Modeling the depletion of dissolved oxygen in a lake due to submerged macrophytes

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Abstract. In this paper a non-linear mathematical model for depletion of dissolved oxygen in a lake due to submerged macrophytes is proposed and analyzed. It is assumed that nutrients are continuously coming to the lake with a constant rate through water run off. In the modeling process five variables are considered, namely concentration of nutrients, density of algal population, density of macrophytes, density of detritus and concentration of dissolved oxygen. Equilibria of the model have been obtained and their stability discussed. The numerical simulation is also performed to support the obtained analytical results.

Keywords: mathematical model, algae, macrophytes, dissolved oxygen, stability.

1 Introduction

Modern agriculture depends on chemical fertilizers, pesticides, etc., to produce high-quality crops for animal and human consumption. To maximize the crop yield, farmers spread nitrogen, phosphorus based fertilizers in their farming land. To improve production, they also spread herbicides to kill weeds and insecticides to kill insects to croplands. Some amount of these fertilizers, pesticides, herbicides and insecticides have been used by the crops however the remaining part of these chemical reaches to the nearer lake through water run off. These chemical contains a large amount of nutrients. Some amount of nutrients also comes in the lake through domestic drainage. Due to the presence of these nutrients into the lake, the algae and macrophytes grow very fast causing eutrophication. Eutrophication is a process by which a lake becomes enriched in dissolved nutrients (e.g. nitrogen, phosphates, etc.) that stimulate the growth of aquatic plant life and resulting in the depletion of dissolved oxygen.

Algae and macrophytes grow very fast in a lake when it becomes rich of nutrients by domestic drainage [1], water run off from agricultural fields, etc. [2]. Algae and macrophytes die out and sink to the bottom of the lake forming detritus. This detritus is being converted into nutrients through biochemical processes occurring in the lake. In these biochemical processes a huge amount of dissolved oxygen is utilized [3–6]. The

level of dissolved oxygen in the lake increases due to the surface re-aeration as well as by photosynthesis process of algae and submerged macrophytes. Since most of the algae and macrophytes float on the water surface, they reduce the transfer of oxygen from air to water through diffusion and the photosynthesis process [3, 7–11].

Several investigators have studied algal bloom and eutrophication problems caused by nutrients in the lakes, [6, 12–21]. Amemiyaa et al. [12] and Arnold and Voss [13] studied the eutrophication in lakes by considering nonlinear interactions between the variables used in their models, but they have done only simulation analysis. They have neither considered the concentration of dissolved oxygen nor macrophytes involved in the formation of nutrients from detritus. Voinov and Tonkikh [6] have presented a nonlinear mathematical model for eutrophication in macrophyte lakes. In this model they have assumed that the nutrient is supplied only by detritus of algae and macrophytes. They have not considered the additional input of nutrients either from domestic drainage or from water run off from agricultural fields. Dachs et al. [9] investigated the influence of eutrophication on air - water exchange, vertical oxygen flux, etc. in the lake Ontario. The nitrification in the water column and sediment of a lake and adjoining river system has also been studied [20]. Jayaweera and Asaeda [17] investigated biomanipulation in shallow eutrophic lakes by using a mathematical model involving phytoplankton, zooplankton, detritus, bacteria and fish population but they did not consider the supply of nutrients from outside. Some other ecological modeling studies involving phytoplankton, zooplankton and nutrients, relevant to our work, have also been conducted by Busenberg et al. [14] and Hallam [22]. However they have not considered the density of dissolved oxygen in the modeling process. Jorgenson [2] presented an eutrophication model for a lake using ecological concepts. In most of the above mentioned models, the density of macrophytes is not considered, however some studies have been conducted to study the depletion of dissolved oxygen in lakes due to excessive growth of algae [23–26]. Misra [23,24] studied the depletion of dissolved oxygen and survival of aquatic population in a lake due to presence of algae and zooplankton. Shukla et al. [26] presented a nonlinear mathematical model for the depletion of dissolved oxygen in a lake caused by algal bloom. In this model, we have not considered the role of macrophytes on the depletion of dissolved oxygen.

In view of the above, a nonlinear mathematical model for depletion of dissolved oxygen in a lake is proposed and analyzed by assuming that nutrients are continuously supplied to the lake from outside through water run off from agricultural fields, domestic drainage, etc.

2 Mathematical model

In this paper, we consider a lake, where the eutrophication process is governed by nutrients, algae, macrophytes, detritus and concentration of dissolved oxygen. Let n be the cumulative concentration of nutrients, a be the density of algae, m be the density of macrophytes, S be the density of detritus and C be the concentration of dissolved oxygen. We assume that the cumulative rate of discharge of nutrients into the lake is q, which is

depleted with rate $\alpha_0 n$. It is further assumed that the growth rate of nutrients by detritus is $\pi\delta S$ and the rate of depletion of nutrients by algae and macrophytes is proportional to the density of algae as well as cumulative concentration of nutrients (i.e. na) and the density of macrophytes and the cumulative concentration of nutrients (i.e. nm) respectively. Thus the growth rate of algae is proportional to na and the growth rate of macrophytes is proportional to nm as they are assumed to be wholly dependent on nutrients. The natural depletion rate of algae is assumed to be proportional to its density a and its depletion rate due to crowding is proportional to a^2. Also the natural depletion rate of macrophytes is assumed to be proportional to its density m and its depletion rate due to crowding is proportional to m^2. Since some part of natural deaths of algae and macrophytes are converted into detritus, hence we assume that the growth rate of detritus is proportional to a and m. Since biochemical processes, which converts detritus into nutrients occur inside the lake, thus we assume that the natural depletion rate of detritus is proportional to S. Let the rate of growth of dissolved oxygen by various sources is q_c (assumed constant) and its natural depletion rate is proportional to its concentration C. It is assumed that the rate of growth of dissolved oxygen by algae and macrophytes is proportional to density of algae a and density of macrophytes m respectively. The depletion rate of dissolved oxygen caused by conversion of detritus into nutrients is assumed to be proportional to the density of detritus S.

In view of the above considerations, the system is governed by the following differential equations:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = q + \pi\delta S - \alpha_0 n - \beta_1 na - \beta_2 nm,$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \theta_1\beta_1 na - \alpha_1 a - \beta_{10}a^2,$$
$$\frac{\mathrm{d}m}{\mathrm{d}t} = \theta_2\beta_2 nm - \alpha_2 m - \beta_{20}m^2, \tag{1}$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = \pi_1\alpha_1 a + \pi_2\alpha_2 m - \delta S,$$
$$\frac{\mathrm{d}C}{\mathrm{d}t} = q_c - \alpha_3 C + \lambda_{11}a + \lambda_{22}m - \delta_1 S,$$

where $n(0) \geq 0$, $a(0) \geq 0$, $m(0) \geq 0$, $S(0) \geq 0$, $C(0) \geq 0$.

Here, the coefficients α_0 and α_3 represents the natural depletion rate of nutrients and dissolved oxygen respectively and are positive. The constants α_1 and α_2 represents the natural death rate of algae and macrophytes respectively whereas constant δ represents the depletion rate of detritus due to the biochemical processes occurring in the lake. The coefficients β_1, β_2, θ_1, θ_2 are proportionality constants, which are also positive. The constants β_{10} and β_{20} are coefficient corresponding to crowding terms(flaking off coefficients [4]) of algae and macrophytes with respect to the aquatic habitat. π, π_1 and π_2 are positive proportionality constants and are less than 1. The coefficients λ_{11} and λ_{22} represents the growth rate proportionality constants of dissolved oxygen due to algae and macrophytes respectively whereas δ_1 represents the depletion rate proportionality constant of dissolved oxygen due to conversion of detritus into nutrients through biochemical

processes.

3 Equilibrium analysis

System (1) has following four non-zero equilibria.

1. $E_1(\frac{q}{\alpha_0}, 0, 0, 0, \frac{q_c}{\alpha_3})$ always exists.
This equilibrium states that algae and macrophytes both are not present in the lake. As detritus is formed due to the natural death of algae and macrophytes, which are absent and so detritus is also absent. In this case the concentrations of nutrients and dissolved oxygen reach their respective equilibrium values.

2. $E_2(n_2^*, a_2^*, 0, S_2^*, C_2^*)$ exists, provided

$$\theta_1\beta_1 q - \alpha_0\alpha_1 > 0, \tag{2}$$
$$q_c + \lambda_{11}a_2^* - \delta_1 S_2^* > 0. \tag{3}$$

This equilibrium states that only algae is present however macophytes are absent in the lake. In this case detritus will be formed due to the natural death of algae only and the concentration dissolved oxygen decreases due to the growth of algae.

3. $E_3(n_3^*, 0, m_3^*, S_3^*, C_3^*)$ exists, provided

$$\theta_2\beta_2 q - \alpha_0\alpha_2 > 0, \tag{4}$$
$$q_c + \lambda_{22}m_3^* - \delta_1 S_3^* > 0. \tag{5}$$

This equilibrium states that macrophytes are present and algae is absent in the lake. In this case also detritus will be formed due to the natural death of macrophytes only and the concentration of dissolved oxygen decreases due to the growth of macrophytes.

4. $E_4(n^*, a^*, m^*, S^*, C^*)$ exists, provided

$$q - \pi\left(\frac{\pi_1\alpha_1{}^2}{\beta_{10}} + \frac{\pi_2\alpha_2{}^2}{\beta_{20}}\right) > 0, \tag{6}$$
$$\theta_1\beta_1 n^* - \alpha_1 > 0, \tag{7}$$
$$\theta_2\beta_2 n^* - \alpha_2 > 0, \tag{8}$$
$$q_c + \lambda_{11}a^* + \lambda_{22}m^* - \delta_1 S^* > 0. \tag{9}$$

This is the most interesting equilibrium in which all the system variables are present. In this case the concentration of nutrients will be less in comparison to the equilibrium E_2 and E_3, since nutrients will be utilized for the growth of both algae and macrophytes. In this case the concentration of dissolved oxygen will be less in comparison to the above cases. Here it may be also noted that for very small values of rate of input of nutrients (i.e. $q \approx 0$), condition (6) may be violated and we will not get the positive value of n^* and thus positive values of a^*, m^*, S^*.

The existence of equilibria $E_1(\frac{q}{\alpha_0}, 0, 0, 0, \frac{q_c}{\alpha_3})$ is obvious. In the following, we show the existence of equilibria $E_2(n_2^*, a_2^*, 0, S_2^*, C_2^*)$, $E_3(n_3^*, 0, m_3^*, S_3^*, C_3^*)$ and $E_4(n^*, a^*, m^*, S^*, C^*)$ in detail. In the equilibrium $E_2(n_2^*, a_2^*, 0, S_2^*, C_2^*)$, the values of n_2^*, a_2^*, S_2^* and C_2^* are obtained by solving the following algebraic equations:

$$q + \pi\delta S - \alpha_0 n - \beta_1 na = 0, \tag{10}$$

$$\theta_1\beta_1 n - \alpha_1 - \beta_{10}a = 0, \tag{11}$$

$$\pi_1\alpha_1 a - \delta S = 0, \tag{12}$$

$$q_c - \alpha_3 C + \lambda_{11}a - \delta_1 S = 0. \tag{13}$$

Solving (11) and (12) for n and S in terms of a and substituting in equation (10), we get the following quadratic equation in a,

$$\beta_1\beta_{10}a^2 + (\beta_{10}\alpha_0 + \beta_1\alpha_1 - \pi\pi_1\theta_1\beta_1\alpha_1)a - (\theta_1\beta_1 q - \alpha_0\alpha_1) = 0. \tag{14}$$

The above quadratic equation (14) will have a unique positive root if the condition (2) is satisfied.

Let the condition (2) be satisfied and a_2^* be the positive value of a in equation (14). Using this value of a_2^* in equations (11) and (12), we get positive values of n and S, say n_2^* and S_2^* respectively. Finally using a_2^* and S_2^* in equation (13), we get positive value of C, say C_2^*, provided condition (3) is satisfied.

Since existence of equilibrium E_3 is similar as E_2, hence omitted.

The values of n^*, a^*, m^*, S^* and C^* in $E_4(n^*, a^*, m^*, S^*, C^*)$ are obtained by solving the following algebraic equations

$$q + \pi\delta S - \alpha_0 n - \beta_1 na - \beta_2 nm = 0, \tag{15}$$

$$\theta_1\beta_1 n - \alpha_1 - \beta_{10}a = 0, \tag{16}$$

$$\theta_2\beta_2 n - \alpha_2 - \beta_{20}m = 0, \tag{17}$$

$$\pi_1\alpha_1 a + \pi_2\alpha_2 m - \delta S = 0, \tag{18}$$

$$q_c - \alpha_3 C + \lambda_{11}a + \lambda_{22}m - \delta_1 S = 0. \tag{19}$$

Using equations (16) and (17) in equation (18), we get

$$S = \frac{1}{\delta}\left[\left(\frac{\pi_1\theta_1\beta_1\alpha_1}{\beta_{10}} + \frac{\pi_2\theta_2\beta_2\alpha_2}{\beta_{20}}\right)n - \left(\frac{\pi_1\alpha_1^2}{\beta_{10}} + \frac{\pi_2\alpha_2^2}{\beta_{20}}\right)\right]. \tag{20}$$

Substituting the values of a, m and S from equations (16), (17) and (20) in equation (15), we get the following quadratic equation in n:

$$\left(\frac{\theta_1\beta_1^2}{\beta_{10}} + \frac{\theta_2\beta_2^2}{\beta_{20}}\right)n^2 - \left[\pi\left(\frac{\pi_1\theta_1\beta_1\alpha_1}{\beta_{10}} + \frac{\pi_2\theta_2\beta_2\alpha_2}{\beta_{20}}\right) + \frac{\beta_1\alpha_1}{\beta_{10}} + \frac{\beta_2\alpha_2}{\beta_{20}} - \alpha_0\right]n$$
$$- \left[q - \pi\left(\frac{\pi_1\alpha_1^2}{\beta_{10}} + \frac{\pi_2\alpha_2^2}{\beta_{20}}\right)\right] = 0. \tag{21}$$

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This quadratic equation will have a unique positive root (say n^*), provided condition (6) is satisfied.

Using this value of n^* in equations (16), (17) and (20) respectively, we get positive values of a^*, m^* and S^*, provided conditions (7) and (8) are satisfied.

Further using a^*, m^* and S^* in equation (19), we get positive value of C, say C^*, provided condition (9) is satisfied.

Keeping in view the model (1) it is noted that growth rate of nutrients in absence of macrophytes is greater than in its presence. Therefore using a comparison theorem, [27] it can be easily concluded that the cumulative concentration of nutrients is greater in absence of macrophytes than when macrophytes is present. Thus $n_2^* > n^*$. Similarly we can also show that $n_3^* > n^*$.

Remark 1. From equation (21) it is easy to note that $\frac{dn^*}{dq} > 0$. Using this, from equations (16), (17) and (18) we easily get that $\frac{da^*}{dq} > 0$, $\frac{dm^*}{dq} > 0$ and $\frac{dS^*}{dq} > 0$. This implies that as the rate of input of nutrients through water run off q increases, the equilibrium levels of density of algae, macrophytes and detritus increases. Now from equation (19), we get

$$\frac{dC^*}{dq} = \frac{1}{\alpha_3} \left[\theta_1 \beta_1 \alpha_1 \left(\lambda_{11} - \frac{\pi_1 \delta_1 \alpha_1}{\delta} \right) + \theta_2 \beta_2 \alpha_2 \left(\lambda_{22} - \frac{\pi_2 \delta_1 \alpha_2}{\delta} \right) \right] \frac{dn^*}{dq}. \quad (22)$$

Since most of algae float on the surface of the water body and therefore the oxygen formed by algae due to photosynthesis will go to the atmosphere and will have very little chance to get dissolve into the water below the water surface, therefore λ_{11} would be very small. Similarly oxygen formed by emerging macrophytes due to photosynthesis will go to the atmosphere, thus λ_{22} will also be very small. Hence we note that $\frac{dC^*}{dq}$ is negative. Thus, the concentration of dissolved oxygen into the water body decreases as the rate of introduction of nutrients by water run off q increases.

4 Stability analysis

In this section, we discuss the stability behaviors of E_1, E_2, E_3 and E_4 in detail. The general variational matrix of model system (1) is given as follows:

$$M = \begin{pmatrix} -r_{11} & -\beta_1 n & -\beta_2 n & \pi\delta & 0 \\ \theta_1 \beta_1 a & r_{22} & 0 & 0 & 0 \\ \theta_2 \beta_2 m & 0 & r_{33} & 0 & 0 \\ 0 & \pi_1 \alpha_1 & \pi_2 \alpha_2 & -\delta & 0 \\ 0 & \lambda_{11} & \lambda_{22} & -\delta_1 & -\alpha_3 \end{pmatrix}$$

where $r_{11} = \alpha_0 + \beta_1 a + \beta_2 m$, $r_{22} = \theta_1 \beta_1 n - \alpha_1 - 2\beta_{10} a$ and $r_{33} = \theta_2 \beta_2 n - \alpha_2 - 2\beta_{20} m$.

Let M_i be the variational matrix M evaluated at equilibrium E_i, $(i = 1, 2, 3, 4)$.

From the matrix M_1, it is easy to note that the eigenvalues of M_1 are $-\alpha_0$, $\frac{\theta_1 \beta_1 q - \alpha_0 \alpha_1}{\alpha_0}$, $\frac{\theta_2 \beta_2 q - \alpha_0 \alpha_2}{\alpha_0}$, $-\delta$ and $-\alpha_3$.

Since three eigenvalues of M_1 are clearly negative, so stability of E_1 will depend on the sign of $(\theta_1 \beta_1 q - \alpha_0 \alpha_1)$ and $(\theta_2 \beta_2 q - \alpha_0 \alpha_2)$. As we know that $(\theta_1 \beta_1 q - \alpha_0 \alpha_1) > 0$

whenever E_2 exists (see condition (2)) and $(\theta_2\beta_2 q - \alpha_0\alpha_2) > 0$ whenever E_3 exists (see condition (4)).

Thus if E_2 or E_3 exists, then E_1 is a saddle point with stable manifold locally in the $n - S - C$ space and with unstable manifold locally in the $a - m$ plane.

From the matrix M_2, we may easily note that one of the eigenvalues of M_2 is $\theta_2\beta_2 n_2^* - \alpha_2$.

But we have already noted that $n_2^* > n^*$. This implies that $\theta_2\beta_2 n_2^* - \alpha_2 > \theta_2\beta_2 n^* - \alpha_2$, which is definitely positive whenever E_4 exists (see condition (8)).

By using Routh–Hurwitz criterion it is easy to show that all the other four eigenvalues of matrix M_2 will be either negative or with negative real part. Thus if E_4 exists, then E_2 is a saddle point with stable manifold locally in the $n - a - S - C$ space and with unstable manifold locally in the m-direction.

Similarly from matrix M_3, we may also note that one of the eigenvalues of M_3 is $\theta_1\beta_1 n_3^* - \alpha_1$, which is definitely positive whenever E_4 exists (as $n_3^* > n^*$ and condition (7)).

Again by using Routh–Hurwitz criterion, we can easily show that the remaining four eigenvalues of matrix M_3 will be either negative or with negative real part. Thus if E_4 exists, then E_3 is a saddle point with stable manifold locally in the $n - m - S - C$ space and with unstable manifold locally in the a-direction.

As we cannot say much about the stability behavior of E_4 from the corresponding variational matrix M_4, we study the stability behavior of this equilibria by using Liapunov's method.

Theorem 1. *The equilibrium $E_4(n^*, a^*, m^*, S^*, C^*)$ is locally stable provided the following condition is satisfied*

$$\frac{\pi^2}{\alpha_0 + \beta_1 a^* + \beta_2 m^*} < \frac{1}{2}n^* \min\left[\frac{\beta_{10}}{\theta_1\pi_1^2\alpha_1^2}, \frac{\beta_{20}}{\theta_2\pi_2^2\alpha_2^2}\right]. \tag{23}$$

Proof. We linearize model (1) by using the following transformations:

$$n = n^* + n_1, \quad a = a^* + a_1, \quad m = m^* + m_1,$$
$$S = S^* + s, \quad C = C^* + c,$$

where n_1, a_1, m_1, s and c are small perturbations around the equilibria E_4.

Now choosing the following positive definite function

$$V = \frac{1}{2}\left(n_1^2 + p_1 a_1^2 + p_2 m_1^2 + p_3 s^2 + p_4 c^2\right), \tag{24}$$

where p_1, p_2, p_3 and p_4 are positive constants to be chosen suitably.

Differentiating V with respect to t along the solutions of linearized system of model (1) and choosing $p_1 = \frac{n^*}{\theta_1 a^*}$, $p_2 = \frac{n^*}{\theta_2 m^*}$, we get

$$\frac{dV}{dt} = -(\alpha_0 + \beta_1 a^* + \beta_2 m^*)n_1^2 - \frac{\beta_{10}n^*}{\theta_1}a_1^2 - \frac{\beta_{20}n^*}{\theta_2}m_1^2 - p_3\delta s^2 - p4\alpha_3 c^2$$
$$+ (\pi\delta)n_1 s + (p_3\pi_1\alpha_1)a_1 s + (p_3\pi_2\alpha_2)m_1 s$$
$$+ (p_4\lambda_{11})a_1 c + (p_4\lambda_{22})m_1 c - (p_4\delta_1)sc. \tag{25}$$

We note that $\frac{\mathrm{d}V}{\mathrm{d}t}$ will be negative definite if the following inequalities are satisfied:

$$p_3 > \frac{\pi^2 \delta}{\alpha_0 + \beta_1 a^* + \beta_2 m^*}, \tag{26}$$

$$p_3 < \frac{\beta_{10} \delta n^*}{2\theta_1 \pi_1^2 \alpha_1^2}, \tag{27}$$

$$p_3 < \frac{\beta_{20} \delta n^*}{2\theta_2 \pi_2^2 \alpha_2^2}, \tag{28}$$

$$p_4 < \frac{2}{3} \frac{\beta_{10} \alpha_3 n^*}{\theta_1 \lambda_{11}^2}, \tag{29}$$

$$p_4 < \frac{2}{3} \frac{\beta_{20} \alpha_3 n^*}{\theta_2 \lambda_{22}^2}, \tag{30}$$

$$p_4 < \frac{1}{3} p_3 \frac{\delta \alpha_3}{\delta_1^2}. \tag{31}$$

From inequalities (26)–(28) we may choose a positive $p_3 = p$ (say) provided condition (23) is satisfied. $\qquad\square$

Now from inequalities (29)–(31) we may choose positive p_4 as follows:

$$0 < p_4 < \frac{2}{3} \alpha_3 \min \left[\frac{\beta_{10} n^*}{\theta_1 \lambda_{11}^2}, \frac{\beta_{20} n^*}{\theta_2 \lambda_{22}^2}, \frac{p\delta}{2\delta_1^2} \right]. \tag{32}$$

In the following we show the nonlinear stability of equilibrium E_4. For this we need the following lemma, which is stated without proof, following [23, 28].

Lemma 1. *The set*

$$\Omega := \left\{ (n, a, m, S, C) \in \Re_5^+ : \ 0 \le n + a + m + S \le \frac{q}{\delta_m}, \ 0 \le C \le R_C \right\} \tag{33}$$

is a region of attraction for all solutions initiating in the interior of positive octant, where

$$\delta_m = \min\big\{\alpha_0, (1 - \pi)\delta, (1 - \pi_1)\alpha_1, (1 - \pi_2)\alpha_2\big\} \ \ and \ \ R_C = \frac{q_c \delta_m + \lambda_{11} q}{\delta_m \alpha_3}.$$

Theorem 2. *The equilibrium $E_4(n^*, a^*, m^*, S^*, C^*)$ is nonlinearly stable inside the region of attraction Ω provided the following condition is satisfied*

$$\frac{\pi^2}{\alpha_0} < \frac{1}{2} n^* \min \left[\frac{\beta_{10}}{\theta_1 \pi_1^2 \alpha_1^2}, \frac{\beta_{20}}{\theta_2 \pi_2^2 \alpha_2^2} \right]. \tag{34}$$

Proof. To prove this theorem, we consider the following positive definite function

$$V = \frac{1}{2}(n - n^*)^2 + k_1 \left(a - a^* - a^* \ln \frac{a}{a^*} \right) + k_2 \left(m - m^* - m^* \ln \frac{m}{m^*} \right)$$

$$+ \frac{1}{2} k_3 (S - S^*)^2 + \frac{1}{2} k_4 (C - C^*)^2, \tag{35}$$

where k_1, k_2, k_3 and k_4 are positive constants to be chosen suitably.

Differentiating V with respect to t along the solutions of model (1) and choosing $k_1 = \frac{n^*}{\theta_1}$, $k_2 = \frac{n^*}{\theta_2}$, we get

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = &- (\beta_1 a + \beta_2 m)(n - n^*)^2 \\
&- \frac{1}{2}g_{11}(n - n^*)^2 + g_{14}(n - n^*)(S - S^*) - \frac{1}{2}g_{44}(S - S^*)^2 \\
&- \frac{1}{2}g_{22}(a - a^*)^2 + g_{24}(a - a^*)(S - S^*) - \frac{1}{2}g_{44}(S - S^*)^2 \\
&- \frac{1}{2}g_{33}(m - m^*)^2 + g_{34}(m - m^*)(S - S^*) - \frac{1}{2}g_{44}(S - S^*)^2 \\
&- \frac{1}{2}g_{22}(a - a^*)^2 + g_{25}(a - a^*)(C - C^*) - \frac{1}{2}g_{55}(C - C^*)^2 \\
&- \frac{1}{2}g_{33}(m - m^*)^2 + g_{35}(m - m^*)(C - C^*) - \frac{1}{2}g_{55}(C - C^*)^2 \\
&- \frac{1}{2}g_{44}(S - S^*)^2 + g_{45}(S - S^*)(C - C^*) - \frac{1}{2}g_{55}(C - C^*)^2, \qquad (36)
\end{aligned}
$$

where $g_{11} = 2\alpha_0$, $g_{22} = \frac{\beta_{10}n^*}{\theta_1}$, $g_{33} = \frac{\beta_{20}n^*}{\theta_2}$, $g_{44} = \frac{1}{2}k_3\delta$, $g_{55} = \frac{2}{3}k_4\alpha_3$, $g_{14} = \pi\delta$, $g_{24} = k_3\pi_1\alpha_1$, $g_{34} = k_3\pi_2\alpha_2$, $g_{25} = k_4\lambda_{11}$, $g_{35} = k_4\lambda_{22}$, $g_{45} = k_4\delta_1$.

Now, the sufficient conditions for $\frac{dV}{dt}$ to be negative definite inside region of attraction Ω are $g_{ij}^2 < g_{ii}g_{jj}$. This gives the following inequalities:

$$
k_3 > \frac{\pi^2\delta}{\alpha_0}, \tag{37}
$$

$$
k_3 < \frac{\beta_{10}\delta n^*}{2\theta_1\pi_1^2\alpha_1^2}, \tag{38}
$$

$$
k_3 < \frac{\beta_{20}\delta n^*}{2\theta_2\pi_2^2\alpha_2^2}, \tag{39}
$$

$$
k_4 < \frac{2}{3}\frac{\beta_{10}\alpha_3 n^*}{\theta_1\lambda_{11}^2}, \tag{40}
$$

$$
k_4 < \frac{2}{3}\frac{\beta_{20}\alpha_3 n^*}{\theta_2\lambda_{22}^2}, \tag{41}
$$

$$
k_4 < \frac{1}{3}k_3\frac{\delta\alpha_3}{\delta_1^2}. \tag{42}
$$

From inequalities (37)–(39) we may easily choose a positive $k_3 = k$ (say) provided condition (34) is satisfied. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

After choosing k_3 we may choose positive k_4 from inequalities (40)–(42) as follows:

$$
0 < k_4 < \frac{2}{3}\alpha_3 \min\left[\frac{\beta_{10}n^*}{\theta_1\lambda_{11}^2}, \frac{\beta_{20}n^*}{\theta_2\lambda_{22}^2}, \frac{k\delta}{2\delta_1^2}\right]. \tag{43}
$$

It is easy to note that the conditions (23) and (34) will be easily satisfied for small values of π and large values of α_0, β_{10} and β_{20}. The above two theorems imply that the rate of conversion of detritus into nutrients has destabilizing effect on the system. However natural depletion rate of nutrients and rate of removal/ food utilization of algae and macrophytes due to crowding have stabilizing effect on the system.

5 Numerical example

To check the feasibility of our analysis regarding the existence of interior equilibrium E_4 and corresponding stability conditions, we conduct some numerical computation by choosing the following values of the parameters in model (1). Some of these parameter values have been taken from Amemiyaa et al. [12] and its cross-references.

$$q = 0.5 \text{ mg} \, l^{-1} \text{day}^{-1}, \quad \alpha_0 = 0.005 \text{ day}^{-1}, \quad \beta_1 = 0.4 \, l \, mg^{-1} \text{day}^{-1}, \quad \pi = 0.02,$$
$$\beta_2 = 0.6 \, l \, mg^{-1} \text{day}^{-1}, \quad \delta = 0.04 \text{ day}^{-1}, \quad \theta_1 = 0.9, \quad \alpha_1 = 0.025 \text{ day}^{-1},$$
$$\beta_{10} = 0.002 \, l \, mg^{-1} \text{day}^{-1}, \quad \alpha_2 = 0.02 \text{ day}^{-1}, \quad \beta_{20} = 0.004 \, l \, mg^{-1} \text{day}^{-1}, \quad (44)$$
$$\theta_2 = 0.9, \quad \pi_1 = 0.9, \quad \pi_2 = 0.9, \quad q_c = 0.2 \text{ mg} \, l^{-1} \text{day}^{-1}, \quad \alpha_3 = 0.01 \text{ day}^{-1},$$
$$\lambda_{11} = 0.02 \text{ day}^{-1}, \quad \lambda_{22} = 0.02 \text{ day}^{-1}, \quad \delta_1 = 0.06 \text{ day}^{-1}.$$

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_4(n^*, a^*, m^*, S^*, C^*)$ are satisfied and E_4 is given by

$$n^* = 0.0892 \text{ mg} \, l^{-1}, \quad a^* = 3.5553 \text{ mg} \, l^{-1}, \quad m^* = 7.0415 \text{ mg} \, l^{-1},$$
$$S^* = 5.1685 \text{ mg} \, l^{-1}, \quad C^* = 10.1825 \text{ mg} \, l^{-1}.$$

The eigenvalues of the variational matrix M_4 corresponding to this equilibrium E_4 are obtained as, $-0.01, -0.016, -0.039, -0.083, -7.754$, which are all negative. Hence the equilibrium E_4 of model system (1) is locally stable.

With the above values of parameters, we note that conditions for nonlinear stability (23) and (34) are satisfied. Further, for the above set of parameters, the solution trajectories of a verses C and m verses C for different initial starts have been drawn in Figs. 1, 2. From Fig. 1, it is clear that all trajectories are attracted by (a^*, C^*), which indicates the global stability behavior of positive equilibrium of the two dimensional model consisting of algae and dissolved oxygen that can be obtained from the five-dimensional model (1) in $a - C$ plane. Similarly in Fig. 2, we can see that all trajectories are attracted by (m^*, C^*), which indicates the global stability behavior of positive equilibrium of the two dimensional model consisting of macrophytes and dissolved oxygen that can be obtained from the five-dimensional model (1) in $m - C$ plane. The effect of rate of inflow of nutrients q in the lake on variables a, m, S and C is presented in the Figs. 3–6. Here, the values of other parameters are as given in (44) except q. From these figures, it is clear that as rate of input of nutrients through water run off q increases, the equilibrium levels of densities of algae, macrophytes and detritus increase, whereas concentration of dissolved oxygen decreases. From the Fig. 3, Fig. 4 and Fig. 5 we also see that if the rate of input of

nutrients in the lake is zero, i.e. $q = 0$, then the density of algae, macrophytes and detritus tend to zero after a short period of time and in this case concentration of dissolved oxygen tends to its maximum value, i.e. $\frac{q_c}{\alpha_3}$, which is shown in Fig. 6. In Fig. 6, we may observe that initially the concentration of dissolved oxygen increases then starts to decrease for $q = 0.5$ and $q = 0.8$. This is because of the fact that initially the density of detritus is less and as density of detritus increases, the concentration of dissolved oxygen starts to decrease. Fig. 7 depicts the effect of rate of conversion of detritus into nutrients on concentration of dissolved oxygen. It is noted that as the conversion rate of detritus into nutrients increases, concentration of dissolved oxygen decreases. It is therefore, speculated that suitable control mechanism should be applied to reduce the conversion rate of detritus into nutrients to maintain the level of dissolved oxygen.
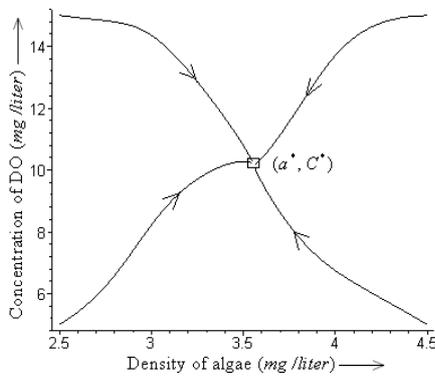
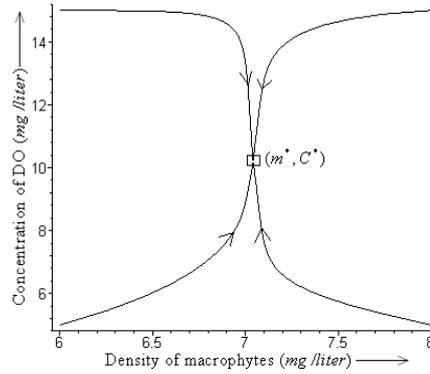Fig. 1. Nonlinear stability in $a - C$ plane.
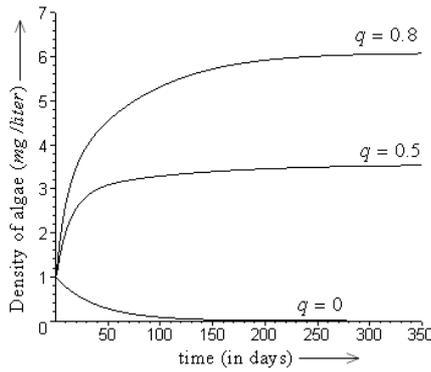

Fig. 2. Nonlinear stability in $m - C$ plane.

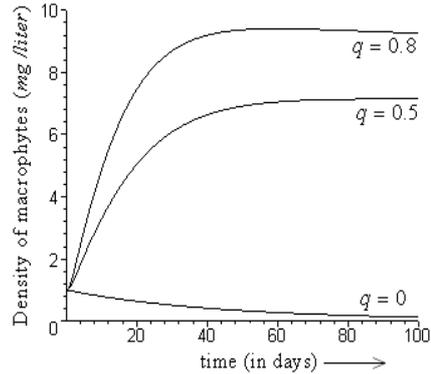Fig. 3. Variation of algae with time for different values of q.

Fig. 4. Variation of macrophytes with time for different values of q.

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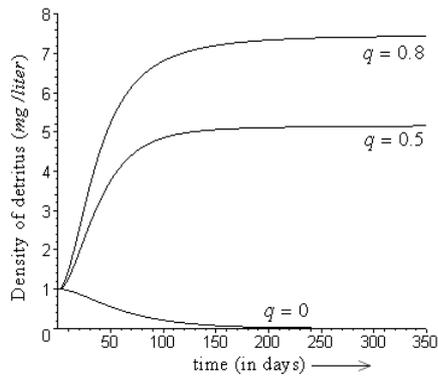
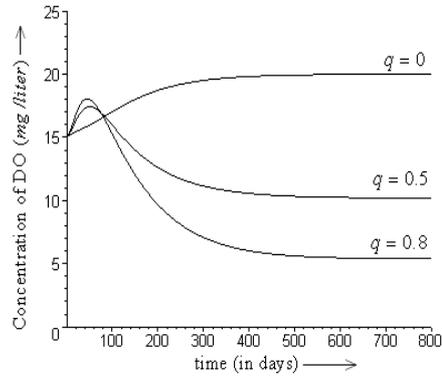Fig. 5. Variation of detritus with time for different values of q.

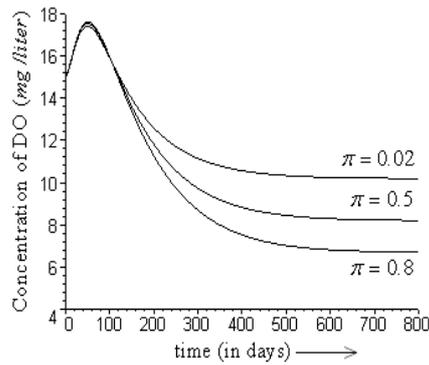Fig. 6. Variation of dissolved oxygen with time for different values of q.

Fig. 7. Variation of dissolved oxygen with time for different values of π.

6 Conclusion

In this paper, a nonlinear mathematical model for depletion of dissolved oxygen has been proposed and analyzed. The model exhibits four nonzero equilibria. It is shown that conversion of detritus into nutrients has destabilizing effect on the system, however natural depletion of nutrients and removal/food utilization of algae and macrophytes have stabilizing effect on the system. It is also shown that as the rate of input of nutrients into the lake increases, the equilibrium levels of concentration of nutrients, density of algae, macrophytes and detritus increase whereas the concentration of dissolved oxygen decreases. These results have also been shown by numerical simulation. By numerical simulation it is shown that as the conversion rate of detritus into nutrients increases, the equilibrium level of dissolved oxygen in the lake decreases. It is also noted that if the rate of input of nutrients into the lake (i.e. q) is 0, then the densities of algae, macrophytes

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and detritus are tending towards zero whereas the concentration of dissolved oxygen in the lake tends towards q_c/α_3. Thus if one wants to reduce the impact of eutrophication on the lake, then some control mechanism should be applied to reduce the input load of nutrients into the lake.

Acknowledgement

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