

Analysis of a delay nonautonomous predator-prey system with disease in the prey

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Received: 2009-08-01 **Revised:** 2010-01-14 **Published online:** 2010-03-04

Abstract. In this paper we have considered a nonautonomous predator-prey model with time delay due to gestation, in which a disease that can be transmitted by contact spreads among the prey only. Here, we have established some sufficient conditions on the permanence of the system by using inequality analytical technique. By Lyapunov functional method, we have also obtained some sufficient conditions for global asymptotic stability of this model. We have observed that the time delay has no effect on the permanence of the system but it has an effect on the global asymptotic stability of this model. The aim of the analysis of this model is to identify the parameters of interest for further study, with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

Keywords: ecoepidemiology, susceptible and infected prey, predator, permanence, Lyapunov functional, global stability.

1 Introduction

The mathematical epidemic models have received much attention from researchers after the pioneering work of Kermack-McKendrick [1] on SIRS (susceptible-infective-removal-susceptible) systems, in which the evolution of a disease which gets transmitted upon contact is described. Ecology and epidemiology are two major and distinct fields of research in their own right. Lotka [2] and Volterra [3] established seminal works on the mathematical modelling of predator-prey and competing species in terms of simultaneous non-linear differential equations, making the first breakthrough in modern mathematical ecology. Ecoepidemiology is the study of interacting species in which a disease spreads. The study of Ecoepidemiology has important ecological significance. Ecoepidemiology research is becoming important as it involves persistence-extinction threshold of each population in systems of two or more interacting species subjected to parasitism [4–8]. In the natural world, however, species do not exist alone while species spreads disease, it also competes with the other species for space or food, or is predated by other species. Understanding how parasites affect biodiversity and ecosystem dynamics is an important

question in conservation biology as infectious disease can be a factor in regulating host population. On the other hand, successful invasion of a parasite into a host population and resulting host-parasite dynamics can depend essentially on other members of a host's community such as predators. Predation can dramatically shape the structures of community and ecosystem and becomes particularly interesting in host-parasite systems because predation itself can strongly alter dynamics of hosts and parasites. Predators may also prevent proper invasion of parasites into host population. Therefore, it is very significant biologically to consider the effect of interacting species when we study the dynamical behaviours of epidemiological systems. Scientists have paid lots of attention to merge these two important areas of research [4–13].

Nonautonomous phenomenon often occurs in many realistic ecoepidemiological models. The nonautonomous phenomenon occurs mainly due to the seasonal variations, which make the population behave periodically. Pathogen contact rate and infectivity vary seasonally and is generally larger in spring and autumn than in summer and winter. This seasonal pattern is related to moderate temperatures in spring and autumn which improves pathogen survival and favors high insect activity. Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. To investigate this kind of phenomenon, in the model, the coefficients should be periodic functions, then the system is called periodic system. The nonautonomous ecoepidemiological models can be regarded as an extension of the periodic ecoepidemiological models. Therefore, the research on the nonautonomous ecoepidemiological dynamical models is also very important.

Considering the above facts, in this paper we have considered a nonautonomous predator-prey model with time delay due to gestation, in which a disease that can be transmitted by contact spreads among the prey only. In the proposed system, all the coefficients are time-dependent, i.e., it is nonautonomous. Usually, such systems do not have any disease-free equilibrium and endemic equilibrium. There are many methods to deal with autonomous systems, but they may not be suitable to nonautonomous systems. Therefore, it is more difficult to study the dynamical behaviours in nonautonomous case. Here, we have established some sufficient conditions on the permanence (uniformly persistent) of the system by using inequality analytical technique. By Lyapunov functional method, we have also obtained some sufficient conditions for the global asymptotic stability of this system.

2 The basic mathematical model

Here, we have considered an ecoepidemiological system consisting of three species, namely, the sound prey (which is susceptible), the infected prey (which becomes infective by some viruses) and the predator population. Our mathematical model is formulated as the following system of nonautonomous delay differential equations:

$$\frac{dx_1(t)}{dt} = x_1(t) [r(t) - k_1(t)(x_1(t) + x_2(t)) - a_1(t)x_3(t) - \beta(t)x_2(t)], \quad (1a)$$

$$\frac{dx_2(t)}{dt} = x_2(t) [\beta(t)x_1(t) - k_2(t)(x_1(t) + x_2(t)) - a_2(t)x_3(t)], \quad (1b)$$

$$\begin{aligned} \frac{dx_3(t)}{dt} = & -d(t)x_3(t) - b(t)x_3^2(t) + c_1(t)x_3(t - \tau)x_1(t - \tau) \\ & + c_2(t)x_3(t - \tau)x_2(t - \tau). \end{aligned} \quad (1c)$$

Here, $(x_1(t), x_2(t), x_3(t))$ are the densities of the sound prey (which is susceptible), the infected prey (which becomes infective by some viruses) and the predator population, respectively at time t .

The model is derived under following assumptions.

The quantities $r(t), b(t), k_1(t), k_2(t), d(t), \beta(t), a_1(t), a_2(t), c_1(t), c_2(t)$ are:

- $r(t)$: intrinsic birth rate function of the susceptible prey population;
- $d(t)$: intrinsic death rate function of the predator population;
- $k_1(t)$: the rate of crowding effects on the susceptible prey population;
- $k_2(t)$: the rate of crowding effects on the infected prey population;
- $b(t)$: the rate of crowding effects on the predator population;
- $\beta(t)$: the transmission rate function of infection when susceptible prey contact with infected prey and the rate of transmission is of the form $\beta(t)x_1(t)x_2(t)$;
- $a_1(t)$: the capturing rate function of susceptible prey by the predator;
- $a_2(t)$: the capturing rate function of infected prey by the predator;
- $c_1(t)$: the growth rate function of the predator due to predation of susceptible prey;
- $c_2(t)$: the growth rate function of the predator due to predation of infected prey.

- (A1) We assume that only susceptible prey $x_1(t)$ is capable of reproducing with logistic law. The mortality terms for susceptible and infected prey are of density dependence [14], say, the rate $k_1(t)$ for the susceptible prey and $k_2(t)$ for the infected prey. The infected prey $x_2(t)$ is removed by death or by predation before having the possibility of reproducing. However, the infected prey population $x_2(t)$ still contributes with $x_1(t)$ to population growth toward the carrying capacity.
- (A2) We assume that the disease spreads among the prey population only and the disease is not genetically inherited. The infected prey population does not recover nor becomes immune. The incidence is assumed to be the simple mass action form $\beta(t)x_1(t)x_2(t)$, where $\beta(t)$ is called the transmission rate function.
- (A3) The predators hunt with possibly different predation rate functions $a_1(t)$ and $a_2(t)$ on susceptible and infected preys respectively. This is in accordance with the fact that the infected individuals can be caught more easily. For example, Peterson and Page [15] have indicated that wolf attacks on moose are more often successful if the moose is heavily infected by "Echinococcus granucosus". $c_1(t)$ and $c_2(t)$ denote the growth rate functions of the predator due to predation on susceptible and infected preys respectively; and the rate of growth are respectively of the forms:

$$c_1(t)x_3(t - \tau)x_1(t - \tau), \quad c_2(t)x_3(t - \tau)x_2(t - \tau),$$

$\tau > 0$ is the fixed time delay due to gestation lag.

Here we assume that the functions $r(t), b(t), k_1(t), k_2(t), d(t), \beta(t), a_1(t), a_2(t), c_1(t), c_2(t)$ are positive continuous bounded and have positive lower bounds.

The initial conditions of (1) are given as

$$x_1(\theta) = \varphi_1(\theta), \quad x_2(\theta) = \varphi_2(\theta), \quad x_3(\theta) = \varphi_3(\theta), \quad -\tau \leq \theta \leq 0, \quad (2)$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i(\theta) \geq 0$ ($i = 1, 2, 3$), $\forall \theta \in [-\tau, 0]$, and C denotes the Banach space $C([-\tau, 0], \mathbb{R}^3)$ of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^3 and denotes the norm of an element φ in C by $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \{|\varphi_1(\theta)|, |\varphi_2(\theta)|, |\varphi_3(\theta)|\}$. For a biological meaning, we further assume that $\varphi_i(0) > 0$, $i = 1, 2, 3$.

Theorem 1. *Every solution of system (1) with initial conditions (2) exists and is unique in the interval $[0, \infty)$ and $x_1(t) > 0, x_2(t) > 0, x_3(t) > 0$, for all $t \geq 0$.*

Proof. Since the right hand side of system (1) is completely continuous and locally Lipschitzian on C , the solution $(x_1(t), x_2(t), x_3(t))$ of (1) with initial conditions (2) exists and is unique on $[0, \alpha)$, where $0 < \alpha \leq +\infty$ [16, Chapter 2]. Now, from the first two equations of system (1), we have

$$x_1(t) = x_1(0) \exp \int_0^t [r(s) - k_1(s)(x_1(s) + x_2(s)) - a_1(s)x_3(s) - \beta(s)x_2(s)] ds > 0, \quad \forall t \geq 0,$$

$$x_2(t) = x_2(0) \exp \int_0^t [\beta(s)x_1(s) - k_2(s)(x_1(s) + x_2(s)) - a_2(s)x_3(s)] ds > 0, \quad \forall t \geq 0.$$

Next, we prove that $x_3(t) > 0$ for all $t \geq 0$.

If not, then there exists a $t_1 > 0$, such that $x_3(t_1) = 0$, and $x_3(t) \geq 0, \forall t \in [-\tau, t_1]$. Furthermore,

$$\frac{dx_3(t)}{dt} \geq -d(t)x_3(t) - b(t)x_3^2(t), \quad \forall t \in [0, t_1].$$

Then

$$x_3(t) \geq x_3(0) \exp \int_0^t [-d(s) - b(s)x_3(s)] ds > 0, \quad \forall t \in [0, t_1],$$

$$\Rightarrow x_3(t_1) \geq x_3(0) \exp \int_0^{t_1} [-d(s) - b(s)x_3(s)] ds > 0,$$

which is a contradiction. Hence, $x_3(t) > 0$, for all $t \geq 0$. This completes the proof. \square

3 Permanence of system (1)

Here we wish to discuss the permanence (uniformly persistent) of system (1) with initial conditions (2), which demonstrates how this system will be uniformly persistent, this means that the long-term survival (i.e., will not vanish in time) of all components of the system (1) with initial conditions (2), under some conditions. Let $f^l = \inf_{t \geq 0} f(t)$, $f^u = \sup_{t \geq 0} f(t)$, for a continuous and bounded function $f(t)$ defined on $[0, +\infty)$.

Definition 1. System (1) is said to be uniformly persistent, i.e., the long-term survival (will not vanish in time) of all components of the system (1), if there are positive constants v_i and w_i ($i = 1, 2, 3$) such that:

$$\begin{aligned} v_1 &\leq \liminf_{t \rightarrow \infty} x_1(t) \leq \limsup_{t \rightarrow \infty} x_1(t) \leq w_1, \\ v_2 &\leq \liminf_{t \rightarrow \infty} x_2(t) \leq \limsup_{t \rightarrow \infty} x_2(t) \leq w_2, \\ v_3 &\leq \liminf_{t \rightarrow \infty} x_3(t) \leq \limsup_{t \rightarrow \infty} x_3(t) \leq w_3, \end{aligned}$$

hold for any solution $(x_1(t), x_2(t), x_3(t))$ of (1) with initial conditions (2). Here v_i and w_i ($i = 1, 2, 3$) are independent of (2).

Theorem 2 ([17]). *Consider the following equation:*

$$\dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t),$$

where $a, b, c, \tau > 0$; $x(t) > 0$, for $-\tau \leq t \leq 0$. We have

- (i) if $a > b$, then $\lim_{t \rightarrow \infty} x(t) = \frac{a-b}{c}$;
- (ii) if $a < b$, then $\lim_{t \rightarrow \infty} x(t) = 0$.

Theorem 3. Let $X(t) = (x_1(t), x_2(t), x_3(t))$ denote any solution of system (1) and (2). Suppose system (1) satisfies

$$(c_1 + c_2)^u M^* - d^l > 0, \quad \text{where } M^* = \max \left\{ \left(\frac{r^u}{k_1^l} \right), \left(\frac{\beta^u}{k_2^l} \right) \left(\frac{r^u}{k_1^l} \right) \right\}. \quad (3)$$

Then \exists a $T_3 > 0$ such that

$$x_1(t), x_2(t) \leq M_3 \text{ and } x_3(t) \leq M_4, \quad \forall t \geq T_3, \quad (4)$$

where $M_3 > M^*$ and $M_4 > \frac{(c_1 + c_2)^u M^* - d^l}{b^l}$.

Proof. Let, $M_1 > \left(\frac{r^u}{k_1^l} \right)$. From (1a),

$$\dot{x}_1(t) \leq x_1(t)[r(t) - k_1(t)x_1(t)] \leq x_1(t)[r^u - k_1^l x_1(t)].$$

Therefore, if $x_1(0) \leq M_1$, then $x_1(t) \leq M_1$, for all $t \geq 0$.

If $x_1(0) > M_1$, and let $-\alpha_1 = M_1(r^u - k_1^l M_1)$, $\alpha_1 > 0$, then \exists an $\epsilon_1 > 0$, s.t. if $t \in [0, \epsilon_1)$, $x(t) > M_1$, and we have $\dot{x}_1(t) < -\alpha_1 < 0$.

Therefore, \exists a $T_1 > 0$ s.t. $x_1(t) \leq M_1$, $\forall t \geq T_1$, where $M_1 > (\frac{r^u}{k_1^l})$.

From (1b) we have, $\dot{x}_2(t) \leq x_2(t)[\beta^u M_1 - k_2^l x_2(t)]$, $\forall t \geq T_1$.

Therefore, \exists a $T_2 \geq T_1 > 0$ s.t. $x_2(t) \leq M_2$, $\forall t \geq T_2$, where $M_2 > (\frac{\beta^u}{k_2^l})(\frac{r^u}{k_1^l})$; M_1 can be chosen sufficiently close to $(\frac{r^u}{k_1^l})$. Hence, $x_1(t), x_2(t) \leq M_3$, where $M_3 > M^* = \max\{(\frac{r^u}{k_1^l}), (\frac{\beta^u}{k_2^l})(\frac{r^u}{k_1^l})\}$, $\forall t \geq T_2$. In addition, from (1c) we obtain

$$\dot{x}_3(t) \leq -d^l x_3(t) - b^l x_3^2(t) + (c_1 + c_2)^u M_3 x_3(t - \tau), \quad \forall t \geq T_2 + \tau.$$

By condition (3) and Theorem 2, we conclude that

$$\exists \text{ a } T_3 \geq T_2 + \tau \text{ s.t. } x_3(t) \leq M_4, \quad \forall t \geq T_3, \quad \text{where } M_4 > \frac{(c_1 + c_2)^u M^* - d^l}{b^l},$$

since M_3 can be chosen sufficiently close to M^* .

This completes the proof. \square

Theorem 4. Suppose that system (1) and (2) satisfies the following conditions:

$$\begin{aligned} \frac{d^l}{(c_1 + c_2)^u} &< M^* = \max \left\{ \left(\frac{r^u}{k_1^l} \right), \left(\frac{\beta^u}{k_2^l} \right) \left(\frac{r^u}{k_1^l} \right) \right\} \\ &< \min \left\{ \frac{b^l r^l + a_1^u d^l - (\frac{k_1^u d^u b^l}{(c_1 + c_2)^l})}{(k_1 + \beta)^u b^l + a_1^u (c_1 + c_2)^u}, \right. \\ &\quad \left. \frac{\beta^l b^l r^l + a_2^u d^l k_1^u + a_1^u \beta^l d^l - (\frac{k_1^u k_2^u d^u b^l}{(c_1 + c_2)^l})}{\beta^l \{ (k_1 + \beta)^u b^l + a_1^u (c_1 + c_2)^u \} + b^l k_1^u k_2^u + k_1^u a_2^u (c_1 + c_2)^u} \right\}. \end{aligned} \quad (5)$$

Then system (1) and (2) is uniformly persistent.

Proof. Suppose $X(t) = (x_1(t), x_2(t), x_3(t))$ be a solution of (1) and (2). Therefore,

$$\dot{x}_1(t) \geq x_1(t)[r^l - \{(k_1 + \beta)^u M_3 + a_1^u M_4\} - k_1^u x_1(t)], \quad \forall t \geq T_3,$$

(using Theorem 3 and T_3 is defined there). By condition (5), we have

$$\begin{aligned} b^l r^l - M^* \{ (k_1 + \beta)^u b^l + a_1^u (c_1 + c_2)^u \} + a_1^u d^l &> 0 \\ \Rightarrow r^l - \left\{ (k_1 + \beta)^u M^* + a_1^u \frac{(c_1 + c_2)^u M^* - d^l}{b^l} \right\} &> 0 \\ \Rightarrow r^l - \{ (k_1 + \beta)^u M_3 + a_1^u M_4 \} &> 0, \end{aligned} \quad (6)$$

since M_3 can be chosen sufficiently close to M^* and M_4 can be chosen sufficiently close to $\frac{(c_1+c_2)^u M^* - d^l}{b^l}$. Let us choose m_1 in such a way that,

$$\begin{aligned} 0 < m_1 &< \frac{r^l - \{(k_1 + \beta)^u M_3 + a_1^u M_4\}}{k_1^u} \\ \Rightarrow r^l - \{(k_1 + \beta)^u M_3 + a_1^u M_4\} - k_1^u m_1 &> 0. \end{aligned} \quad (7)$$

If $x_1(T_3) \geq m_1$, then $x_1(t) \geq m_1, \forall t \geq T_3$. If $x_1(T_3) < m_1$, and let $\mu_1 = x_1(T_3) \{r^l - \{(k_1 + \beta)^u M_3 + a_1^u M_4\} - k_1^u m_1\} > 0$, then \exists an $\epsilon_1 > 0$, s.t. $x_1(t) < m_1$, and $\dot{x}_1(t) > \mu_1 > 0, \forall t \in [T_3, T_3 + \epsilon_1)$. Therefore,

$$\exists a \ T_4 > T_3 > 0, \text{ s.t. } x_1(t) \geq m_1, \quad \forall t \geq T_4. \quad (8)$$

From the second equation of system (1) and using Theorem 3, we have $\dot{x}_2(t) \geq x_2(t) \times [\beta^l m_1 - (k_2^u M_3 + a_2^u M_4) - k_2^u x_2(t)], \forall t \geq T_4$, using (8). By condition (5), after some simplifications, we have

$$\beta^l m_1 - (k_2^u M_3 + a_2^u M_4) > 0, \quad (9)$$

since m_1 can be chosen sufficiently close to $\frac{r^l - \{(k_1 + \beta)^u M_3 + a_1^u M_4\}}{k_1^u}$ and M_3, M_4 can be chosen sufficiently close to $M^*, \frac{(c_1+c_2)^u M^* - d^l}{b^l}$ respectively. Let us choose m_2 in such a way that,

$$\begin{aligned} 0 < m_2 &< \frac{\beta^l m_1 - (k_2^u M_3 + a_2^u M_4)}{k_2^u} \\ \Rightarrow \beta^l m_1 - (k_2^u M_3 + a_2^u M_4) - k_2^u m_2 &> 0. \end{aligned} \quad (10)$$

If $x_2(T_4) \geq m_2$, then $x_2(t) \geq m_2, \forall t \geq T_4$. If $x_2(T_4) < m_2$, and let $\mu_2 = x_2(T_4) \times \{\beta^l m_1 - (k_2^u M_3 + a_2^u M_4) - k_2^u m_2\} > 0$, then \exists an $\epsilon_2 > 0$, s.t. $x_2(t) < m_2$, and $\dot{x}_2(t) > \mu_2 > 0, \forall t \in [T_4, T_4 + \epsilon_2)$. Therefore,

$$\exists a \ T_5 > T_4 > 0, \text{ s.t. } x_2(t) \geq m_2, \forall t \geq T_5. \quad (11)$$

Hence,

$$x_1(t), x_2(t) \geq m_3, \forall t \geq T_5, \text{ where } m_3 < m^* = \min\{m_1, m_2\}. \quad (12)$$

From the third equation of (1), we have

$$\dot{x}_3(t) \geq -d^u x_3(t) - b^u x_3^2(t) + (c_1 + c_2)^l m_3 x_3(t - \tau), \forall t \geq T_5 + \tau.$$

By using condition (5) and after some simplifications, we have $(c_1 + c_2)^l m_3 > d^u$. Therefore by Theorem 2, we conclude that

$$\exists a \ T_6 \geq T_5 + \tau \text{ s.t. } x_3(t) \geq m_4, \quad \forall t \geq T_6, \quad (13)$$

where $m_4 < \frac{(c_1+c_2)^l m_3 - d^u}{b^u}$.

From the above discussions, we conclude that $\exists T_6 > 0$ s.t. every solution of system (1) and (2) eventually enters and remains in the region $\Omega = \{(x_1, x_2, x_3) | m \leq x_i \leq M, i = 1, 2, 3\}$, $\forall t \geq T_6$, where $m = \min\{m_3, m_4\}$ and $M = \max\{M_3, M_4\}$. This completes the proof. \square

4 Global asymptotic stability

In this section, we derive sufficient conditions for global asymptotic stability of system (1) with initial conditions of type (2). We now state a definition of global asymptotic stability of solutions of system (1).

Definition 2. System (1) with initial conditions (2) is said to be globally asymptotically stable if

$$\lim_{t \rightarrow \infty} |x_1(t) - u_1(t)| = 0, \quad \lim_{t \rightarrow \infty} |x_2(t) - u_2(t)| = 0, \quad \lim_{t \rightarrow \infty} |x_3(t) - u_3(t)| = 0,$$

hold for any two solutions $(x_1(t), x_2(t), x_3(t))$ and $(u_1(t), u_2(t), u_3(t))$ of (1) with initial conditions of type (2).

Theorem 5. *If there exist $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_3 > 0$ such that the functions $B_i(t)$ ($i = 1, 2, 3$) are nonnegative on $[0, \infty)$ and for any interval sequence $\{[e_i, f_i]\}_1^\infty, [e_i, f_i] \cap [e_j, f_j] = \emptyset$ and $f_i - e_i = f_j - e_j > 0$, for all $i, j = 1, 2, \dots$ and $i \neq j$, one has $\sum_{k=1}^\infty \int_{e_k}^{f_k} B_i(t) dt = \infty$, then system (1) with initial conditions (2) is globally asymptotically stable. Here,*

$$\begin{aligned} B_1(t) &= \alpha_1 k_1(t) - \alpha_2 |k_2(t) - \beta(t)| - \alpha_3 M c_1(t + \tau), \\ B_2(t) &= \alpha_2 k_2(t) - \alpha_1 (k_1(t) + \beta(t)) - \alpha_3 M c_2(t + \tau), \\ B_3(t) &= \alpha_3 (d(t) + 2mb(t)) - \alpha_1 a_1(t) - \alpha_2 a_2(t) \\ &\quad - \alpha_3 M (c_1(t + \tau) + c_2(t + \tau)), \end{aligned} \tag{14}$$

where m, M are given in Theorem 4.

Proof. Assume that $(x_1(t), x_2(t), x_3(t))$ and $(u_1(t), u_2(t), u_3(t))$ are any two solutions of system (1) with initial conditions of type (2).

Define $V_1(t) = |\ln x_1(t) - \ln u_1(t)|$, $V_2(t) = |\ln x_2(t) - \ln u_2(t)|$ and $V_3(t) = |x_3(t) - u_3(t)|$. Then the right-upper derivative of $V_1(t)$, $V_2(t)$ and $V_3(t)$ along the solution of system (1) and (2) are given below:

$$\begin{aligned} D^+ V_1(t) &= \left(\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{u}_1(t)}{u_1(t)} \right) \operatorname{sgn}(x_1(t) - u_1(t)) \\ &\leq -k_1(t) |x_1(t) - u_1(t)| + (k_1(t) + \beta(t)) |x_2(t) - u_2(t)| \\ &\quad + a_1(t) |x_3(t) - u_3(t)|, \end{aligned} \tag{15}$$

$$\begin{aligned}
 D^+V_2(t) &= \left(\frac{\dot{x}_2(t)}{x_2(t)} - \frac{\dot{u}_2(t)}{u_2(t)} \right) \operatorname{sgn}(x_2(t) - u_2(t)) \\
 &\leq |k_2(t) - \beta(t)| |x_1(t) - u_1(t)| - k_2(t) |x_2(t) - u_2(t)| \\
 &\quad + a_2(t) |x_3(t) - u_3(t)|, \tag{16}
 \end{aligned}$$

$$\begin{aligned}
 D^+V_3(t) &= (\dot{x}_3(t) - \dot{u}_3(t)) \operatorname{sgn}(x_3(t) - u_3(t)) \\
 &\leq -d(t) |x_3(t) - u_3(t)| - b(t) (x_3(t) + u_3(t)) |x_3(t) - u_3(t)| \\
 &\quad + c_1(t) \{ x_3(t - \tau) |x_1(t - \tau) - u_1(t - \tau)| \\
 &\quad \quad + u_1(t - \tau) |x_3(t - \tau) - u_3(t - \tau)| \} \\
 &\quad + c_2(t) \{ x_3(t - \tau) |x_2(t - \tau) - u_2(t - \tau)| \\
 &\quad \quad + u_2(t - \tau) |x_3(t - \tau) - u_3(t - \tau)| \}. \tag{17}
 \end{aligned}$$

Define

$$\begin{aligned}
 V_4(t) &= \int_{t-\tau}^t c_1(s + \tau) x_3(s) |x_1(s) - u_1(s)| \, ds \\
 &\quad + \int_{t-\tau}^t c_1(s + \tau) u_1(s) |x_3(s) - u_3(s)| \, ds \\
 &\quad + \int_{t-\tau}^t c_2(s + \tau) x_3(s) |x_2(s) - u_2(s)| \, ds \\
 &\quad + \int_{t-\tau}^t c_2(s + \tau) u_2(s) |x_3(s) - u_3(s)| \, ds. \tag{18}
 \end{aligned}$$

Calculating the right-upper derivative of $V_4(t)$ along the solution of system (1) and (2), we have

$$\begin{aligned}
 D^+V_4(t) &= c_1(t + \tau) x_3(t) |x_1(t) - u_1(t)| - c_1(t) x_3(t - \tau) |x_1(t - \tau) - u_1(t - \tau)| \\
 &\quad + c_1(t + \tau) u_1(t) |x_3(t) - u_3(t)| - c_1(t) u_1(t - \tau) |x_3(t - \tau) - u_3(t - \tau)| \\
 &\quad + c_2(t + \tau) x_3(t) |x_2(t) - u_2(t)| - c_2(t) x_3(t - \tau) |x_2(t - \tau) - u_2(t - \tau)| \\
 &\quad + c_2(t + \tau) u_2(t) |x_3(t) - u_3(t)| - c_2(t) u_2(t - \tau) |x_3(t - \tau) - u_3(t - \tau)|. \tag{19}
 \end{aligned}$$

Let $V(t) = \alpha_1 V_1(t) + \alpha_2 V_2(t) + \alpha_3 (V_3(t) + V_4(t))$, then by using (15)–(19), we have

$$\begin{aligned}
 D^+V(t) &\leq -B_1(t) |x_1(t) - u_1(t)| - B_2(t) |x_2(t) - u_2(t)| \\
 &\quad - B_3(t) |x_3(t) - u_3(t)|, \quad \forall t \geq T_6, \tag{20}
 \end{aligned}$$

where T_6 is defined in Theorem 4 and $B_i(t)$, ($i = 1, 2, 3$) are defined in (14).

Integrating (20) from T_6 to t , we have

$$\begin{aligned} & \int_{T_6}^t \{B_1(t)|x_1(t) - u_1(t)| + B_2(t)|x_2(t) - u_2(t)| \\ & \quad + B_3(t)|x_3(t) - u_3(t)|\} dt \leq V(T_6) - V(t) \\ \Rightarrow & \int_{T_6}^t \{B_1(t)|x_1(t) - u_1(t)| \\ & \quad + B_2(t)|x_2(t) - u_2(t)| + B_3(t)|x_3(t) - u_3(t)|\} dt < \infty. \end{aligned} \quad (21)$$

By assumptions (14) about $B_i(t)$, ($i = 1, 2, 3$) and the boundedness of $(x_1(t), x_2(t), x_3(t))$ and $(u_1(t), u_2(t), u_3(t))$ on $[0, \infty)$, we obtain from system (1) that $|x_1(t) - u_1(t)|$, $|x_2(t) - u_2(t)|$ and $|x_3(t) - u_3(t)|$ are bounded and uniformly continuous on $[0, \infty)$. It follows from (21) that,

$$\lim_{t \rightarrow \infty} |x_1(t) - u_1(t)| = 0, \quad \lim_{t \rightarrow \infty} |x_2(t) - u_2(t)| = 0, \quad \lim_{t \rightarrow \infty} |x_3(t) - u_3(t)| = 0.$$

This shows that system (1) with initial conditions (2) is globally asymptotically stable. This completes the proof. \square

Corollary 1. *If there exist $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_3 > 0$ such that*

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \{\alpha_1 k_1(t) - \alpha_2 |k_2(t) - \beta(t)| - \alpha_3 M c_1(t + \tau)\} > 0, \\ & \liminf_{t \rightarrow \infty} \{\alpha_2 k_2(t) - \alpha_1 (k_1(t) + \beta(t)) - \alpha_3 M c_2(t + \tau)\} > 0, \\ & \liminf_{t \rightarrow \infty} \{\alpha_3 (d(t) + 2mb(t)) - \alpha_1 a_1(t) - \alpha_2 a_2(t) \\ & \quad - \alpha_3 M (c_1(t + \tau) + c_2(t + \tau))\} > 0, \end{aligned}$$

then system (1) with initial conditions (2) is globally asymptotically stable.

Assume that system (1) is ω -periodic, i.e. all coefficients are ω -periodic functions. Then system (1) has a positive ω -periodic solution if system (1) is uniformly persistent [18]. Thus, we have the following corollary.

Corollary 2. *If system (1) is ω -periodic and conditions in Theorems 4 and 5 are valid, then there exists a unique positive ω -periodic solution which is globally asymptotically stable.*

5 Conclusions

In this paper we have considered a nonautonomous predator-prey model with time delay due to gestation, in which a disease that can be transmitted by contact spreads among

the prey only. The most basic and important questions to ask for biological systems in the theory of mathematical epidemiology are the persistence, extinctions, the existence of periodic solutions, global stability, etc. [1, 19–23]. Here, we have established some sufficient conditions for the permanence (uniformly persistent) of the above system by using inequality analytical technique. By Lyapunov functional method, we have also obtained some sufficient conditions for the global asymptotic stability of this model. We have observed that the time delay has no effect on the permanence of the system but it has an effect on the global asymptotic stability of this model. The aim of the analysis of this model is to identify the parameters of interest for further study, with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

Acknowledgement

I am very grateful to the anonymous referees and Journal Secretary (Dr. Romas Baronas) for their careful reading, valuable comments and helpful suggestions, which have helped me to improve the presentation of this work significantly. I am grateful to Prof. A. Dvurecenskij, Dr. I. Vrto, Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovak Republic and Dr. D.N. Garain, P.G. Department of Mathematics, S.K.M. University, Dumka-814101, Jharkhand, India for their helps and encouragements. I am thankful to Slovak Academic Information Agency (SAIA), Bratislava, Slovak Republic for financial support.

References

1. W.O. Kermack, A.G. McKendrick, Contributions to the Mathematical Theory of Epidemics, Part I, *Proc. R. Soc. A*, **115**(5), pp. 700–721, 1927.
2. A. Lotka, *Elements of Physical Biology*, Williams and Wilkins, Baltimore, 1925.
3. V. Volterra, Variazioni e fluttuazioni del numero di individui in specie animali conviventi, *Mem. Accad. Lincei*, **2**, pp. 31–113, 1926.
4. K.P. Hadeler, H.I. Freedman, Predator-prey population with parasite infection, *J. Math. Biol.*, **27**, pp. 609–631, 1989.
5. E. Beltrami, T.O. Carroll, Modelling the role of viral disease in recurrent phytoplankton blooms, *J. Math. Biol.*, **32**, pp. 857–863, 1994.
6. E. Venturino, Epidemics in predator-prey model: disease in the prey, in: *Mathematical Population Dynamics: Analysis of Heterogeneity*, Vol.1, O. Arino, O. Axelrod, M. Kimmel, M. Langlais (Eds.), pp. 381–393, 1995.
7. J. Chattopadhyay, O. Arino, A predator-prey model with disease in the prey, *Nonlinear Analysis*, **36**, pp. 747–766, 1999.
8. X. Zhou, X. Shi, X. Song, Analysis of a delay prey-predator model with disease in the prey species only, *J. Korean Math. Soc.*, **46**, pp. 713–731, 2009.

9. Y. Xiao, L. Chen, Modeling and analysis of a predator-prey model with disease in the prey, *Math. Biosci.*, **171**, pp. 59–82, 2001.
10. Y. Xiao, L. Chen, Analysis of a three species eco-epidemiological model, *J. Math. Anal. Appl.*, **258**, pp. 733–754, 2001.
11. Y. Xiao, L. Chen, A ratio-dependent predator-prey model with disease in the prey, *Appl. Math. Comput.*, **131**, pp. 397–414, 2002.
12. H.W. Hethcote, W. Wang, L. Han, Z. Ma, A predator-prey model with infected prey, *Theor. Pop. Biol.*, **66**, pp. 259–268, 2004.
13. M. Haque, J. Zhen, E. Venturino, An ecoepidemiological predator-prey model with standard disease incidence, *Mathematical Methods in the Applied Sciences*, **32**, pp. 875–898, 2009.
14. Y. Xiao, F.V.D. Bosch, The dynamics of an eco-epidemic model with biological control, *Ecol. Model.*, **168**, pp. 203–214, 2003.
15. R.O. Peterson, R.E. Page, Wolf density as a predictor of predation rate, *Swedish Wildlife Research Suppl.*, **1**, pp. 771–773, 1987.
16. J. Hale, *Theory of functional differential equations*, Springer-Verlag, Heidelberg, 1977.
17. X.Y. Song, L.S. Chen, Optimal harvesting and stability with stage-structure for a two species competitive system, *Math. Biosci.*, **170**, pp. 173–186, 2001.
18. Z. Teng, L. Chen, The positive periodic solutions of periodic Kolmogorov type systems with delays, *Acta Math. Appl. Sin.*, **22**, pp. 446–456, 1999.
19. R.M. Anderson, R.M. May, Population Biology of Infectious Diseases, Part I, *Nature*, **280**, pp. 361–367, 1979.
20. V. Capasso, *Mathematical Structures of Epidemic Systems*, Lectures Notes in Biomathematics, Vol. 97, Springer-Verlag, Berlin, 1993.
21. O. Diekmann, J.A.P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases: Model Building Analysis, and Interpretation*, John Wiley and Sons Ltd., Chichester, New York, 2000.
22. Z. Ma, Y. Zhou, W. Wang, Z. Jin, *Mathematical Modelling and Research of Epidemic Dynamical Systems*, Science Press, Beijing, 2004.
23. X. Meng, L. Chen, H. Cheng, Two profitless delays for the SEIRS epidemic disease model with nonlinear incidence and pulse vaccination, *Appl. Math. Comput.*, **186**, pp. 516–529, 2007.