Run up flow of a couple stress fluid between parallel plates

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Received: 2009-05-31 **Revised:** 2009-11-16 **Published online:** 2010-03-04

Abstract. Consider the flow of an incompressible fluid between two parallel plates, initially induced by a constant pressure gradient. After steady state is attained, the pressure gradient is suddenly with drawn while the plates are impulsively started simultaneously. The arising flow is referred to as run up flow and the present paper aims at studying this flow in the context of a couple stress fluid. Using Laplace transform technique, the expression for velocity is obtained in Laplace transform domain which is later inverted to the space time domain using a numerical approach. The variation of velocity with respect to various flow parameters is presented through graphs.

Keywords: run-up flow, couple stress fluid, Laplace transform, numerical inversion.

1 Introduction

The growing importance of the use of non-Newtonian fluids in modern technology and industries has led various researchers to attempt diverse flow problems related to several non-Newtonian fluids. One such fluid that has attracted the attention of research workers in fluid mechanics during the last four decades is the couple stress fluid. The theory of couple stress fluids initiated by Stokes [1], is a generalization of the classical theory of viscous fluids, which allows for the presence of couple stresses and body couples in the fluid medium. The concept of couple stresses arises due to the way in which the mechanical interactions in the fluid medium are modeled. In this theory the rotational field is defined in terms of the velocity field itself and the rotation vector equals one half the curl of the velocity vector. The stress tensor here is no longer symmetric. An excellent introduction to this theory is available in the monograph “Theories of Fluids with Microstructure – An Introduction” written by Stokes [2] himself. Rao and Iyengar [3] have made analytical and computational studies of diverse couple stress fluid flows dealing with a class of axisymmetric problems. Nabil et.al [4] discussed the effects of couple stresses on pulsatile hydro magnetic Poiseuille flow. Some salient references to couple stress flows through tubes and channels can be seen in Stokes [2]. This theory has several industrial and scientific applications as well, which comprise pumping fluids such as synthetic fluids, polymer thickened oils, liquid crystal, animal blood, synovial fluid present in synovial joints and the theory of lubrication (Naduvinamani et al. [5–7], Naduvinamani et al. [8, 9], Lin and Hung [10]).

In lubrication theory and in many physical situations where we come across slip flows, there arises a class of problems referred to as “run up and spin up flows”. Kazakia and Rivlin [11] initiated the study of these flows and later Rivlin [12–14] elaborately studied the run-up and spin-up flows of viscoelastic fluids between rigid parallel plates and in circular geometries. Ramacharyulu and Raju [15] investigated the run-up flow of a viscous incompressible fluid in a long circular cylinder of porous material. Ramakrishna [16] discussed the run up and spin up flows related to a dusty viscous fluid.

Prompted by the recent researches in couple stress fluid flows cited earlier, in the present paper, we examine the run-up flow of an incompressible couple stress fluid between two infinite rigid parallel plates. The flow is assumed to be initially induced by a constant pressure gradient between two infinite rigid parallel plates. After the steady state is attained, the pressure gradient is suddenly withdrawn and the parallel plates are set to move instantaneously with different velocities in the direction of the applied pressure gradient. The time dependence of the resultant flow is investigated.

2 Mathematical formulation of the problem

The equations of motion that characterize a couple stress fluid [1] flow are similar to the Navier-Stokes equations and are given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \,\mathrm{div}\,(\bar{q}) = 0, \tag{1}$$

$$\rho\frac{\mathrm{d}\bar{q}}{\mathrm{d}t} = \rho\bar{f} + \frac{1}{2}\,\mathrm{curl}(\rho\bar{c}) + \,\mathrm{div}\,\big(\tau^{(s)}\big) + \frac{1}{2}\,\mathrm{curl}\big(\,\mathrm{div}\,(\bar{M})\big), \tag{2}$$

where ρ is the density of the fluid, $\tau^{(s)}$ is the symmetric part of the force stress diad and $\bar{M}$ is the couple stress diad and $\bar{f}$, $\bar{c}$ are the body force per unit mass and body couple per unit mass respectively.

The constitutive equations concerning the force stress t_{ij} and the rate of deformation tensor d_{ij} are given by:

$$t_{ij} = -p\delta_{ij} + \lambda\,\mathrm{div}\,(\bar{q})\delta_{ij} + 2\mu d_{ij} - \frac{1}{2}\varepsilon_{ijk}[m_{,k} + 4\eta w_{k,rr} + \rho c_k]. \tag{3}$$

The couple stress tensor that arises in the theory has the linear constitutive relation

$$m_{ij} = \frac{1}{3}m\delta_{ij} + 4\eta w_{j,i} + 4\eta' w_{i,j}. \tag{4}$$

In the above $w = \frac{1}{2}\,\mathrm{curl}(\bar{q})$ is the spin vector, $w_{i,j}$ is the spin tensor, p is the fluid pressure and ρc_k is the body couple vector. The quantities λ and μ are the viscosity coefficients and

η, η' are the couple stress viscosity coefficients. These material constants are constrained by the inequalities,

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \eta \geq 0, \quad |\eta'| \leq \eta. \tag{5}$$

There is a length parameter $l = \sqrt{\eta/\mu}$, which is a characteristic measure of the polarity of the fluid model and this parameter is identically zero in the case of non-polar fluids.

If the fluid is incompressible, in the absence of body forces and body couples the above field equations (1) and (2) reduce to

$$\text{div}\,(\bar{q}) = 0, \tag{6}$$

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q}.\nabla\bar{q})\right] = -\,\text{grad}\,(p) - \mu\,\text{curl}\big(\text{curl}(\bar{q})\big) - \eta\,\text{curl}\big(\text{curl}\big(\text{curl}\big(\text{curl}(\bar{q})\big)\big)\big). \tag{7}$$

The boundary conditions usually employed in the solution of these equations are that the velocity $\bar{q}$ at the boundary equals to the velocity $\bar{q}_B$ of the boundary and the couple stresses vanish on the boundary [2].

Consider the flow of an incompressible couple stress fluid between two infinite rigid parallel plates $y = -k$ and $y = k$ along the direction of x-axis. Since the flow is along the x-direction, we take the velocity $\bar{q} = (u(y,t), 0, 0)$, which satisfies the continuity equation (6). It is seen that the equation governing $u(y,t)$ is given by

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} - \eta\frac{\partial^4 u}{\partial y^4}. \tag{8}$$

The fluid is set in motion by a constant pressure gradient G between two infinite rigid parallel plates. When the flow is fully developed, the pressure gradient is suddenly withdrawn and at the same time the parallel plates are impulsively set to move with different velocities W_1, W_0 respectively, in the direction of the applied pressure gradient. Using the following non-dimensional variables,

$$u = \frac{\mu}{\rho k}u', \quad y = ky', \quad x = kx', \quad t = \frac{\rho k^2}{\mu}t', \quad p = \frac{\mu^2}{\rho k^2}p', \quad a^2 = \frac{k^2}{l^2} \tag{9}$$

equation (8) reduces to,

$$\frac{\partial u'}{\partial t'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2} - \frac{1}{a^2}\frac{\partial^4 u'}{\partial y'^4}, \tag{10}$$

where $l^2 = \frac{\eta}{\mu}$. Neglecting the primes, the equation (10) takes the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^4 u}{\partial y^4}. \tag{11}$$

The run up flow problem considered will be solved presuming that the couple stresses vanish on the boundary in addition to the no-slip condition on the boundary.

3 Initial state for the run-up flow

In the initial state, we consider the steady flow of an incompressible couple stress fluid between two infinite rigid parallel plates under constant pressure gradient G. Therefore

$$u = u(y) \quad \text{and} \quad -\frac{\partial p}{\partial x} = G. \tag{12}$$

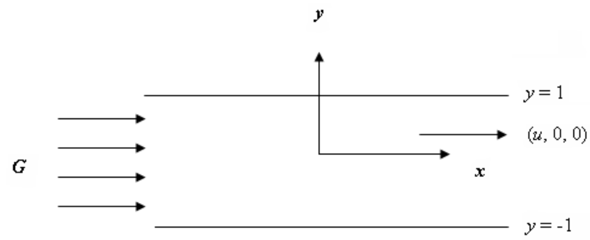

Fig. 1. Initial state of run up flow.

The flow is characterized by the momentum equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}y^4} - a^2 \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - Ga^2 = 0. \tag{13}$$

The general solution to the equation (13) is seen to be

$$u(y) = A + By + Ce^{-ay} + De^{ay} - \frac{G}{2}y^2, \tag{14}$$

where A, B, C, D are arbitrary constants to be determined using the boundary conditions.

$$\begin{aligned} &u(\pm 1) = 0 && \text{(no-slip condition)}, \\ &\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0 \text{ at } y = \pm 1 && \text{(vanishing of couple stresses on the boundary)}. \end{aligned} \tag{15}$$

The initial state for run-up flow (14) employing boundary conditions (15) is

$$u(y) = \frac{G}{2}\left[\left(1 - y^2\right) - \frac{2}{a^2}\left(1 - \frac{\cosh(ay)}{\cosh(a)}\right)\right]. \tag{16}$$

4 Run-up flow

The bounding plates which are hitherto stationary are impulsively set in motion along the direction of applied pressure gradient with different velocities W_0 and W_1 respectively while the applied pressure gradient is instantaneously withdrawn. This allows us to take,

$$u = u(y) \quad \text{and} \quad -\frac{\partial p}{\partial x} = G.$$

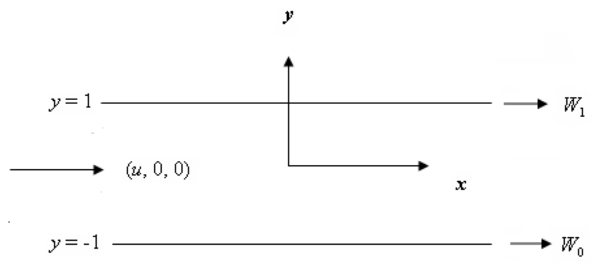

Fig. 2. Run up flow.

Now the velocity field $u(y,t)$ satisfies the equation

$$\frac{\partial^4 u}{\partial y^4} - a^2 \frac{\partial^2 u}{\partial y^2} = -a^2 \frac{\partial u}{\partial t} \tag{17}$$

together with the boundary conditions

$$\begin{aligned} &u(-1,t) = W_0 H(t), \quad u(1,t) = W_1 H(t) && \text{(no-slip condition)}, \\ &\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0 \ \text{ at } \ y = \pm 1 && \text{(vanishing of couple stresses on the boundary)}, \end{aligned} \tag{18}$$

where $H(t)$ is the Heaviside function given by

$$H(t) = \begin{cases} 1 & \text{for } \ t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Taking Laplace transforms with respect to t, of the equations (17) and (18), we get

$$\frac{\mathrm{d}^4 \bar{u}}{\mathrm{d}y^4} - a^2 \frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}y^2} + a^2 s \bar{u} = a^2 u(y,0) \tag{19}$$

with conditions

$$\begin{aligned} &\bar{u}(-1,s) = \frac{W_0}{s}, \quad \bar{u}(1,s) = \frac{W_1}{s} && \text{(no-slip condition)}, \\ &\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}y^2} = 0 \ \text{ at } \ y = \pm 1 && \text{(vanishing of couple stresses on the boundary)}. \end{aligned} \tag{20}$$

As $u(y,0) =$ initial (steady state) velocity for run-up flow $= u(y)$, using (16) we obtain,

$$\frac{\mathrm{d}^4 \bar{u}}{\mathrm{d}y^4} - a^2 \frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}y^2} + a^2 s \bar{u} = a^2 \frac{G}{2} \left[(1 - y^2) - \frac{2}{a} \left(\coth(a) - \frac{\cosh(ay)}{\sinh(a)} \right) \right]. \tag{21}$$

The solution of equation (21) with conditions (20) is given by

$$\bar{u}(y,s) = \frac{1}{\beta^2-\alpha^2}\Bigg\{\frac{W_0+W_1}{2s}\left[\beta^2\frac{\cosh(\alpha y)}{\cosh\alpha} - \alpha^2\frac{\cosh(\beta y)}{\cosh\beta}\right] - \frac{W_1-W_0}{2s}\left[\beta^2\frac{\sinh(\alpha y)}{\sinh\alpha} - \alpha^2\frac{\sinh(\beta y)}{\sinh\beta}\right] + G\left(\frac{1}{s^2}+\frac{2}{s(s+2a^2)}\right)\left[\beta^2\frac{\cosh(\alpha y)}{\cosh\alpha} - \alpha^2\frac{\cosh(\beta y)}{\cosh\beta}\right] - \frac{2Ga^2}{s(s+2a^2)}\left[\frac{\cosh(\alpha y)}{\cosh\alpha} - \frac{\cosh(\beta y)}{\cosh\beta}\right]\Bigg\} + \frac{G}{2}\left[\left(\frac{1}{s}-\frac{y^2}{s}-\frac{2}{s^2}\right) - \frac{2}{a^2}\left(\frac{1}{s}-\frac{1}{s+2a^2}\frac{\cosh(ay)}{\cosh a}\right)\right], \tag{22}$$

where

$$\alpha = \frac{a}{\sqrt{2}}\sqrt{1+\sqrt{1-\frac{4s}{a^2}}},\quad \beta = \frac{a}{\sqrt{2}}\sqrt{1-\sqrt{1-\frac{4s}{a^2}}}\quad \text{and}\quad a^2=\frac{k^2}{l^2}. \tag{23}$$

As the expression $\bar{u}(y,s)$ is in terms of α and β each one of which depends upon s, the analytical inversion seems to be difficult. Hence, we have used a standard numerical inversion procedure suggested by Honig and Hirdes [17] to determine the velocity in space time domain.

Numerical inversion procedure

In order to invert $\bar{u}(y,s)$, we adopt a numerical inversion technique due to Honig and Hirdes [17]. Using this method, the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{e^{bt}}{t_1}\left[\frac{1}{2}\bar{f}(b) + \mathrm{Re}\left(\sum_{k=1}^{N}\bar{f}\left(b+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right)\right],\quad 0<t_1\le 2t, \tag{24}$$

where N is sufficiently large integer chosen such that,

$$e^{bt}\mathrm{Re}\left[\bar{f}\left(b+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right] < \varepsilon, \tag{25}$$

where ε is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter b is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of b was obtained according to the criteria described in [17].

5 Results and discussion

We have inverted the function $\bar{u}(y,s)$ to get $u(y,t)$ for any given time t and y between -1 and 1 using the numerical approach proposed by Honig and Hirdes [17] taking $\varepsilon = 10^{-6}$. The choice of b has to be guided by the parameters G, a as well as time t and the spacial coordinate y. For each one of the sets of parameters under consideration, the quantity b is chosen to be sufficiently large so that the difference between two successive values of u obtained for two consecutive choices of b is sufficiently small.

We observe from Fig. 3 that the velocity increases as time increases at any y for a fixed value of a, G and with $W_0 = W_1 = 10$. In view of the equality of the velocities of the upper and lower plates, as expected, the velocity profiles are symmetric about y-axis.

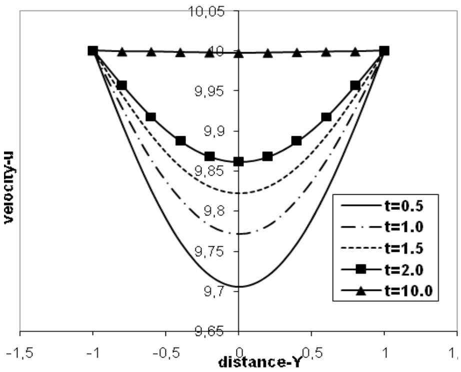

Fig. 3. Variation of velocity with distance at different times for $a = 0.5$, $W_0 = W_1 = 10$, $G = 2$.

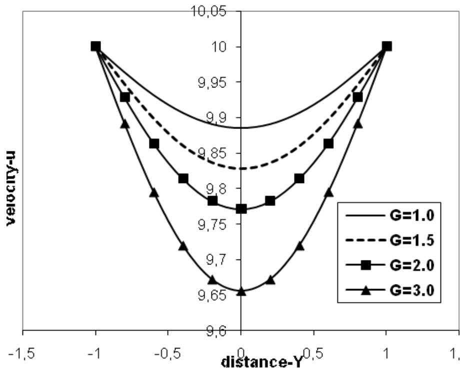

Fig. 4. Variation of velocity with distance for $t = 1.0$, $a = 0.5$, $W_0 = W_1 = 10$.

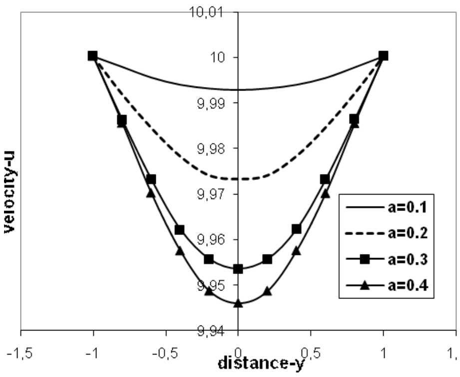

Fig. 5. Variation of velocity with distance for $t = 1.0$, $G = 2$, $W_0 = W_1 = 10$.

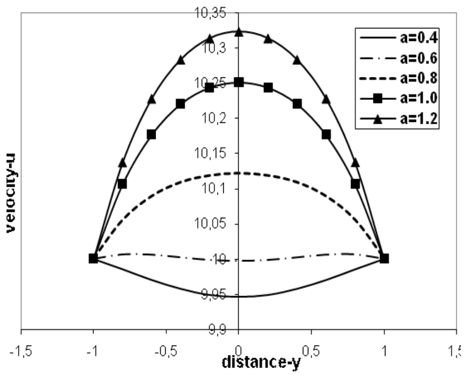

Fig. 6. Variation of velocity with distance for $t = 1.0$, $G = 2$, $W_0 = W_1 = 10$.

Fig. 4 indicates that for a fixed t, a and $W_0 = W_1 = 10$, as G increases the velocity decreases for any y. This is in accordance with the fact that an increase in G implies a decrease in pressure which naturally results in a decrease of velocity.

From Figs. 5, 6, 7 and 8 we notice that the velocity decreases as a increases from 0.1 to 0.4. When a increases from 0.4 to 1.2, for any y, the velocity increases. From that stage onwards, as a increases from 1.2 to higher values, the velocity is showing a decreasing trend. The values of $a = 0.4$ and $a = 1.2$ seem to be critical values where the trend of the velocity appears to be changing.

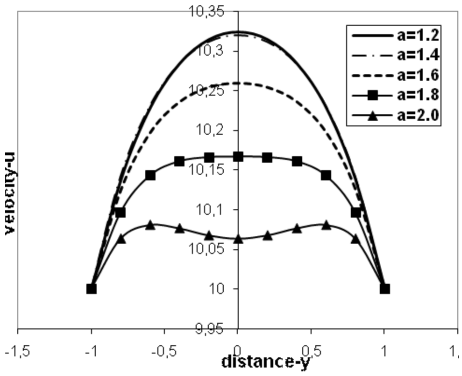

Fig. 7. Variation of velocity with distance for $t = 1.0$, $G = 2$, $W_0 = W_1 = 10$.

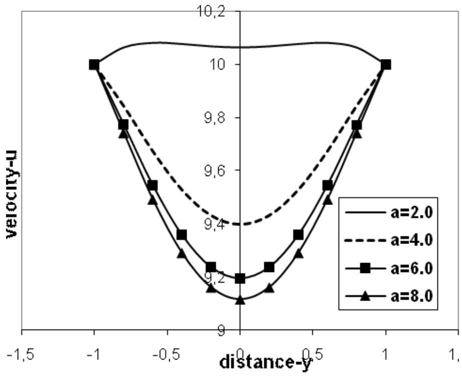

Fig. 8. Variation of velocity with distance for $t = 1.0$, $G = 2$, $W_0 = W_1 = 10$.

6 Conclusions

The run-up flow of an incompressible couple stress fluid between two infinite parallel plates is studied using Laplace transform technique. Analytical expressions for the fluid velocity field is obtained in Laplace transform domain. Using a standard numerical inversion procedure, the velocity field is obtained for the space time domain numerically. It is interesting to note that there is a critical interval of the couple stress parameter a, wherein, as a increases the velocity increases; outside this critical interval, for the range of values of a taken, as a increases, the velocity decreases.

Acknowledgement

The authors thank the referees for their suggestions which have resulted in the improvement of the presentation.

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