\|.
\end{aligned} \tag{18}$$

Hence F is a contraction. It follows that equation (8) has a unique k pseudo almost periodic sequence x which satisfies

$$\|x - \hat{x}\| \le \frac{\gamma(1+r)}{1-r}.$$

Let y be any sequence satisfying equation (8). Consider $Q(n) = x^k(n) - y^k(n)$, then we get

$$Q_i(n+1) = C_i(n)Q_i(n) + \sum_{j=1}^{m} D_{ij}(n)\big(f_j\big(x_j^k(n)\big) - f_j\big(y_j^k(n)\big)\big)$$
$$+ E_{ij}(n)\big(f_j\big(x_j^k(n-k^*)\big) - f_j\big(y_j^k(n-k^*)\big)\big). \quad (19)$$

Taking modulus both side one have

$$|Q_i(n+1)| \le C_i^*|Q_i(n)| + \sum_{j=1}^{m} D_{ij}^* L_j |Q_j(n)| + \sum_{j=1}^{m} E_{ij}^* L_j |Q_j(n-k^*)|.$$

Define $Q(n) = \max_{1\le i\le m} |Q_i(n)|$, we have

$$|Q(n+1)| \le C_i^*|Q(n)| + \sum_{j=1}^{m} D_{ij}^* L_j Q(n) + \sum_{j=1}^{m} E_{ij}^* L_j Q(n) \le rQ(n). \quad (20)$$

By induction we have

$$Q(n) \le r^n Q(0).$$

Hence

$$|x_i^k(n) - y_i^k(n)| \to 0 \quad as\ n \to \infty.$$

Thus x is a unique k-pseudo almost periodic sequence solution of (8) which is globally attractive. □

Theorem 2. *Suppose the assumptions* (A1), (A2) *holds. There exists a unique k-pseudo almost periodic sequence solution of* (8) *which is exponentially attractive, if*

$$\max_{1\le i\le m}\left\{e^{-ka_i^*} + \sum_{j=1}^{m} k(b_{ij}^* + c_{ij}^*)L_j\right\} < 1.$$

Proof. Let x be the unique k-pseudo almost periodic sequence solution and y be a arbitrary sequence solution of (8). Denote $\Phi(n) = x^k(n) - y^k(n)$, we get

$$\Phi_i(n+1) = C_i(n)\Phi_i(n) + \sum_{j=1}^{m}\big[D_{ij}(n)\big(f_j\big(x_j^k(n)\big) - f_j\big(y_j^k(n)\big)\big)$$
$$+ E_{ij}(n)\big(f_j\big(x_j^k(n-k_{ij})\big) - f_j\big(y_j^k(n-k_{ij})\big)\big)\big].$$

Let $g_j(\Phi_j(n)) = f_j(x_j^k(n)) - f_j(y_j^k(n))$, then we have

$$\Phi_i(n+1) = C_i(n)\Phi_i(n) + \sum_{j=1}^{m} D_{ij}(n)g_j\big(\Phi_j(n)\big) + E_{ij}(n)g_j\big(\Phi_j(n-k^*)\big).$$

Using induction we have,

$$\begin{aligned}
&\Phi_i(n+1)\\
&= \prod_{k=0}^{n} C_i(k)\Phi_i(0) + \sum_{j=1}^{m}\sum_{l=0}^{n}\prod_{k=n-l+1}^{n} C_i(n)\big[D_{ij}(n-l)g_j\big(\Phi_j(n-l)\big)\\
&\qquad\qquad + E_{ij}(n-l)g_j\big(\Phi_j(n-l-k^*)\big)\big]\\
&= e^{-\int_0^{(n+1)k} a_i(u)\,\mathrm{d}u}\Phi_i(0) + \sum_{j=1}^{m}\sum_{l=0}^{n} e^{-\int_{(n-l+1)k}^{(n+1)k} a_i(u)\,\mathrm{d}u}\big[D_{ij}(n-l)g_j\big(\Phi_j(n-l)\big)\\
&\qquad\qquad + E_{ij}(n-l)g_j\big(\Phi_j(n-l-k^*)\big)\big]\\
&= e^{-\int_0^{(n+1)k} a_i(u)\,\mathrm{d}u}\Phi_i(0)\\
&\quad + \sum_{j=1}^{m}\sum_{l=0}^{n}\Bigg[\Bigg(\int_{(n-l)k}^{(n-l+1)k} b_{ij}(s)e^{-\int_s^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s\Bigg)g_j\big(\Phi_j(n-l)\big)\\
&\qquad + \Bigg(\int_{(n-l)k}^{(n-l+1)k} c_{ij}(s)e^{-\int_s^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s\Bigg)g_j\big(\Phi_j(n-l-k^*)\big)\Bigg]. \qquad (21)
\end{aligned}$$

Taking the norm both side we have,

$$\begin{aligned}
|\Phi_i(n+1)| &\le e^{-\int_0^{(n+1)k} a_i(u)\,\mathrm{d}u}|\Phi_i(0)|\\
&\quad + \sum_{j=1}^{m}\sum_{l=0}^{n}\Bigg[\Bigg(\int_{(n-l)k}^{(n-l+1)k} b_{ij}^* e^{-\int_{(n-l+1)k}^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s\Bigg)L_j|\Phi_j(n-l)|\\
&\qquad + \Bigg(\int_{(n-l)k}^{(n-l+1)k} c_{ij}^* e^{-\int_{(n-l+1)k}^{(n+1)k} a_i(u)\,\mathrm{d}u}\,\mathrm{d}s\Bigg)L_j|\Phi_j(n-l-k^*)|\Bigg]\\
&\le e^{-a_i^*(n+1)k}|\Phi_i(0)| + \sum_{j=1}^{m} kb_{ij}^*L_j\sum_{l=0}^{n} e^{-a_i^* lk}|\Phi_i(n-l)|\\
&\quad + \sum_{j=1}^{m} kc_{ij}^*L_j\sum_{l=0}^{n} e^{-a_i^* lk}|\Phi_i(n-l-k^*)|. \qquad (22)
\end{aligned}$$

Defining

$$a = \min_{1\le i\le m}\{a_i^*\}, \quad \alpha_1 = k\max_{1\le i\le m}\left\{\sum_{j=1}^{m} b_{ij}^*L_j\right\}, \quad \alpha_2 = k\max_{1\le i\le m}\left\{\sum_{j=1}^{m} c_{ij}^*L_j\right\},$$

we get

$$\sum_{i=1}^{m} |\Phi_i(n+1)| \leq e^{-a(n+1)k} \sum_{i=1}^{m} |\Phi_i(0)| + \sum_{i=1}^{m} \sum_{l=0}^{n} \big(\alpha_1 e^{-alk} |\Phi_i(n-l)| + \alpha_2 e^{-alk} |\Phi_i(n-l-k^*)|\big).$$

Assuming that

$$z(n+1) = \sum_{i=1}^{m} |\Phi_i(n+1)|,$$

we have

$$z(n+1) \leq e^{-a(n+1)k} \sum_{i=1}^{m} z(0) + \sum_{l=0}^{n} \big(\alpha_1 e^{-alk} z(n-l) + \alpha_2 e^{-alk} z(n-l-k^*)\big).$$

Define

$$V(n+1) = e^{-a(n+1)k} \sum_{i=1}^{m} V(0) + \sum_{l=0}^{n} \big(\alpha_1 e^{-alk} z(n-l) + \alpha_2 e^{-alk} z(n-l-k^*)\big)$$

and $V(0) = z(0)$ for $n > 0$. It is easy to note that

$$V(n) \geq z(n), \quad V(n) \geq z(n-k^*) \quad \text{for all } \; n \geq 0.$$

Thus we have

$$\begin{aligned} V(n+1) &= e^{-ak} V(n) + \alpha_1 z(n) + \alpha_2 z(n-k^*) \\ &\leq \big(e^{-ak} + \alpha_1 + \alpha_2\big) V(n) \leq \big(e^{-ak} + \alpha_1 + \alpha_2\big)^{n+1} V(0). \end{aligned} \tag{23}$$

Assuming $\zeta_0 = (e^{-ak} + \alpha_1 + \alpha_2)$, one get $z(n+1) \leq \zeta_0^{n+1} z(0)$, finally we have

$$\sum_{i=1}^{m} \big|x_i^k(n) - y_i^k(n)\big| \leq \sum_{i=0}^{m} \big|x_i^k(0) - y_i^k(0)\big| \zeta_0^n.$$

Thus y converges exponentially to the unique k-pseudo almost periodic sequence solution x if

$$\max_{1 \leq i \leq m} \left\{ e^{-ka_i^*} + \sum_{j=1}^{m} k(b_{ij}^* + c_{ij}^*) L_j \right\} < 1.$$

□

For $k = 1$ we can easily observe that the discrete analogue of (3) is as follows,

$$x_i(n+1) = x_i(n)e^{-\int\limits_n^{(n+1)} a_i(u)\,\mathrm{d}u} + \int\limits_n^{(n+1)} \left(\sum_{j=1}^m b_{ij}(s) f_j\big(x_j(n)\big) + \sum_{j=1}^m c_{ij}(s) f_j\big(x_j(n-k^*)\big) + I_i(s) \right) e^{-\int\limits_s^{(n+1)} a_i(u)\,\mathrm{d}u} \,\mathrm{d}s. \tag{24}$$

From Theorem 1 and Theorem 2 it is easy to generalize the results for pseudo almost periodic sequence solution.

Theorem 3. *Suppose* (A1) *and* (A2) *holds. There exists a unique pseudo almost periodic sequence solution of* (24) *which is globally attractive, if*

$$\max_{1\le i\le m} \left\{ e^{-a_i^*} + \sum_{j=1}^m (b_{ij}^* + c_{ij}^*) \frac{(1-e^{-a_i^*})}{a_i^*} L_j \right\} < 1.$$

Theorem 4. *Suppose* (A1) *and* (A2) *holds. There exists a unique pseudo almost periodic sequence solution of* (24) *which is exponentially attractive, if*

$$\max_{1\le i\le m} \left\{ e^{-a_i^*} + \sum_{j=1}^m (b_{ij}^* + c_{ij}^*) L_j \right\} < 1.$$

4 Example

We consider the following pseudo almost periodic cellular neural network,

$$\begin{aligned} \dot{x}_1(t) &= -\left(2 + \sin\sqrt{2}t + \frac{1}{1+t^2}\right) x_1(t) + 0.2\sin t f\big(x_2(t)\big) \\ &\quad + 0.2 f\big(x_1(t-0.1)\big) + 2\sin\sqrt{2}t, \\ \dot{x}_2(t) &= -(4+\cos t)x_2(t) + 0.2\left(\cos t + \cos\sqrt{2}t + \frac{1}{1+t^2}\right) f\big(x_1(t)\big) \\ &\quad + 0.2 f\big(x_2(t-0.1)\big) + 2\cos\sqrt{3}t, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &a_1(t) = 2 + \sin\sqrt{2}t + \frac{1}{(1+t^2)}, \quad a_2(t) = 4 + \cos t, \\ &b_{12} = 0.2\sin t, \quad b_{21} = 0.2\left(\cos t + \cos\sqrt{2}t + \frac{1}{1+t^2}\right), \\ &c_{11} = c_{22} = 0.2, \quad I_1 = 2\sin\sqrt{2}t, \quad 2\cos\sqrt{3}t. \end{aligned}$$

It is easy to verify that these functions are pseudo almost periodic. Consider the following nonlinear activation functions $f_i(x) = f(x) = \tanh(x),\ i = 1,2$. The discrete time analogues are

$$\begin{aligned} x_1\big((n+1)k\big) &= C_1(n)x_1(nk) + D_{12}\tanh x_2(nk) \\ &\qquad + E_{11}\tanh x_1\left(\left(n - \left[\frac{0.1}{k}\right]\right)k\right) + F_1(n), \\ x_2\big((n+1)k\big) &= C_2(n)x_2(nk) + D_{21}\tanh x_1(nk) \\ &\qquad + E_{22}\tanh x_2\left(\left(n - \left[\frac{0.1}{k}\right]\right)k\right) + F_2(n). \end{aligned} \tag{26}$$

We have our $L_i = 1, i = 1,2$, and the constants are as follows,

$$\begin{aligned} C_1(n) &= e^{-\int_{nk}^{(n+1)k}(2+\sin\sqrt{2}t+\frac{1}{(1+t^2)})\,\mathrm{d}t}, \quad C_2(n) = e^{-\int_{nk}^{(n+1)k}(4+\cos t)\,\mathrm{d}t}, \\ D_{12}(n) &= \int_{nk}^{(n+1)k} 0.2\sin s e^{-\int_{s}^{(n+1)k}(2+\sin\sqrt{2}t+\frac{1}{(1+t^2)})\,\mathrm{d}t}\,\mathrm{d}s, \\ E_{11}(n) &= \int_{nk}^{(n+1)k} 0.2 e^{-\int_{s}^{(n+1)k}(2+\sin\sqrt{2}t+\frac{1}{(1+t^2)})\,\mathrm{d}t}\,\mathrm{d}s, \\ F_1(n) &= \int_{nk}^{(n+1)k} 2\sin\sqrt{2}s e^{-\int_{s}^{(n+1)k}(2+\sin\sqrt{2}t+\frac{1}{(1+t^2)})\,\mathrm{d}t}\,\mathrm{d}s, \\ D_{21}(n) &= \int_{nk}^{(n+1)k} 0.2\left(\cos s + \cos\sqrt{2}s + \frac{1}{(1+s^2)}\right)e^{-\int_{s}^{(n+1)k}(4+\cos t)\,\mathrm{d}t}\,\mathrm{d}s, \\ E_{22}(n) &= \int_{nk}^{(n+1)k} 0.2 e^{-\int_{s}^{(n+1)k}(4+\cos t)\,\mathrm{d}t}\,\mathrm{d}s, \\ F_2(n) &= \int_{nk}^{(n+1)k} 2\cos\sqrt{3}s e^{-\int_{s}^{(n+1)k}(4+\cos t)\,\mathrm{d}t}\,\mathrm{d}s. \end{aligned}$$

We also have

$$\begin{aligned} C_1^* &\le e^{-2k}, \quad C_2^* \le e^{-3k}, \\ D_{12}^* &\le \frac{1}{10}\big(1 - e^{-2k}\big), \quad D_{21}^* \le \frac{1}{5}\big(1 - e^{-3k}\big), \\ E_{11}^* &\le \frac{1}{10}\big(1 - e^{-2k}\big), \quad E_{22}^* \le \frac{1}{15}\big(1 - e^{-3k}\big), \end{aligned}$$

$$F_1^* \le \left(1 - e^{-2k}\right), \quad F_2^* \le \frac{2}{3}\left(1 - e^{-3k}\right).$$

We can easily observe that

$$C_1^* + D_{12}^* + E_{11}^* \le 1, \quad C_2^* + D_{21}^* + E_{22}^* \le 1.$$

Thus from Theorem 1, there exists a unique k-pseudo almost periodic sequence solution of (16). Next we calculate

$$e^{-ka_1^*} + k(b_{12} + c_{12}) \le e^{-2k} + k(0.2 + 0.2) = e^{-2k} + 0.4k$$

and

$$e^{-ka_2^*} + k(b_{21} + c_{22}) \le e^{-3k} + k(0.6 + 0.2) = e^{-3k} + 0.8k.$$

Thus

$$\max\left\{e^{-2k} + 0.4k, e^{-3k} + 0.8k\right\} < 1$$

for sufficiently small k. Thus from Theorem 2, there exists a unique exponentially attractive k-pseudo almost periodic sequence solution of the neural network model (26).

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