On the eigenvalue problems for differential operators with coupled boundary conditions

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Abstract. In the paper, the eigenvalue problems for one- and two-dimensional second order differential operators with nonlocal coupled boundary conditions are considered. Conditions for the existence of zero, positive, negative or complex eigenvalues are proposed and analytical expressions of eigenvalues are provided.

Keywords: coupled boundary conditions, eigenvalue problem, differential operator.

1 Introduction

The present paper deals with the eigenvalue problems for one- and two-dimensional second order differential operators with given nonlocal coupled boundary conditions. The corresponding finite-difference (discrete) problems have been investigated in the paper [1].

First of all, we will consider the eigenvalue problem for one-dimensional differential operator with given nonlocal coupled boundary conditions,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0, \quad 0 < x < 1, \tag{1}$$

$$u(0) = \gamma_0 u(1), \tag{2}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = \gamma_1 \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1}, \tag{3}$$

where $\gamma_0, \gamma_1 \in \mathbb{R}$, $\gamma_0 + \gamma_1 \neq 0$. We will also briefly discuss the similar two-dimensional problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0, \quad 0 < x < 1,\ \ 0 < y < 1, \tag{4}$$

with the classical boundary conditions

$$u(x,0) = u(x,1) = 0, \quad 0 < x < 1, \tag{5}$$

and the coupled boundary conditions

$$u(0,y) = \gamma_0 u(1,y), \tag{6}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \gamma_1 \left.\frac{\partial u}{\partial x}\right|_{x=1}, \quad 0 < y < 1. \tag{7}$$

Such values of λ that the problem (1)–(3) or (4)–(7) has the non-trivial solution are called eigenvalues, and the set of all eigenvalues is called the spectrum of the problem.

Since conditions (2), (3) and (6), (7) are nonlocal, the corresponding differential operators are non-self-adjoint. Therefore, the analysis of the spectra of these problems leads to the problems on the existence of both real and complex eigenvalues.

Let us introduce a parameter γ,

$$\gamma = \frac{1 + \gamma_0\gamma_1}{\gamma_0 + \gamma_1}.$$

The main aim of this paper is to investigate the dependence of the qualitative structure of the spectra of the differential problems (1)–(3) and (4)–(7) on the parameters γ_0, γ_1 (to be precise, on the generalized parameter γ), i.e., to formulate conditions for the existence of zero, positive, negative or complex eigenvalues, and (when it is possible) to provide analytical expressions of eigenvalues. The eigenvalue problems for differential operators with nonlocal conditions can be investigated numerically [2]. We use technique and argument which are used, for example, in the papers [3,4] to investigate similar problems with other types of nonlocal conditions.

2 The one-dimensional problem

Let us consider the one-dimensional differential problem (1)–(3) and four qualitative cases of possible values of λ: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ and $\lambda \in \mathbb{C}$.

Case 1: $\boldsymbol{\lambda < 0}$. If a number $\lambda < 0$ is an eigenvalue of the problem (1)–(3), the general solution of the equation (1) can be expressed as

$$u(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x), \quad \alpha = \sqrt{-\lambda} > 0,$$

where c_1 and c_2 are arbitrary constants. By substituting this expression into nonlocal conditions (2), (3), we get a system of two linear algebraic equations with unknowns c_1 and c_2:

$$\begin{cases} (1 - \gamma_0 \cosh\alpha)c_1 - \gamma_0 \sinh\alpha \cdot c_2 = 0, \\ \gamma_1 \alpha \sinh\alpha \cdot c_1 + \alpha(1 - \gamma_1 \cosh\alpha)c_2 = 0. \end{cases}$$

This system has a non-trivial solution if its determinant is equal to zero, i.e.,

$$D_1 = \begin{vmatrix} 1-\gamma_0\cosh\alpha & -\gamma_0\sinh\alpha \\ \gamma_1\alpha\sinh\alpha & \alpha(1-\gamma_1\cosh\alpha) \end{vmatrix} = \alpha\bigl(1+\gamma_0\gamma_1-(\gamma_0+\gamma_1)\cosh\alpha\bigr) = 0.$$

Since $\alpha > 0$, after simple rearrangements we get the equation

$$\cosh\alpha = \gamma, \quad \alpha > 0, \tag{8}$$

and the following proposition is valid.

Proposition 1. *The inequality $\gamma > 1$ is the necessary and sufficient condition for the existence of one and only one negative eigenvalue of the problem* (1)–(3)*:*

$$\lambda_{-1} = -(\operatorname{arccosh}\gamma)^2 = -\Bigl\{\ln\Bigl(\gamma+\sqrt{\gamma^2-1}\Bigr)\Bigr\}^2.$$

Case 2: $\boldsymbol{\lambda = 0}$. The general solution of the equation (1) in this case is $u(x) = c_1 + c_2x$. Similarly as in Case 1, we get a system of two linear algebraic equations with unknowns c_1 and c_2:

$$\begin{cases} (1-\gamma_0)c_1 - \gamma_0c_2 = 0, \\ (1-\gamma_1)c_2 = 0. \end{cases}$$

There exists a non-trivial solution to this system, if

$$D_2 = \begin{vmatrix} 1-\gamma_0 & -\gamma_0 \\ 0 & 1-\gamma_1 \end{vmatrix} = 1+\gamma_0\gamma_1-(\gamma_0+\gamma_1) = 0.$$

Proposition 2. *The number $\lambda_0 = 0$ is an eigenvalue of the problem* (1)–(3) *if and only if $\gamma = 1$.*

Case 3: $\boldsymbol{\lambda > 0}$. In this case, the general solution of the equation (1) is

$$u(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x), \quad \alpha = \sqrt{\lambda} > 0.$$

Using the same technique as in Case 1 and Case 2, we get a system of two linear algebraic equations

$$\begin{cases} \bigl(1-\gamma_0\cos\alpha\bigr)c_1 - \gamma_0\sin\alpha\cdot c_2 = 0, \\ \gamma_1\alpha\sin\alpha\cdot c_1 + \alpha\bigl(1-\gamma_1\cos\alpha\bigr)c_2 = 0. \end{cases}$$

By equating the determinant of this system with zero, we obtain

$$D_3 = \begin{vmatrix} 1-\gamma_0\cos\alpha & -\gamma_0\sin\alpha \\ \gamma_1\alpha\sin\alpha & \alpha(1-\gamma_1\cos\alpha) \end{vmatrix} = \alpha\bigl(1+\gamma_0\gamma_1-(\gamma_0+\gamma_1)\cos\alpha\bigr) = 0,$$

i.e.,

$$\cos\alpha = \gamma, \quad \alpha > 0. \tag{9}$$

Hence, we can prove the following

Proposition 3. *The inequality $|\gamma| \leq 1$ is the necessary and sufficient condition for the existence of infinitely many positive solutions to the equation* (9)*, i.e., for the existence of infinitely many (countable set) positive eigenvalues for the problem* (1)–(3):

$$\lambda_{2k-1} = (2k\pi - \arccos\gamma)^2, \quad \lambda_{2k} = (2k\pi + \arccos\gamma)^2, \quad k \in \mathbb{N}.$$

Remark 1. When $|\gamma| < 1$, all positive eigenvalues are simple. However, when $|\gamma| = 1$, all positive eigenvalues (except $\lambda_1 = \pi^2$, when $\gamma = -1$) are multiple (double).

Remark 2. We can observe the qualitative behaviour of real eigenvalues of the problem (1)–(3) from Fig. 1, where graphs of functions $\gamma(\alpha) = \cosh\alpha$ and $\gamma(\alpha) = \cos\alpha$, $\alpha > 0$, are exhibited.

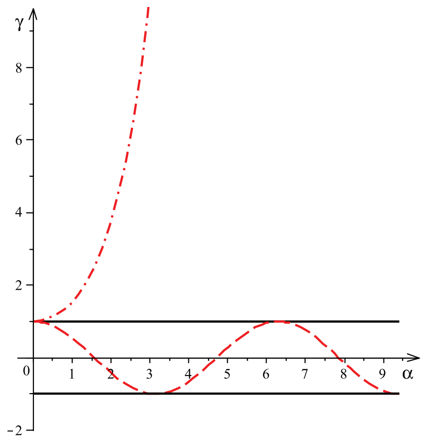

Fig. 1. The graphs of functions $\gamma(\alpha) = \cosh\alpha$ (dash-dot line) and $\gamma(\alpha) = \cos\alpha$ (dashed line), $\alpha > 0$.

Case 4: $\boldsymbol{\lambda} \in \mathbb{C}$. We represent the general solution of the equation (1) in the form

$$u(x) = c_1 \mathrm{e}^{\mathrm{i}qx} + c_2 \mathrm{e}^{-\mathrm{i}qx}, \quad q = \sqrt{\lambda} = \alpha \pm \mathrm{i}\beta, \quad \mathrm{i} = \sqrt{-1}.$$

We assume that $\alpha \neq 0$ and $\beta \neq 0$. If $\alpha = 0$, $\beta \neq 0$ or $\alpha \neq 0$, $\beta = 0$, then this case coincides with Case 1 or Case 3, respectively. However, when $\alpha = \beta = 0$, a situation is the same as in Case 2.

By substituting the expression of general solution into nonlocal conditions (2) and (3), we get

$$\begin{cases} \left(1 - \gamma_0 \mathrm{e}^{\mathrm{i}q}\right)c_1 + \left(1 - \gamma_0 \mathrm{e}^{-\mathrm{i}q}\right)c_2 = 0, \\ \mathrm{i}q\left(1 - \gamma_1 \mathrm{e}^{\mathrm{i}q}\right)c_1 - \mathrm{i}q\left(1 - \gamma_1 \mathrm{e}^{-\mathrm{i}q}\right)c_2 = 0. \end{cases}$$

The determinant of this system is equal to zero, if

$$D_4 = \begin{vmatrix} 1-\gamma_0 \mathrm{e}^{\mathrm{i}q} & 1-\gamma_0 \mathrm{e}^{-\mathrm{i}q} \\ \mathrm{i}q(1-\gamma_1 \mathrm{e}^{\mathrm{i}q}) & -\mathrm{i}q(1-\gamma_1 \mathrm{e}^{-\mathrm{i}q}) \end{vmatrix} = 2\mathrm{i}q\big(1+\gamma_0\gamma_1 - (\gamma_0+\gamma_1)\cosh(\mathrm{i}q)\big) = 0,$$

i.e.,

$$\cosh(\mathrm{i}q) = \gamma, \quad q = \alpha \pm \mathrm{i}\beta, \quad \alpha \neq 0, \quad \beta \neq 0.$$

Since $\cosh(\mathrm{i}q) = \cos(q)$, the condition for the existence of a non-trivial solution is

$$\cos q = \gamma, \quad q = \alpha \pm \mathrm{i}\beta, \quad \alpha \neq 0, \quad \beta \neq 0.$$

By separating the real and imaginary parts in the latter relation, we obtain the equations

$$\begin{cases} \cos\alpha \cdot \cosh(\pm\beta) = \gamma, \\ \sin\alpha \cdot \sinh(\pm\beta) = 0. \end{cases} \tag{10}$$

Taking into account assumption that $\alpha \neq 0$ and $\beta \neq 0$ allow us to prove the following statement:

Proposition 4. *When* $|\gamma| > 1$, *there exists the series of non-trivial solutions to the system* (10), $(\alpha_k, \pm\beta)$, $k \in \mathbb{Z} \setminus \{0\}$, *where*

$$\alpha_k = \begin{cases} (2k+1)\pi, & \text{if } \gamma < -1, \\ 2k\pi, & \text{if } \gamma > 1; \end{cases} \qquad \beta = \operatorname{arccosh}|\gamma|,$$

i.e., the problem (1)–(3) *has infinitely many (countable set) complex eigenvalues* $\lambda_{\mathrm{c},k}$. *Distinct pairs of conjugate complex eigenvalues can be calculated by the formula*

$$\lambda_{\mathrm{c},k} = (\alpha_k^2 - \beta^2) \pm \mathrm{i}(2\alpha_k\beta), \quad k \in \mathbb{N}.$$

Remark 3. When $\alpha_k^2 < \beta^2$, i.e.,

$$1 \le k < \frac{\operatorname{arccosh}|\gamma|}{2\pi} - \frac{1}{2}, \quad \text{if } \gamma < -1,$$
$$1 \le k < \frac{\operatorname{arccosh}\gamma}{2\pi}, \qquad \text{if } \gamma > 1,$$

then real parts of complex eigenvalues $\lambda_{\mathrm{c},k}$ are negative.

3 The two-dimensional problem

Now let us consider the real part of the spectrum of the two-dimensional differential eigenvalue problem (4)–(7). By separating variables, i.e., by representing the solution of the problem (4)–(7) in the form

$$u(x, y) = v(x)w(y),$$

we get two one-dimensional eigenvalue problems:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \eta v = 0, \quad 0 < x < 1, \quad v(0) = \gamma_0 v(1), \quad \left.\frac{\mathrm{d}v}{\mathrm{d}x}\right|_{x=0} = \gamma_1 \left.\frac{\mathrm{d}v}{\mathrm{d}x}\right|_{x=1}, \tag{11}$$

and

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \mu w = 0, \quad 0 < y < 1, \quad w(0) = w(1) = 0, \tag{12}$$

where $\eta + \mu = \lambda$. The problem (11) was considered in Section 2, while the problem (12) is classic. It is well-known, that all the eigenvalues of the problem (12) are real, positive, algebraically simple, and can be computed by the formula

$$\mu_l = (\pi l)^2, \quad l \in \mathbb{N}. \tag{13}$$

Let us denote

$$\lambda_{kl} = \eta_k + \mu_l. \tag{14}$$

It is easy to see, that the positivity of λ_{kl} is conditioned by the positivity of η_k. Therefore, the following statement is valid.

Proposition 5. *If* $|\gamma| \leq 1$*, then the problem* (4)–(7) *has infinitely many (countable set) positive eigenvalues:*

$$\lambda_{2k-1,l} = (2k\pi - \arccos\gamma)^2 + (\pi l)^2, \quad \lambda_{2k,l} = (2k\pi + \arccos\gamma)^2 + (\pi l)^2, \quad k, l \in \mathbb{N}.$$

Now let us investigate the existence of zero and negative eigenvalues of the problem (4)–(7). If $\lambda_{kl} = 0$, then the equations (13) and (14) imply that $\eta_k = -(\pi l)^2$. However, the negative eigenvalue of the problem (11) has the form $\eta_k = -\alpha^2$, where α is a positive root of the equation (8). Since the equation (8) has a unique positive root, $\lambda_{kl} = 0$ provided that $\alpha = \pi l$. The number $\alpha = \pi l$ is a root of the equation (8) if $\gamma = \cosh(\pi l)$. We can prove the following

Proposition 6. *If* $\gamma_l = \cosh(\pi l)$*,* $l \in \mathbb{N}$*, then, for each* $l \in \mathbb{N}$*, the problem* (4)–(7) *has an algebraically simple eigenvalue* $\lambda_{kl} = 0$*.*

If $\gamma = \cosh(\pi s)$, then the problem (11) has a negative eigenvalue $\eta_s = -(\pi s)^2$. Hence, problem (4)–(7) has $s - 1$ negative eigenvalues,

$$\lambda_{sl} = -(\pi s)^2 + (\pi l)^2, \quad l = 1, 2, \ldots, s - 1,$$

and an algebraically simple eigenvalue $\lambda_{ss} = 0$.

Proposition 7. *The number of negative eigenvalues of the problem* (4)–(7) *depends on* γ*. If*

$$\cosh(\pi s) < \gamma < \cosh(\pi(s + 1)),$$

where $s \in \mathbb{N}$*, then there exist exactly* s *negative eigenvalues of the problem* (4)–(7)*. Consequently, when* $\gamma < \cosh\pi \approx 11.59195\ldots$*, then all real eigenvalues of the problem* (4)–(7) *are positive.*

4 Concluding remarks

The qualitative information about the spectral structure of differential operator and especially its finite-difference counterpart is useful, for example, in order to analyse the stability of finite-difference schemes [5–15] or justify the convergence of iterative methods for finite-difference equations [16–18].

As a rule, any nonlocal condition, implies that, depending on nonlocal condition parameters, both real numbers (positive or non-positive) and complex numbers (with positive or non-positive real parts) can be the eigenvalues of the corresponding differential problem. Using the quite simple technique allow us to investigate the qualitative structure of the spectra of the differential problems (1)–(3) and (4)–(7).

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