

Stability and Hopf-Bifurcation Analysis of Delayed BAM Neural Network under Dynamic Thresholds

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Abstract. In this paper the dynamics of a three neuron model with self-connection and distributed delay under dynamical threshold is investigated. With the help of topological degree theory and Homotopy invariance principle existence and uniqueness of equilibrium point are established. The conditions for which the Hopf-bifurcation occurs at the equilibrium are obtained for the weak kernel of the distributed delay. The direction and stability of the bifurcating periodic solutions are determined by the normal form theory and central manifold theorem. Lastly global bifurcation aspect of such periodic solutions is studied. Some numerical simulations for justifying the theoretical analysis are also presented.

Keywords: BAM neural network, distributed delay, Hopf-bifurcation, stability and direction of Hopf-bifurcation, global Hopf-bifurcation.

1 Introduction

In many areas of science, for example biology, population dynamics, neuroscience, economics network with connection delays arise [1–3]. However in neural networks delays occur in the signal transmission between neurons or electronic model neurons due to finite propagation velocity of action potentials (axonal delay) non negligible time of a signal from a neuron to reach the receiving site of a postsynaptic neuron (synaptic delay) and some finite switching speed.

In some artificial neural network information is stored as stable equilibrium points of the system. Retrieval occurs when the system is initialized within the basin of attraction of one of the equilibria and the network is allowed to stabilize in its steady state [1, 4]. Delay may render such networks more versatile [5, 6]. Nevertheless uncontrolled delays may degrade network performance by rendering the equilibria unstable, and makes the retrieval of the corresponding information impossible [7]. Thus delay is an important controlled parameter in neural system. Delays may be discrete or continuous in nature. We have considered here a three neuron BAM network with distributed delay.

In 1980, Hopfield proposed a simplified neuron network model in the development of memory in human [1]. In this model each neuron is represented by a linear circuit consisting of a resistor and a capacitor and is connected to the other neuron via nonlinear sigmoidal activation function. Since then hopfield model have been widely developed and studied both in theory and application including both continuous and discrete time delay.

Since the complexity found in simple models can often be carried over to large scale networks in some way thereby yielding much better understanding of the later from a careful study of the former, most network has focussed on the neural networks where all connection terms have same time delay [8].

Recently many two neuron neural network models with discrete or distributed delay are proposed and their bifurcation, stability properties, have been analysed [8–22].

X. Liao, S. Guo, C. Li [23] considered a simple delayed three neuron network model and obtained sufficient delay dependent criteria to ensure global asymptotic stability of the equilibrium. They also paid attention to the double Hopf-bifurcation associated with resonance. Das Gupta, Majee, and Roy [24] studied stability, bifurcation and global existence of a Hopf-bifurcating periodic solution for a tri neuron Hopfield type general model. They derived sufficient condition for linear stability, instability and occurrence of Hopf-bifurcation with respect to delay parameter about the trivial equilibrium. They also studied asymptotic stability, orbital stability of Hopf-bifurcating periodic solution for a three neuron network with distributed delay [25]. Baldi and Atiya [26] investigated the effect of delays on the dynamics of a n -neuron ring network and discussed its oscillating properties. Recently Wei and Velarde [27] studied stability and other properties of delay induced Hopf-bifurcation for Baldi and Atiya model with three neurons. There are many other research works [28–30] on three neuron network.

Bidirectional associate memorial (BAM) neural networks are a type of network with the neurons arrayed in two layers. Networks with such a bidirectional structure have practical applications in storing paired patterns or memories and possess the ability of searching the desired patterns via both directions: forward and backward [31–35]. A BAM neural network can be described by the following system of ordinary differential equations.

$$\begin{aligned}\dot{x}_i(t) &= -x_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t - \tau_{ij})) + I_i, \quad i = 1, 2, \dots, m, \\ \dot{y}_i(t) &= -y_i(t) + \sum_{j=1}^n b_{ij} g_j(x_j(t - r_{ij})) + J_i, \quad i = 1, 2, \dots, m.\end{aligned}$$

Realizing the ubiquitous existences of delay in neural networks Gopalsamy and He [31] incorporated time delays into the model and considered the following system of delay differential equations

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t - \tau_{ij})) + I_i, \quad i = 1, 2, \dots, m,$$

$$\dot{y}_i(t) = -y_i(t) + \sum_{j=1}^n b_{ij}g_j(x_j(t - r_{ij})) + J_i, \quad i = 1, 2, \dots, m.$$

A diagonally dominant and delay independent criterion for global stability of above model was established [31]. Recently Mohammad [35] addressed the exponential stability of above model. More recently Wang and Zou [36] considered a special case of above model where all delays in each layer are identical, and performed local stability and Hopf bifurcation analysis of that.

Liao et al. [8, 9] proposed the following two neuron system with distributed delays and no self-connection:

$$\begin{aligned} \dot{x}(t) &= -x(t) + a_1 f \left[y(t) - b_2 \int_0^\infty F(s)y(t-s) ds \right], \\ \dot{y}(t) &= -y(t) + a_2 f \left[x(t) - b_1 \int_0^\infty F(s)x(t-s) ds \right] \end{aligned}$$

and found that Hopf-bifurcation occurred for the weak kernel. However as far as we know, there is few works dealing with self connected delayed neural network systems.

In this paper we have investigated local asymptotic stability; existence, uniqueness of equilibrium, existence and directional stability of Hopf bifurcating periodic solution for a BAM three-neuron network with distributed delay. In this network neurons 1, 2 and neurons 1, 3 are coupled. Each neuron possesses non-linear self feedback under dynamic threshold (Fig. 1).

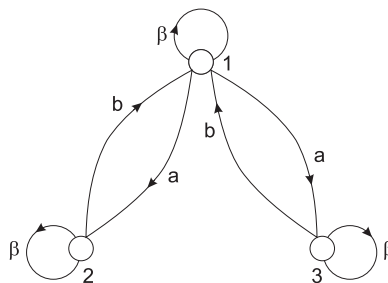


Fig. 1. Directed graph of a three-neuron network model.

The paper is structured as follows:

In Section 2 the model is presented and a sufficient condition for existence and uniqueness of equilibrium is given using degree theory. In Section 3 local stability analysis of trivial equilibrium has been done. In Section 4 existence of Hopf-bifurcating periodic solution about origin is studied. In Section 5 direction, period and stability of that bifurcating periodic solution is studied. In Section 6, existence of global Hopf-bifurcation has been studied. To verify the theoretical analysis, numerical simulations are demonstrated in Section 7. Finally a conclusion has been drawn in Section 8.

2 Model description and existence and uniqueness of the equilibrium

In this paper our aim is to consider a three neuron system with distributed delay and having self connections under dynamic threshold. Such a model can be expressed in following form.

$$\begin{aligned}
 \frac{dx_1}{dt} &= -x_1(t) + \beta' f \left[x_1(t) - \gamma \int_0^\infty F(s)x_1(t-s) ds - c_1 \right] \\
 &\quad + b' f \left[x_2(t) - \gamma \int_0^\infty F(s)x_2(t-s) ds - c_2 \right] \\
 &\quad + b' f \left[x_3(t) - \gamma \int_0^\infty F(s)x_3(t-s) ds - c_3 \right], \\
 \frac{dx_2}{dt} &= -x_2(t) + a' f \left[x_1(t) - \gamma \int_0^\infty F(s)x_1(t-s) ds - c_1 \right] \\
 &\quad + \beta' f \left[x_2(t) - \gamma \int_0^\infty F(s)x_2(t-s) ds - c_2 \right], \\
 \frac{dx_3}{dt} &= -x_3(t) + a' f \left[x_1(t) - \gamma \int_0^\infty F(s)x_1(t-s) ds - c_1 \right] \\
 &\quad + \beta' f \left[x_3(t) - \gamma \int_0^\infty F(s)x_3(t-s) ds - c_3 \right].
 \end{aligned} \tag{1}$$

In this model x_i ($i = 1, 2, 3$) denotes the mean soma potential of the neuron i . The non-negative constant β' , corresponds to the strength of neurons to itself. b' corresponds to the strength of neuron 2 and 3 on neuron 1. a' represents the strength of neuron 1 on neuron 2 and 3. $\gamma \neq 0$ is the measure of the inhibitory influence of the past history. c_i ($i = 1, 2, 3$) > 0 denotes the neuronal threshold. The term x_i in the argument of function f represents local positive feedback. In biological literature, such a feedback is known as reverberation, while in the literature of artificial neural network it is known as excitation from other neurons. The weight function $F(s)$ is a non-negative bounded function defined on $[0, \infty)$ to reflect the influence of the past states on the current dynamics. $F(s)$ is called the delay kernel.

Let us assume that:

(H1) $f \in C^4(\mathbb{R})$, $f(0) = 0$, and $\mu f(\mu) > 0$ for $\mu \neq 0$;

(H2) $f: \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz with Lipschitz constant $L > 0$, that is

$$|f(u) - f(v)| \leq L|u - v| \quad \forall u, v \in \mathbb{R}. \tag{2}$$

The general form of delay kernel $F(s)$ is as follows:

$$F(s) = \alpha^{n+1} \frac{s^n e^{-\alpha s}}{n!}, \quad s \in (0, \infty), \quad n = 0, 1, 2,$$

where α is a parameter denoting the rate of decay of the effects of past memories and it is a positive real number. It is also known as exponentially fading memory. $n = 0$ represents weak kernel whereas $n = 1$ represents strong kernel.

In this paper we study the effect of weak kernel only, that is

$$F(s) = \alpha e^{-\alpha s}, \quad s \in (0, \infty), \quad \alpha > 0. \tag{3}$$

Therefore $\int_0^\infty F(s) ds = 1, \int_0^\infty sF(s) ds < \infty$.

In this section we are interested in the existence and uniqueness of the equilibrium of the system (1). The initial condition associated with (1) is of the form

$$x_i(t) = \varphi_i(t), \quad \phi_i \in C((-\infty, 0], R), \quad t \in (-\infty, 0], \quad i = 1, 2, 3,$$

$\phi_i(t)$ is bounded on $(-\infty, 0]$ and the norm of $C((-\infty, 0])$ is denoted by $\|\phi(t)\| = \sup_{t \in (-\infty, 0]} (|\phi_1(t)| + |\phi_2(t)| + |\phi_3(t)|)$ where $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$.

We shall use here topological degree and homotopy invariance principle to establish the existence and uniqueness of the equilibrium of the system (1).

Definition 1 ([37]). Let $f(x): \Omega \rightarrow \mathbb{R}^n$ is continuous and differentiable function. If $p \notin f(\partial\Omega)$ and $J_f(x) \neq 0$ for all $x \in f^{-1}(p)$ then

$$\text{deg}(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \text{Sgn } J_f(x)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Suppose $f(x): \Omega \rightarrow \mathbb{R}^n$ is a continuous function, $g(x): \Omega \rightarrow \mathbb{R}^n$ is a continuous and differentiable function, if $p \in f(\partial\Omega)$ and $\|f(x) - g(x)\| < p(p, f(\partial\Omega))$ then

$$\text{deg}(f, \Omega, p) = \text{deg}(g, \Omega, p).$$

Homotopy invariance principle ([37]). Assuming that $H: \bar{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$ is a continuous function, let $h_t(x) = H(x, t)$ and let $p: [0, 1] \rightarrow \mathbb{R}^n$ be a continuous function satisfying $p(t) \notin h_t(\partial\Omega)$ if $t \in [0, 1]$. Then $\text{deg}(h_t, \Omega, p(t))$ is independent of t .

Theorem 1. If (H2) holds and $F(s)$ is of form (3) and there exists positive constants $\xi_i > 0, i = 1, 2, 3$ such that

$$\left(\xi_i - \xi_i \beta' L |1 - \gamma| - \xi_i \sum_{j=1, j \neq i}^3 \mu_{ij} L |1 - \gamma| \right) > 0, \quad i = 1, 2, 3, \tag{4}$$

then system (1) has a unique equilibrium x^* .

Proof. The system (1) can be rewritten as

$$\begin{aligned} \frac{dx_i}{dt} = & -x_i(t) + \beta' f \left[x_i(t) - \gamma \int_0^\infty F(s)x_i(t-s) ds - c_i \right] + \\ & + \sum_{j=1, j \neq i}^3 \mu_{ij} f \left[x_j(t) - \gamma \int_0^\infty F(s)x_j(t-s) ds - c_j \right]. \end{aligned} \quad (5)$$

Here $\mu_{11} = \mu_{22} = \mu_{33} = \beta'$, $\mu_{12} = \mu_{21} = a'$, $\mu_{23} = \mu_{32} = 0$.

As $\int_0^\infty F(s) ds = 1$, it is easy to see that $x^* = (x_1^*, x_2^*, x_3^*)$ is an equilibrium of system (5) if and only if the following condition holds:

$$x_i^* = \beta' f[(1 - \gamma)x_i^* - c_i] + \sum_{j=1, j \neq i}^3 \mu_{ij} f[(1 - \gamma)x_j^* - c_j], \quad i = 1, 2, 3, \quad (6)$$

μ is given in (5).

Let $h(x) = (h_1(x), h_2(x), h_3(x))$ where

$$h_i(x) = x_i - \beta' f[(1 - \gamma)x_i - c_i] - \sum_{j=1, j \neq i}^3 \mu_{ij} f[(1 - \gamma)x_j - c_j], \quad i = 1, 2, 3. \quad (7)$$

Obviously the solution of $h(x) = 0$ are equilibrium of system (5). We define a homotopic mapping

$$F(x, \lambda) = \lambda h(x) + (1 - \lambda)x \quad (8)$$

where $\lambda \in [0, 1]$, $F(x, \lambda) = (F_1(x, \lambda), F_2(x, \lambda), F_3(x, \lambda))$ and

$$F_i(x, \lambda) = \lambda h_i(x) + (1 - \lambda)x_i. \quad (9)$$

Then it follows that

$$\begin{aligned} |F_i(x, \lambda)| &= |\lambda h_i(x) + (1 - \lambda)x_i| \\ &= \left| x_i - \lambda \beta' f[(1 - \gamma)x_i - c_i] - \lambda \sum_{j=1, j \neq i}^3 \mu_{ij} f[(1 - \gamma)x_j - c_j] \right| \\ &\geq |x_i| - \lambda |\beta' f[(1 - \gamma)x_i - c_i]| - \lambda \sum_{j=1, j \neq i}^3 |\mu_{ij} f[(1 - \gamma)x_j - c_j]| \\ &= |x_i| - \lambda \beta' |f[(1 - \gamma)x_i - c_i] - f(-c_i) + f(-c_i)| \\ &\quad - \lambda \sum_{j=1, j \neq i}^3 \mu_{ij} |f[(1 - \gamma)x_j - c_j] - f(-c_j) + f(-c_j)| \end{aligned}$$

$$\begin{aligned}
 &\geq |x_i| - \lambda\beta' L|1 - \gamma||x_i| - \lambda\beta'|f(-c_i)| \\
 &\quad - \lambda \sum_{j=1, j \neq i}^3 \mu_{ij} L|1 - \gamma||x_j| - \lambda \sum_{j=1, j \neq i}^3 \mu_{ij} |f(-c_j)| \\
 &\geq \lambda \left[|x_i| - \beta' L|1 - \gamma||x_i| - \sum_{j=1, j \neq i}^3 \mu_{ij} L|1 - \gamma||x_j| \right] \\
 &\quad - \lambda\beta'|f(-c_i)| - \lambda \sum_{j=1, j \neq i}^3 \mu_{ij} |f(-c_j)|. \tag{10}
 \end{aligned}$$

By (5) we have

$$\begin{aligned}
 \sum_{i=1}^3 \xi_i |F_i(x, \lambda)| &\geq \lambda \sum_{i=1}^3 \xi_i \left[|x_i| - \beta' L|1 - \gamma||x_i| - L \sum_{j=1, j \neq i}^3 \mu_{ij} |1 - \gamma||x_j| \right] \\
 &\quad - \lambda\beta' \sum_{i=1}^3 \xi_i |f(-c_i)| - \lambda \sum_{i=1}^3 \xi_i \left(\sum_{j=1, j \neq i}^3 \mu_{ij} |f(-c_j)| \right) \\
 &= \lambda \sum_{i=1}^3 \left[\xi_i - \xi_i \beta' L|1 - \gamma| - \xi_i L \sum_{j=1, j \neq i}^3 \mu_{ij} |1 - \gamma||x_j| \right] |x_i| \\
 &\quad - \lambda \left[\beta' \sum_{i=1}^3 \xi_i |f(-c_i)| + \sum_{i=1}^3 \xi_i \left(\sum_{i=1, i \neq j}^3 \mu_{ij} |f(-c_j)| \right) \right]. \tag{11}
 \end{aligned}$$

Let

$$\begin{aligned}
 \xi_0 &= \min_{1 \leq i \leq 3} \left[\xi_i - \xi_i \beta' L|1 - \gamma| - \xi_i \sum_{j=1, j \neq i}^3 \mu_{ij} L|1 - \gamma| \right], \\
 a_0 &= \max_{1 \leq i \leq 3} \xi_i \left[\beta' |f(-c_i)| + \sum_{j=1, j \neq i}^3 \mu_{ij} |f_j(-c_j)| \right]. \tag{12}
 \end{aligned}$$

Then $\xi_0 > 0$ by (4) and a_0 is a positive constant by (H2).

Let

$$U(0) = \left[|x|/|x_i| < \frac{3(a_0 + 1)}{\xi_0} \right]. \tag{13}$$

It follows from (13) that for any $x \in \partial(U(0))$, $\exists 1 \leq i_0 \leq 3$ such that

$$|x_{i_0}| = \frac{3(a_0 + 1)}{\xi_0}. \tag{14}$$

By (12) we can obtain that for any $\lambda \in (0, 1]$

$$\begin{aligned} & \sum_{i=1}^3 \xi_i |F_i(x, \lambda)| \\ & \geq \lambda \sum_{i=1}^3 \left[\xi_{i0} - \xi_i \beta' L |1 - \gamma| - \sum_{j=1, j \neq 3} \xi_j \mu_{ji0} L |1 - \gamma| \right] |x_{i0}| - \lambda \sum_{i=1}^3 a_0 \\ & \geq \lambda \xi_0 |x_{i0}| - 3\lambda a_0 = 3\lambda > 0 \end{aligned} \tag{15}$$

which implies that $F(x, \lambda) \neq 0$ for any $x \in \partial(U(0))$ and $\lambda \in (0, 1]$.

If $\lambda = 0$ from (9) we have $F(x, \lambda) = x \neq 0$ for any $x \in \partial(U(0))$. Hence $F(x, \lambda) \neq 0$ for any $x \in \partial(U(0))$ and $\lambda \in [0, 1]$. Hence it is easy to prove $\deg(g, U(0), 0) = 1$ where $g(x) = x$ is differentiable and strictly monotonic increasing. Therefore by Homotopy invariance principle

$$\deg(F, U(0), 0) = \deg(g, U(0), 0) = 1.$$

By topological degree theory we can conclude that equation $h(x) = 0$ has at least a solution in $U(0)$. That is to say system (5) has at least one equilibrium x^* implying system (1) has at least one equilibrium.

Now we consider uniqueness of equilibrium x^* of system (5).

Suppose $y^* = (y_1^*, y_2^*, y_3^*)$ is also an equilibrium of system (5). Then we have

$$y_i^* = \beta' f[(1 - \gamma)y_i^* - c_i] + \sum_{j=1, j \neq i}^3 \mu_{ij} f[(1 - \gamma)y_j^* - c_j], \quad i = 1, 2, 3. \tag{16}$$

By (6) and (16)

$$\begin{aligned} x_i^* - y_i^* &= \beta' (f[(1 - \gamma)x_i^* - c_i] - f[(1 - \gamma)y_i^* - c_i]) \\ &+ \sum_{j=1, j \neq i}^3 \mu_{ij} (f[(1 - \gamma)x_j^* - c_j] - f[(1 - \gamma)y_j^* - c_j]), \quad i = 1, 2, 3. \end{aligned} \tag{17}$$

According to (H2)

$$|x_i^* - y_i^*| \leq \beta' L |1 - \gamma| |x_i^* - y_i^*| + \sum_{j=1, j \neq i}^3 \mu_{ij} L |1 - \gamma| |x_j^* - y_j^*|.$$

Therefore

$$\begin{aligned} \sum_{i=1}^3 \xi_i |x_i^* - y_i^*| &\leq \sum_{i=1}^3 \xi_i \left[\beta' L |1 - \gamma| |x_i^* - y_i^*| + \sum_{j=1, j \neq i}^3 \mu_{ij} L |1 - \gamma| |x_j^* - y_j^*| \right] \\ \implies \sum_{i=1}^3 \left[\xi_i - \xi_i \beta' L |1 - \gamma| - \xi_i \sum_{i=1, i \neq j}^3 \mu_{ji} L |1 - \gamma| \right] |x_i^* - y_i^*| &\leq 0. \end{aligned} \tag{18}$$

In view of (4) as $\xi_i > 0$ it is obvious that $|x_i^* - y_i^*| = 0$ implying $x_i^* = y_i^*, i = 1, 2, 3$. Hence $x^* = y^*$. Therefore the system (1) has a unique equilibrium.

Corollary 1. *If in (1) we assume $f(x) \equiv \tanh(x)$ and $F(s) = \alpha e^{-\alpha s}$ then from Theorem 1 follows that corresponding system has unique equilibrium if*

$$[1 - (\beta' + 2b')|1 - \gamma|] > 0, \quad [1 - (\beta' + a')|1 - \gamma|] > 0. \tag{19}$$

3 Local stability analysis of trivial equilibrium

In this section we focus on investigating the local stability of equilibrium and existence of Hopf-bifurcation for system (1).

For convenience we set $\gamma = 1$ and $c_1 = c_2 = c_3 = 0$.

Now let

$$y_1(t) = x_1(t) - \int_0^\infty F(s)x_1(t-s) ds,$$

$$y_2(t) = x_2(t) - \int_0^\infty F(s)x_2(t-s) ds,$$

$$y_3(t) = x_3(t) - \int_0^\infty F(s)x_3(t-s) ds.$$

Then system (1) is equivalent to the following model:

$$\begin{aligned} \frac{dy_1}{dt} &= -y_1(t) + \beta' f[y_1(t)] - \beta' \int_0^\infty F(-s)f[y_1(t+s)] ds + b' f[y_2(t)] \\ &\quad - b' \int_0^\infty F(-s)f[y_2(t+s)] ds + b' f[y_3(t)] - b' \int_0^\infty F(-s)f[y_3(t+s)] ds, \\ \frac{dy_2}{dt} &= -y_2(t) + a' f[y_1(t)] - a' \int_0^\infty F(-s)f[y_1(t+s)] ds + \beta' f[y_2(t)] \\ &\quad - \beta' \int_0^\infty F(-s)f[y_2(t+s)] ds, \\ \frac{dy_3}{dt} &= -y_3(t) + a' f[y_1(t)] - a' \int_0^\infty F(-s)f[y_1(t+s)] ds + \beta' f[y_3(t)] \\ &\quad - \beta' \int_0^\infty F(-s)f[y_3(t+s)] ds. \end{aligned} \tag{20}$$

From (4) it is clear that if $\gamma = 1$, then the corresponding system has a unique steady state and as $c_1 = c_2 = c_3 = 0$ it is obvious that $(0, 0, 0)$ is the unique steady state of (20). The linearization of system (20) about $(0, 0, 0)$ takes the form:

$$\begin{aligned} \frac{dy_1}{dt} &= -y_1(t) + \beta y_1(t) + by_2(t) + by_3(t) - \beta \int_{-\infty}^0 F(-s)y_1(t+s) ds \\ &\quad - b \int_{-\infty}^0 F(-s)y_2(t+s) ds - b \int_{-\infty}^0 F(-s)y_3(t+s) ds, \\ \frac{dy_2}{dt} &= -y_2(t) + \beta y_2(t) + ay_1(t) - a \int_{-\infty}^0 F(-s)y_1(t+s) ds \\ &\quad - \beta \int_{-\infty}^0 F(-s)y_2(t+s) ds, \\ \frac{dy_3}{dt} &= -y_3(t) + \beta y_3(t) + ay_1(t) - a \int_{-\infty}^0 F(-s)y_1(t+s) ds \\ &\quad - \beta \int_{-\infty}^0 F(-s)y_3(t+s) ds \end{aligned} \tag{21}$$

where $\beta = \beta' f'(0)$, $a = a' f'(0)$, $b = b' f'(0)$.

Now the associated characteristic equation of the linearized system (20) after substituting $F(s) = \alpha e^{-\alpha s}$, $\alpha > 0$ is

$$\begin{vmatrix} \lambda + 1 - \beta(1 - \frac{\alpha}{\alpha+\lambda}) & -b(1 - \frac{\alpha}{\alpha+\lambda}) & -b(1 - \frac{\alpha}{\alpha+\lambda}) \\ -a(1 - \frac{\alpha}{\alpha+\lambda}) & \lambda + 1 - \beta(1 - \frac{\alpha}{\alpha+\lambda}) & 0 \\ -a(1 - \frac{\alpha}{\alpha+\lambda}) & 0 & \lambda + 1 - \beta(1 - \frac{\alpha}{\alpha+\lambda}) \end{vmatrix} = 0 \tag{22}$$

$$\implies \lambda^6 + m_1\lambda^5 + m_2\lambda^4 + m_3\lambda^3 + m_4\lambda^2 + m_5\lambda + m_6 = 0 \tag{23}$$

where $m_1, m_2, m_3 \dots m_6$ are given by

$$\begin{aligned} m_1 &= 3(1 + \alpha - \beta), \\ m_2 &= 3(1 + \alpha - \beta)^2 + 3\alpha - 2ab, \\ m_3 &= (1 + \alpha - \beta)[(1 + \alpha - \beta)^2 + 6\alpha - 2ab], \\ m_4 &= \alpha m_2, \\ m_5 &= \alpha^2 m_1, \\ m_6 &= \alpha^3. \end{aligned}$$

Now the trivial equilibrium $(0, 0, 0)$ is locally asymptotically stable iff all the roots of equation (23) have negative real parts.

According to Routh-Hurwitz criteria all the roots of equation

$$y^k + \alpha_1 y^{k-1} + \alpha_2 y^{k-2} + \dots + \alpha_{k-1} y + \alpha_k = 0$$

have negative real part iff

$$D_m = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \dots & \dots & \dots & \dots \\ 1 & \alpha_2 & \alpha_4 & \dots & \dots & \dots & \dots \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 & \dots & \dots & \dots \\ 0 & 1 & \alpha_2 & \alpha_4 & \dots & \dots & \dots \\ \vdots & & & & & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \alpha_m \end{vmatrix} > 0 \quad \forall m = 1, 2, \dots, k.$$

In case of equation (22)

$$\begin{aligned} D_1 &= 3(1 + \alpha - \beta), \\ D_2 &= (1 + \alpha - \beta)[8(1 + \alpha - \beta)^2 + 3\alpha - 4ab], \\ D_3 &= 4(1 + \alpha - \beta)^2[(1 + \alpha - \beta)^2 + 3\alpha - 2ab][2(1 + \alpha - \beta)^2 - ab], \\ D_4 &= 4\alpha(1 + \alpha - \beta)^2[2(1 + \alpha - \beta)^2 - ab] \\ &\quad \times [4a^2b^2 - 2ab\{4\alpha^2 + \alpha(11 - 8\beta) + 4(\beta - 1)^2\} \\ &\quad + 3\{\alpha^4 + \alpha^3(5 - 4\beta) + (\beta - 1)^4 - \alpha(\beta - 1)^2(4\beta - 5) \\ &\quad + \alpha^2(6\beta^2 - 14\beta + 9)\}], \\ D_5 &= 16\alpha^3(1 + \alpha - \beta)^3[(1 + \alpha - \beta)^2 - 2ab][2(1 + \alpha - \beta)^2 - ab]^2, \\ D_6 &= \alpha^3 D_5. \end{aligned}$$

Here

$$\begin{aligned} D_1, D_2, D_3, D_5, D_6 > 0 \quad \text{if } (1 + \alpha - \beta) > 0 \\ \text{and } ab < \min \left[\frac{(1 + \alpha - \beta)^2}{2} + \frac{3}{2}\alpha, (1 + \alpha - \beta)^2 \right], \end{aligned}$$

$$D_4 > 0 \quad \text{if } 4a^2b^2 - 2abA + B > 0$$

where

$$\begin{aligned} A &= [4\alpha^2 + \alpha(11 - 8\beta) + 4(\beta - 1)^2] > 0, \\ B &= 3[\alpha^4 + \alpha^3(5 - 4\beta) + (\beta - 1)^4 - \alpha(\beta - 1)^2(4\beta - 5) + \alpha^2(6\beta^2 - 14\beta + 9)] > 0. \end{aligned}$$

Now

$$4a^2b^2 - 2abA + B > 0 \implies ab < \frac{n_1}{2} \quad \text{or} \quad ab > \frac{n_2}{2}$$

where

$$n_1 = \frac{A - \sqrt{A^2 - 4B}}{2}, \quad n_2 = \frac{A + \sqrt{A^2 - 4B}}{2}$$

or

$$\begin{aligned} \frac{n_1}{2} &= \frac{1}{4} \left[\{4(1 + \alpha - \beta)^2 + 3\alpha\} \right. \\ &\quad \left. - \sqrt{\{4(1 + \alpha - \beta)^2 + 3\alpha\}^2 + 12\left\{(1 + \alpha - \beta)^2 + \frac{\alpha}{2}\right\}^2 + 9\alpha^2} \right], \\ \frac{n_2}{2} &= \frac{1}{4} \left[\{4(1 + \alpha - \beta)^2 + 3\alpha\} \right. \\ &\quad \left. + \sqrt{\{4(1 + \alpha - \beta)^2 + 3\alpha\}^2 + 12\left\{(1 + \alpha - \beta)^2 + \frac{\alpha}{2}\right\}^2 + 9\alpha^2} \right]. \end{aligned}$$

Combining above results we get

$$D_1, D_2, D_3, D_4, D_5, D_6 > 0 \quad \text{if } (1 + \alpha - \beta) > 0,$$

$$ab < \min \left[\frac{n_1}{2}, \frac{(1 + \alpha - \beta)^2}{2} + \frac{3}{2}\alpha, 2(1 + \alpha - \beta)^2 \right] \quad \text{or } (1 + \alpha - \beta) > 0,$$

$$\frac{n_2}{2} < ab < \min \left[\frac{(1 + \alpha - \beta)^2}{2} + \frac{3}{2}\alpha, 2(1 + \alpha - \beta)^2 \right].$$

Now as $n_1 < 0$ and $\frac{n_2}{2} > 2(1 + \alpha - \beta)^2 + \frac{3}{2}\alpha > \min \left[\frac{(1 + \alpha - \beta)^2}{2} + \frac{3}{2}\alpha, 2(1 + \alpha - \beta)^2 \right]$, it can be concluded that $D_1, D_2, D_3, D_4, D_5, D_6 > 0$ implying origin is locally asymptotically stable if $(1 + \alpha - \beta) > 0$ and $ab < \frac{n_1}{2}$.

Hence we have the following conclusion:

Theorem 2. For system (20) the trivial equilibrium $(0, 0, 0)$ is locally asymptotically stable if $(1 + \alpha - \beta) > 0$ and $ab < \frac{n_1}{2}$ where

$$\begin{aligned} \frac{n_1}{2} &= \frac{1}{4} \left[\{4(1 + \alpha - \beta)^2 + 3\alpha\} \right. \\ &\quad \left. - \sqrt{\{4(1 + \alpha - \beta)^2 + 3\alpha\}^2 + 12\left\{(1 + \alpha - \beta)^2 + \frac{\alpha}{2}\right\}^2 + 9\alpha^2} \right]. \end{aligned}$$

4 Existence of Hopf bifurcating periodic solution

Let $\lambda = i\omega$, $\omega > 0$ be a root of equation (23). Substituting $\lambda = i\omega$ in (23) and then separating real and imaginary parts we get

$$\begin{aligned} m_6 - m_4\omega^2 + m_2\omega^4 - \omega^6 &= 0, \\ m_5\omega - m_3\omega^3 + m_1\omega^5 &= 0. \end{aligned} \tag{24}$$

Eliminating ω from equations in (24) and then substituting values of $m_1, m_2, m_3, \dots, m_6$ after simplification it is obtained as

$$(1 + \alpha - \beta)[(1 + \alpha - \beta)^2 - 2ab][2(1 + \alpha - \beta)^2 - ab] = 0. \quad (25)$$

(25) implies that characteristic equation (24) has a purely imaginary root $i\omega$ if

$$(1 + \alpha - \beta) = 0 \text{ or } (1 + \alpha - \beta)^2 = 2ab \text{ or } (1 + \alpha - \beta)^2 = \frac{ab}{2}.$$

To study the existence of Hopf-bifurcating periodic solution first let us assume $0 < 2ab < 3\alpha$.

Case 1. $(1 + \alpha - \beta) = 0$. Then $m_1 = m_3 = m_5 = 0, m_2 = 3\alpha - 2ab > 0$. Therefore characteristic equation (23) reduces to

$$\lambda^6 + m_2\lambda^4 + m_4\lambda^2 + m_6 = 0, \quad m_2, m_4, m_6 > 0. \quad (26)$$

Applying Descarte's rule of sign and relation between roots and coefficients of a polynomial it can be concluded that apart from two purely imaginary roots other roots of (23) can not have real negative part.

Therefore in this case there is no possibility of Hopf-bifurcation.

Case 2. $(1 + \alpha - \beta)^2 = 2ab$. Here $m_1, m_2, m_3, \dots, m_6 > 0$ if $(1 + \alpha - \beta) > 0$. Now applying Descarte's rule of sign and relation between roots and coefficients of a polynomial it can be concluded that (23) has two purely imaginary roots and other four roots have real negative part.

Now $(1 + \alpha - \beta)^2 = 2ab \Rightarrow \alpha = (\beta - 1) + \sqrt{2ab} = \alpha_1$ (say) ($\alpha = (\beta - 1) - \sqrt{2ab}$ is neglected as $(1 + \alpha - \beta) > 0$).

As before we have assumed $\alpha > \frac{2ab}{3}$ therefore $\alpha_1 > \frac{2ab}{3} \Rightarrow (\beta - 1) > \sqrt{2ab}(\frac{\sqrt{2ab}}{3} - 1)$.

At $\alpha = \alpha_1$, if $\pm i\omega_1$ is a pair of purely imaginary roots of characteristic equation (23) then from (24) we get

$$\begin{aligned} \omega_1^2 &= \left[\frac{m_5m_1m_2 - m_5m_3 - m_1^2m_6}{m_1m_5 - m_1^2m_4 + m_1m_2m_3 - m_3^2} \right]_{\alpha=\alpha_1} \\ &= \frac{12\alpha_1^2(1+\alpha_1-\beta)^2[2(1+\alpha_1-\beta)^2-ab]}{4(1+\alpha_1-\beta)^2[(1+\alpha_1-\beta)^2+3\alpha_1-2ab][2(1+\alpha_1-\beta)^2-ab]} = \alpha_1. \end{aligned} \quad (27)$$

Differentiating equation (23) with respect to α implicitly at $\alpha = \alpha_1$ that is at $\lambda = i\omega_1$ we get $(\frac{d\lambda}{d\alpha})_{\alpha=\alpha_1} = \frac{-4ab\alpha_1(1+i\omega_1)}{-8iab\alpha_1\omega_1}$ (substituting $\omega_1^2 = \alpha_1$ and $(1 + \alpha_1 - \beta)^2 = 2ab$). Therefore $\text{Re}(\frac{d\lambda}{d\alpha})_{\alpha=\alpha_1} = \frac{1}{2} \neq 0$. Therefore all the sufficient conditions of Hopf-bifurcation [38–40] are satisfied.

Case 3. $2(1 + \alpha - \beta)^2 = ab \Rightarrow \alpha = (\beta - 1) + \sqrt{\frac{ab}{2}} = \alpha_2$ (say) ($\alpha = (\beta - 1) - \sqrt{\frac{ab}{2}}$ is neglected as $(1 + \alpha - \beta) > 0$).

Let at $\alpha = \alpha_2$, $\pm i\omega_2$ is a pair of purely imaginary roots of characteristic equation (23) then from (24) we get

$$\begin{aligned} \omega_2^2 &= \left[\frac{m_5 m_1 m_2 - m_5 m_3 - m_1^2 m_6}{m_1 m_5 - m_1^2 m_4 + m_1 m_2 m_3 - m_3^2} \right]_{\alpha=\alpha_2} \\ &= \frac{12\alpha_2^2(1 + \alpha_2 - \beta)^2[2(1 + \alpha_2 - \beta)^2 - ab]}{4(1 + \alpha_2 - \beta)^2[(1 + \alpha_2 - \beta)^2 + 3\alpha_2 - 2ab][2(1 + \alpha_2 - \beta)^2 - ab]} \\ &= \frac{0}{0}, \quad \text{as } 2(1 + \alpha_2 - \beta)^2 - ab = 0. \end{aligned}$$

As in this case value of ω_2 is indeterminate, characteristic equation (23) can have no purely imaginary root $\pm i\omega_2$ at $\alpha = \alpha_2$.

So there is no possibility of Hopf-bifurcation.

Hence we have the following theorem:

Theorem 3. *If $0 < 2ab < 3\alpha$, $(1 + \alpha - \beta)^2 = 2ab$, $\beta - 1 > \sqrt{2ab}(\frac{\sqrt{2ab}}{3} - 1)$, then Hopf bifurcation occurs at $\alpha = \alpha_1 = (\beta - 1) + \sqrt{2ab}$.*

5 Direction, period and stability of Hopf-bifurcating periodic solution

In the previous section we obtained the condition for Hopf Bifurcation to occur at the critical value $\alpha = \alpha_1$. In this section we are interested to determine the stability and direction of the periodic solutions bifurcating from the equilibrium $(0, 0, 0)$ following the idea of the Normal form and the centre manifold theory [39].

The nonlinear system (20) can be expanded into first second and higher order terms near origin and then we have the following matrix form.

$$\frac{dy}{dt} = Ly(t) + \int_{-\infty}^0 F(-s)y(t+s) ds + H(s) \tag{28}$$

where

$$L = \begin{pmatrix} -1 + \beta & b & b \\ a & -1 + \beta & 0 \\ a & 0 & -1 + \beta \end{pmatrix}, \tag{29}$$

$$H(y) = \begin{pmatrix} \beta^{(2)}[y_1^2(t) - \int_{-\infty}^0 F(-s)y_1^2(t+s) ds] \\ + b^{(2)}[y_2^2(t) - \int_{-\infty}^0 F(-s)y_2^2(t+s) ds] \\ + b^{(2)}[y_3^2(t) - \int_{-\infty}^0 \beta^{(2)}[y_1^2(t)F(-s)y_3^2(t+s) ds] + \dots \\ a^{(2)}[y_1^2(t) - \int_{-\infty}^0 F(-s)y_1^2(t+s) ds] \\ + \beta^{(2)}[y_2^2(t) - \int_{-\infty}^0 F(-s)y_2^2(t+s) ds] + \dots \\ a^{(2)}[y_1^2(t) - \int_{-\infty}^0 F(-s)y_1^2(t+s) ds] \\ + \beta^{(2)}[y_3^2(t) - \int_{-\infty}^0 F(-s)y_3^2(t+s) ds] + \dots \end{pmatrix} \tag{30}$$

where

$$\beta^{(2)} = \frac{\beta' f''(0)}{2}, \quad a^{(2)} = \frac{a' f''(0)}{2}, \quad b^{(2)} = \frac{b' f''(0)}{2}. \quad (31)$$

For convenience we rewrite the system (28) into an operator form:

$$y_t = A_\mu y_t + R y_t \quad (32)$$

where $y = (y_1, y_2, y_3)^T$, $y_t = y(t + \theta)$, $\theta \in (-\infty, 0)$; $\mu = \alpha - \alpha_1$.

The operators A and R are defined as

$$A_\mu \phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \text{if } \theta \in (-\infty, 0), \\ L\phi(\theta) + \int_{-\infty}^0 K(s)\phi(s) ds, & \text{if } \theta = 0, \end{cases} \quad (33)$$

and

$$R\phi(\theta) = \begin{cases} (0, 0, 0)^T, & \text{if } \theta \in (-\infty, 0), \\ f(\phi, \sigma) = (f_1, f_2, f_3)^T, & \text{if } \theta = 0, \end{cases} \quad (34)$$

where L is defined in (29), K is defined as

$$K(s) = \begin{pmatrix} -\beta F(-s) & -bF(-s) & -bF(-s) \\ -aF(-s) & -\beta F(-s) & 0 \\ -aF(-s) & 0 & -\beta F(-s) \end{pmatrix} \quad (35)$$

and

$$\begin{aligned} f_1 &= \beta^{(2)} \left[\phi_1^2(0) - \int_{-\infty}^0 F(-s)\phi_1^2(s) ds \right] + b^{(2)} \left[\phi_2^2(0) - \int_{-\infty}^0 F(-s)\phi_2^2(s) ds \right] \\ &\quad + b^{(2)} \left[\phi_3^2(0) - \int_{-\infty}^0 F(-s)\phi_3^2(s) ds \right] + \dots, \\ f_2 &= a^{(2)} \left[\phi_1^2(0) - \int_{-\infty}^0 F(-s)\phi_1^2(s) ds \right] + \beta^{(2)} \left[\phi_2^2(0) - \int_{-\infty}^0 F(-s)\phi_2^2(s) ds \right] + \dots, \\ f_3 &= a^{(2)} \left[\phi_1^2(0) - \int_{-\infty}^0 F(-s)\phi_1^2(s) ds \right] + \beta^{(2)} \left[\phi_3^2(0) - \int_{-\infty}^0 F(-s)\phi_3^2(s) ds \right] + \dots \end{aligned} \quad (36)$$

Let us define an adjoint operator A^* of A as

$$A^* \psi(\delta) = \begin{cases} -\frac{d\psi(\delta)}{d\delta}, & \text{if } \delta \in (0, \infty), \\ L^T \psi(0) + \int_{-\infty}^0 K^T(s)\psi(-s) ds, & \text{if } \delta = 0, \end{cases} \quad (37)$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-\infty}^0 \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta)K(\theta)\phi(\xi) d\xi d\theta \quad (38)$$

where $L^T, K^T, \bar{\psi}^T$ are the transpose of $L, K, \bar{\psi}$ respectively. Since A and A^* are adjoint operators, if $\pm i\omega_0$ are eigen values of A , then they are also eigen values of A^* . Let $q(\theta)$ be the eigen vector of A_0 associated with the eigenvalue $i\omega_0$

$$\implies A(0)q(\theta) = i\omega_0q(\theta).$$

This gives $q(\theta) = \begin{pmatrix} q_1 \\ 1 \\ 1 \end{pmatrix} e^{i\omega_0\theta}$ where

$$q_1 = \frac{(i\omega_0 + 1)(\alpha + i\omega_0) - \beta i\omega_0}{ia\omega_0}. \quad (39)$$

Similarly it can be verified that $q^*(\delta)$ is the eigen vector of A^* corresponding to $-i\omega_0$ where

$$q^*(\delta) = (q_1', 1, 1)e^{-i\omega_0\delta} = D(q_1^*, 1, 1)^T e^{i\omega_0\delta},$$

$$q_1^* = \frac{(1 - i\omega_0)(\alpha - i\omega_0) + \beta i\omega_0}{-ia\omega_0}. \quad (40)$$

\bar{D} can be calculated from the relations $\langle q^*(s), q(\theta) \rangle = 1$ and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Now we first compute the coordinates describing the center manifold C_0 at $\mu = 0$. Let y_t be the solution of (22) at $\mu = 0$.

Let us define

$$z(t) = \langle q^*, y_t \rangle \quad (41)$$

and

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) \quad (42)$$

$$= y_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta)$$

$$= y_t(\theta) - 2 \operatorname{Re} z(t)q(\theta). \quad (43)$$

On the centre manifold C_0 we have

$$W(t, \theta) = W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \dots \quad (44)$$

where z and \bar{z} are local coordinates for C_0 in the direction of q^* and \bar{q}^* respectively. W is real if X_t is real. We shall deal with real solutions only. Now for any solution $y_t \in C_0$

of (22)

$$\begin{aligned} \dot{z}(t) &= i\omega_0 z(t) + \langle \bar{q}^*(0), R(0, W(z, \bar{z}, \theta) + 2\text{Re}\{zq(0)\}) \rangle \\ &= i\omega_0 z(t) + \bar{q}^*(0)R(W(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\}) \\ &= i\omega_0 z(t) + \bar{q}^*(0)R_0(z, \bar{z}) \end{aligned} \quad (45)$$

$$= i\omega_0 z(t) + g(z, \bar{z}) \quad (46)$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots \quad (47)$$

By comparing the coefficients in the two sides of (46) and (47) we have

$$\begin{aligned} \frac{g_{20}}{2} &= \bar{D} \left[\bar{q}_1^* q_1^2 \beta^{(2)} + 2b^{(2)} \left(1 - \frac{\alpha}{\alpha + 2i\omega_0} \right) + 2q_1^2 a^{(2)} + \beta^{(2)} \left(1 - \frac{\alpha}{\alpha + 2i\omega_0} \right) \right], \\ \frac{g_{02}}{2} &= \bar{D} \left[\bar{q}_1^* \bar{q}_1^2 \beta^{(2)} + 2b^{(2)} \left(1 - \frac{\alpha}{\alpha + 2i\omega_0} \right) + 2\bar{q}_1^2 a^{(2)} + \beta^{(2)} \left(1 - \frac{\alpha}{\alpha + 2i\omega_0} \right) \right], \\ g_{11} &= 0, \\ \frac{g_{21}}{2} &= \bar{D} \left[\bar{q}_1^* \left\{ \beta^{(2)} (W_{20}^{(1)}(0)\bar{q}_1 + 2W_{11}^{(1)}(0)q_1) + 2b^{(2)} (W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)) \right. \right. \\ &\quad - \beta^{(2)} \int_{-\infty}^0 F(-s) (W_{20}^{(1)}(s)\bar{q}_1 e^{-i\omega_0 s} + 2W_{11}^{(1)}(s)q_1 e^{i\omega_0 s}) ds \\ &\quad - 2b^{(2)} \int_{-\infty}^0 F(-s) (W_{20}^{(2)}(s)e^{-i\omega_0 s} + 2W_{11}^{(2)}(s)e^{i\omega_0 s}) ds \\ &\quad \left. \left. + 3(\beta^{(3)} q_1^2 \bar{q}_1 + 2b^{(3)}) \left(1 - \frac{\alpha}{\alpha + i\omega_0} \right) \right\} \right. \\ &\quad + \left\{ 2a^{(2)} (W_{20}^{(1)}(0)\bar{q}_1 + 2W_{11}^{(1)}(0)q_1) + \beta^{(2)} (W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)) \right. \\ &\quad - 2a^{(2)} \int_{-\infty}^0 F(-s) (W_{20}^{(1)}(s)\bar{q}_1 e^{-i\omega_0 s} + 2W_{11}^{(1)}(s)q_1 e^{i\omega_0 s}) ds \\ &\quad - b^{(2)} \int_{-\infty}^0 F(-s) (W_{20}^{(2)}(s)e^{-i\omega_0 s} + 2W_{11}^{(2)}(s)e^{-i\omega_0 s}) ds \\ &\quad \left. \left. + 3(a^{(3)} q_1^2 \bar{q}_1 + b^{(3)}) \left(1 - \frac{\alpha}{\alpha + i\omega_0} \right) \right\} \right]. \end{aligned} \quad (48)$$

In order to determine g_{21} , we need to compute W_{20} and W_{11} where $W_{20}(\theta) = (W_{20}^{(1)}(\theta), W_{20}^{(2)}(\theta), W_{20}^{(3)}(\theta))^T$ and $W_{11}(\theta) = (W_{11}^{(1)}(\theta), W_{11}^{(2)}(\theta), W_{11}^{(3)}(\theta))^T$.

Now we have

$$\begin{aligned} \dot{W} &= y_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} A_0W - 2\text{Re}\bar{q}^*(0)R_0q(\theta), & \theta \in (-\infty, 0), \\ A_0W - 2\text{Re}\bar{q}^*(0)R_0q(\theta) + R_0, & \theta = 0, \end{cases} \\ &= AW + H(z, \bar{z}, \theta) \end{aligned} \quad (49)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{30}(\theta)\frac{z^3}{6} + \dots \quad (50)$$

On the centre manifold C_0 near the origin

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \quad (51)$$

From (49), (50) and (51) we get

$$(A - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta), \quad (52)$$

$$AW_{11}(\theta) = -H_{11}(\theta). \quad (53)$$

Also

$$H_{20} = -g_{20}q(\theta) - \bar{g}_{20}\bar{q}(\theta), \quad (54)$$

$$H_{11} = g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (55)$$

From (52) and (54) and definition of A

$$\dot{W}_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(0)e^{i\omega_0\theta} + \bar{g}_{20}\bar{q}(0)e^{-i\omega_0\theta}. \quad (56)$$

Similarly from (53) and (55)

$$\dot{W}_{11}(\theta) = g_{11}q(0)e^{i\omega_0\theta} - \bar{g}_{11}\bar{q}(0)e^{-i\omega_0\theta} \quad (57)$$

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{20}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + K_1e^{2i\omega_0\theta}, \\ \Rightarrow W_{11}(\theta) &= \frac{-ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + K_2 \end{aligned}$$

where K_1 and K_2 are both three dimensional vectors that can be determined by setting

$\theta = 0$ in H .

$$\begin{aligned}
 K_1 = & \left[\begin{pmatrix} -1+\beta-2i\omega_0 & b & b \\ a & -1+\beta-2i\omega_0 & 0 \\ a & 0 & -1+\beta-2i\omega_0 \end{pmatrix} - \frac{\alpha}{\alpha+2i\omega_0} \begin{pmatrix} \beta & b & b \\ a & \beta & 0 \\ a & 0 & \beta \end{pmatrix} \right]^{-1} \\
 & \times \left[\begin{pmatrix} -1+\beta-2i\omega_0 & b & b \\ a & -1+\beta-2i\omega_0 & 0 \\ a & 0 & -1+\beta-2i\omega_0 \end{pmatrix} \left(\frac{g_{20}}{i\omega_0} q(0) + \frac{\bar{g}_{02}}{i\omega_0} \bar{q}(0) \right) \right. \\
 & \left. - \begin{pmatrix} \beta & b & b \\ a & \beta & 0 \\ a & 0 & \beta \end{pmatrix} \left(\frac{g_{20}}{i\omega_0} q(0) \frac{\alpha}{\alpha+i\omega_0} + \frac{\bar{g}_{02}}{i\omega_0} \bar{q}(0) \frac{\alpha}{\alpha-i\omega_0} - H_{20} \right) \right], \\
 K_2 = & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \\
 & \times \left[-H_{11}(0) - \begin{pmatrix} -1+\beta & b & b \\ a & -1+\beta & 0 \\ a & 0 & -1+\beta \end{pmatrix} \left(-\frac{ig_{11}}{\omega_0} q(0) + \frac{i\bar{g}_{11}}{\omega_0} \bar{q}(0) \right) \right. \\
 & \left. + \begin{pmatrix} \beta & b & b \\ a & \beta & 0 \\ a & 0 & \beta \end{pmatrix} \left(-\frac{ig_{11}}{\omega_0} q(0) \frac{\alpha}{\alpha+i\omega_0} + \frac{i\bar{g}_{11}}{\omega_0} \bar{q}(0) \frac{\alpha}{\alpha-i\omega_0} \right) \right]
 \end{aligned}$$

where

$$\begin{aligned}
 H_{20} &= (H_{20}^{(1)}, H_{20}^{(2)}, H_{20}^{(3)})^T, \quad H_{11} = (H_{11}^{(1)}, H_{11}^{(2)}, H_{11}^{(3)})^T, \\
 q(0) &= (q^{(1)}(0), q^{(2)}(0), q^{(3)}(0))^T, \quad \bar{q}(0) = (\bar{q}^{(1)}(0), \bar{q}^{(2)}(0), \bar{q}^{(3)}(0))^T.
 \end{aligned}$$

Thus we can calculate the following quantities

$$\begin{aligned}
 E_1(0) &= \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}, \\
 \mu_2 &= -\frac{\operatorname{Re} E_1(0)}{\operatorname{Re} \lambda'(\alpha_1)}, \\
 T_2 &= -\frac{1}{\omega_0} [\operatorname{Im} E_1(0) + \mu_2 \operatorname{Im} \lambda'(\alpha_1)], \\
 b_2 &= 2 \operatorname{Re} E_1(0).
 \end{aligned}$$

Then from conclusion of [39] we obtain the following result:

Theorem 4. (i) *The direction of the Hopf-bifurcating periodic solution is determined by*

$$\mu(\sigma) = \mu_2 \sigma^2 + \dots$$

When $\mu_2 > 0$ (< 0), the Hopf-bifurcation is super critical (subcritical) and the bifurcating periodic solution exists for $\alpha > \alpha_0$ ($< \alpha_0$).

(ii) *The period of the bifurcating periodic solution can be estimated by*

$$T(\sigma) = \frac{2\pi}{\omega_0} (1 + T_2\sigma^2 + \dots).$$

The period increases (decreases) if $T_2 > 0$ (< 0).

(iii) *The stability of the bifurcating periodic solution is determined by*

$$B(\sigma) = b_2\sigma^2 + \dots$$

When $b_2 > 0$ (< 0) the bifurcating periodic solution is unstable (stable).

6 Global Hopf-bifurcation

In this section we shall consider the global existence of Hopf-bifurcating periodic solution that is continuation of the bifurcating periodic solutions as the bifurcation parameter α increases and varies over the interval (α_1, ∞) assuming that the Hopf-bifurcation is supercritical. This phenomenon will be proved by using the technique of Alexander and Aumchmuty [41].

Let P^1 denote the space of all $x: \mathbb{R} \rightarrow \mathbb{R}^6$ which are periodic with period 2π . The space P^1 is a Banach space with the norm

$$\|x\|_\infty^{(1)} = \max_{1 \leq i \leq 6} \max_{0 \leq t \leq 2\pi} \left[|x_i(t)| + \left| \frac{dx_i(t)}{dt} \right| \right].$$

Let Λ denote the open interval $(0, \infty)$. Let $L(P^1)$ denote the set of all continuous linear maps of P^1 into itself with the induced norm topology. Let $F: P^1 \times \Lambda \rightarrow P^1$ be continuous and let us consider the problem of finding the solutions $(y, \alpha, \omega) \in P^1 \times \Lambda \times (0, \infty)$ of the equation

$$\omega \frac{dy}{dt} = F(y, \alpha). \tag{58}$$

If $y(t)$ is a solution of equation (58), and if $x(t) = y(\omega t)$, then $x(t)$ is a solution of

$$\frac{dx}{dt} = F(x, \alpha) \tag{59}$$

where x is periodic with period $T = 2\pi/\omega$.

Now let

$$y_4(t) = \int_0^\infty \alpha e^{\alpha s} \tanh[y_1(t+s)] ds,$$

$$y_5(t) = \int_0^\infty \alpha e^{\alpha s} \tanh[y_2(t+s)] ds,$$

$$y_6(t) = \int_0^\infty \alpha e^{\alpha s} \tanh[y_3(t + s)] ds.$$

Then (22) can be written as

$$\begin{aligned} \frac{dy_1}{dt} &= -y_1(t) + \beta' \tanh[y_1] + b' \tanh[y_2] + b' \tanh[y_3] - \beta'[y_4] - b'[y_5] - b'[y_6], \\ \frac{dy_2}{dt} &= -y_2(t) + a' \tanh[y_1] + \beta' \tanh[y_2] + -a'[y_4] - \beta'[y_5], \\ \frac{dy_3}{dt} &= -y_3(t) + a' \tanh[y_1] + \beta' \tanh[y_3] + -a'[y_4] - \beta'[y_6], \\ \frac{dy_4}{dt} &= \alpha(\tanh[y_1] - y_4), \\ \frac{dy_5}{dt} &= \alpha(\tanh[y_2] - y_5), \\ \frac{dy_6}{dt} &= \alpha(\tanh[y_3] - y_6). \end{aligned} \tag{60}$$

The system of integro differential equations (22) is equivalent to system (60) of ordinary differential equations [42]. Now if we denote the system (60) by

$$\frac{dY}{dt} = F_1(Y), \tag{61}$$

then (61) can be rewritten as

$$\omega \frac{dZ}{dt} = A(\alpha)Z + R(Z, \alpha) \tag{62}$$

where $A(\alpha)$ is variational matrix of (60) about trivial equilibrium $(0, 0, 0, 0, 0, 0)$ and $R(Z, \alpha) = F_1(Z, \alpha) - A(\alpha)Z$.

Now theorem which is used to prove global existence of Hopf-bifurcating prriodic solution is stated under (proof is in [41]).

Theorem 5. *Let F be a Frechet differentiable map of $P^1 \times \Lambda$ into P^1 . There is a global bifurcation of 2π -periodic solutions of equation (58) from a solution $(y^*, \alpha_0, \omega_0)$ provided*

- (i) $A(\alpha) \in L(P^1)$ for $\alpha \in \Lambda$. The mapping $\alpha \rightarrow A(\alpha)$ is continuous and 0 is not in the spectrum of $A(\alpha_0)$.
- (ii) The number of linearly independent solutions in P^1 of $\omega_0 \frac{dW}{dt} = A(\alpha)W$ is finite and congruent to 2 mod 4.
- (iii) There are positive δ and ε such that if $\lambda(\alpha)$ is in the spectrum of $A(\alpha)$ and $\text{Re}\lambda(\alpha) = P(\beta)$, then $|P(\alpha)| > \varepsilon|\alpha - \alpha_0|$ for $|\alpha - \alpha_0| < \delta$.

We will verify the above sufficient conditions of global Hopf-bifurcation for the system (62). We consider the linearized system

$$\omega_0 \frac{dW}{dt} = A(\alpha)W \tag{63}$$

and let us suppose that it has a periodic solution of period 2π ; let it be

$$W = \sum_{k=-\infty}^{\infty} \mathbf{d}_k e^{ikt}.$$

The coefficients \mathbf{d}_k are solutions of the linear system $\omega_0 ik \mathbf{d}_k = A(\alpha) \mathbf{d}_k$, $k = 0, \pm 1, \pm 2, \dots$. Nontrivial periodic solutions of period 2π exist iff $ik\omega_0$ is an eigen value of $A(\alpha)$. That is $ik\omega_0$ is a solution of equation (24). From previous discussion it is clear that non-trivial periodic solutions exist only for $k = \pm 1$ at $\alpha = \alpha_1$, and hence there is only one periodic solution of period 2π for the linearized system (63). It has been shown that

$$\operatorname{Re} \left(\frac{d\lambda}{d\alpha} \right)_{\alpha=\alpha_1} > 0.$$

It follows that if $P(\alpha) = \operatorname{Re}[\lambda(\alpha)]$ then

$$\lim_{\alpha \rightarrow \alpha_1} \frac{\alpha(\beta) - \alpha(\beta_1)}{\beta - \beta_1} > 0$$

which implies that there exists $\varepsilon > 0$, $\delta > 0$ such that

$$\begin{aligned} \implies & \left| \frac{P(\alpha) - P(\alpha_1)}{\alpha - \alpha_1} \right| > \varepsilon \quad \text{if } |\alpha - \alpha_1| < \delta, \\ \implies & |P(\alpha)| > \varepsilon |\alpha - \alpha_1| \quad \text{for } |\alpha - \alpha_1| < \delta \quad (\text{as } P(\alpha_1) = 0). \end{aligned} \tag{64}$$

Also $A(\alpha)$ is continuous in α . Thus all the conditions of Theorem 5 are satisfied. Therefore there is a global bifurcation of 2π periodic solutions from the point $(0, \alpha_1, \omega_1)$.

Thus the global Hopf-bifurcation has been established in a product space of the phase space, parameter space and the frequency space.

7 Numerical simulation

In this section using the Matlab software we carry out the numerical simulation on a particular form of system (1). Let $f(x) \equiv \tanh(x)$; $c_1 = c_2 = c_3 = 0$; $\gamma = 1$. Then (1)

takes the form

$$\begin{aligned}
 \frac{dx_1}{dt} &= -x_1(t) + \beta' \tanh \left[x_1(t) - \int_0^\infty F(s)x_1(t-s) ds \right] \\
 &\quad + b' \tanh \left[x_2(t) - \int_0^\infty F(s)x_2(t-s) ds \right] \\
 &\quad + b' \tanh \left[x_3(t) - \int_0^\infty F(s)x_3(t-s) ds \right], \\
 \frac{dx_2}{dt} &= -x_2(t) + a' \tanh \left[x_1(t) - \int_0^\infty F(s)x_1(t-s) ds \right] \\
 &\quad + \beta' \tanh \left[x_2(t) - \int_0^\infty F(s)x_2(t-s) ds \right], \\
 \frac{dx_3}{dt} &= -x_3(t) + a' \tanh \left[x_1(t) - \int_0^\infty F(s)x_1(t-s) ds \right] \\
 &\quad + \beta' \tanh \left[x_3(t) - \int_0^\infty F(s)x_3(t-s) ds \right].
 \end{aligned} \tag{65}$$

Now first we take $\alpha = 1$, $\beta = 1$, $a = 1.5$, $b = -1.5$. These values satisfy the conditions of Theorem 2 and with these values we get Fig. 2. It shows that in this case origin is locally asymptotically stable.

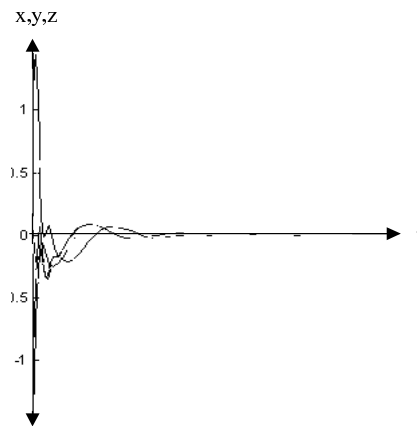


Fig. 2. $a = 1.5$, $b = -1.5$, $\alpha = 1$, $\beta = 1$. Here origin is locally asymptotically stable.

Then to verify Theorem 3 we choose $\beta = 1$, $a = 1$, $b = 0.5$. These parametric values give the critical value of α as $\alpha = 1$. Then with these above mentioned prescribed values of parameters we get Fig. 3, Fig. 4 and Fig. 5

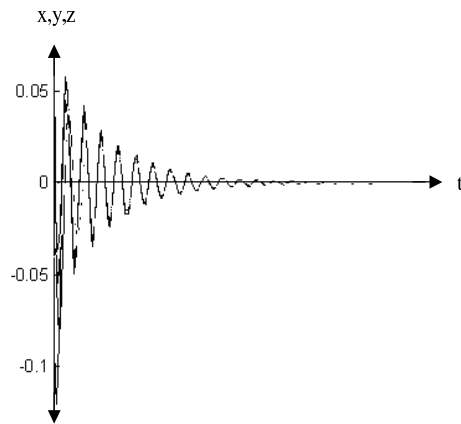


Fig. 3. $a = 1$, $b = 0.5$, $\alpha = 0.8 < 1$, $\beta = 1$. Here origin is locally asymptotically stable.

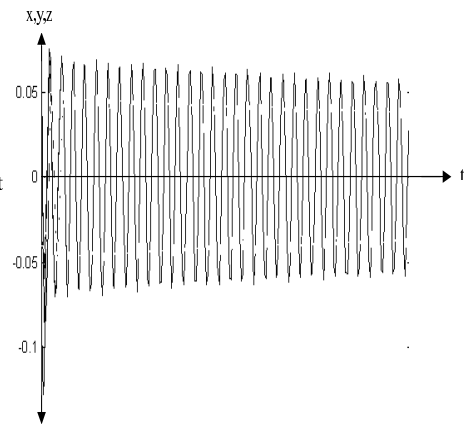


Fig. 4. $a = 1$, $b = 0.5$, $\alpha = 1$, $\beta = 1$. A periodic solution exists near origin.

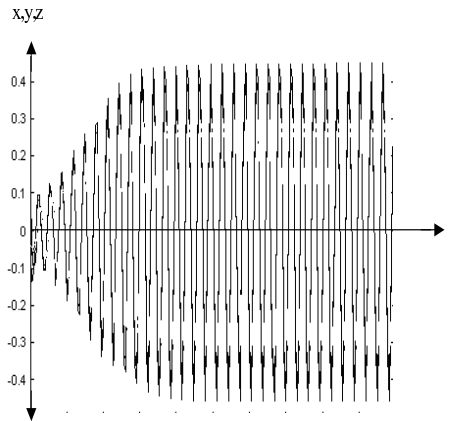


Fig. 5. $a = 1$, $b = 0.5$, $\alpha = 1.2 > 1$, $\beta = 1$. A periodic solution exists near origin.

8 Conclusion

In this paper we have analyzed a BAM neural network model composed of three neurons with distributed delay. It is a generalization of the model studied in [8, 9]. To our knowledge, the stability analysis and bifurcation of a bidirectional associate memorial

network with self-connection have not been investigated in literature. This paper is an attempt to do this. As the distributed delay can become a discrete delay when the delay kernel is a delta function at a certain time, a neural network model with distributed delay is more general than that with discrete delay.

In this paper in Theorem 1 sufficient condition for existence of unique equilibrium has been studied. In Theorem 2 we have obtained the criteria under which the trivial equilibrium remains locally asymptotically stable. In Theorem 3 condition for existence of Hopf-bifurcating periodic solution about origin has been obtained. In Theorem 4 we have studied the direction, period and stability of such Hopf-bifurcating periodic solution. Then the global stability of that Hopf-bifurcating periodic solution has been studied.

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