

Stability analysis of an eco-epidemiological model incorporating a prey refuge

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Abstract. The present paper deals with the problem of a predator-prey model incorporating a prey refuge with disease in the prey-population. We assume the predator population will prefer only infected population for their diet as those are more vulnerable. Dynamical behaviours such as boundedness, permanence, local and global stabilities are addressed. We have also studied the effect of discrete time delay on the model. The length of delay preserving the stability is also estimated. Computer simulations are carried out to illustrate our analytical findings.

Keywords: eco-epidemiological model, permanence, global stability, Hopf bifurcation.

1 Introduction

The dynamic relationship between predator and their prey has long been and will continue to be one of the dominant topics in both applied mathematics and theoretical ecology due to its universal existence and importance. These problems may appear to be simple mathematically at first sight, they are, in fact, often very challenging and complicated.

The formation of classical Lotka–Volterra [1, 2] model was a milestone progress in the study of predator-prey interactions. Similarly, after the pioneering work of Kermack–Mckendric [3], epidemiological studies also received special attention to the researchers. Disease in ecological systems is an important issue. Anderson and May [4] were the pioneers for formulating the mathematical model on this topic. A lot of research articles have already been appeared on this subject [4, 5]. Most of the studies mainly focused on parasite infection in prey population only [6–9]. The dynamics of predator-prey system with infection in prey population is an important study from modelling point of view. Most of predators preferentially consume diseased prey [10]. Predators harm parasites directly by consuming infected prey and also harms parasites indirectly by reducing the density of susceptible hosts. Predators behaviour can magnify this direct effect because

predators often prefer infected prey over uninfected prey [11]. So that infectious disease can be a factor to regulate human and animal population size.

The research of the hiding behaviour of preys has been incorporated as a new ingredient of prey-predator models and its consequences on the dynamics of prey-predator interactions can be recognized as one of the major issues in both applied mathematics and theoretical ecology. In nature, prey populations often access to areas where they are safe from their predators. Such refugia are usually playing two significant roles, serving both to reduce the chance of extinction due to predation and to damp prey-predator oscillations. These are therefore a potentially important means of increasing species richness in natural communities and of stabilizing population sizes, biomass and productivity. It is well known that many more attentions have paid on the effects of a prey refuge for predator-prey system. Predator-prey interactions often exhibit spatial refuge which afford the prey some degree of protection from predation and reduce the chance of extinction due to predation [3, 12]. Hassel [13] showed that adding a large refuge to a model, which exhibited divergent oscillations in the absence of refuge, replaced the oscillatory behaviour with a stable equilibrium. These mathematical models and a number of experiments indicate that refuge have a stabilizing effect on predator-prey interactions.

Time delays of one type or another have been incorporated into epidemiological models by many authors [1, 2, 14]. In general, delay-differential equations exhibit much more complicated dynamics than ordinary differential equations since a time-delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate.

In this paper, we have investigated the dynamical behaviour of a ratio-dependent predator-prey systems with infection in prey population [9], and the effect of refuge in the infected prey. Here we have studied the boundedness, permanence, local and global stabilities of the non-equilibrium points of this system. We have also considered a discrete time-delay in the interaction term of the predator equation.

The rest of the paper is structured as follows: In Section 2, we present a brief sketch of the construction of the model, which may indicate the epidemiological relevance of it. In Section 3, boundedness of the basic deterministic model (3) is discussed. Section 4 deals with the boundary equilibrium points and their stability. In Section 5, we find out the condition for which system (3) is permanent. In Section 6, we find the necessary and sufficient condition for the existence of the interior equilibrium point $E^*(s^*, i^*, y^*)$ and study its stability. Computer simulations of some solutions of the system (3) are presented in Section 7. The effect of discrete time-delay on the system (3) is studied in Section 8. In Section 9, we calculate the length of delay for which the system preserves stability. Section 10 contains the general discussions of the paper.

2 The basic mathematical model

Xiao and Chen [9] have considered a predator-prey model with disease in the prey:

$$\frac{dS}{dT} = r_1 S \left(1 - \frac{S+I}{K} \right) - \beta SI, \quad (1a)$$

$$\frac{dI}{dT} = \beta SI - cI - \frac{bIY}{aY + I}, \quad (1b)$$

$$\frac{dY}{dT} = -dY + \frac{pbIY}{aY + I} \quad (1c)$$

with initial data $S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0$.

They make the following assumptions in formulating the mathematical model of a predator-prey system with disease in the prey population:

1. In the absence of disease, the prey population grows logistically with carrying capacity $K \in \mathfrak{R}_+$ and intrinsic birth rate $r_1 \in \mathfrak{R}_+$.
2. In the presence of virus, the prey population is divided into two groups, namely susceptible prey denoted by $S(T)$ and infected prey denoted by $I(T)$. Therefore at time T , the total population is $N(T) = S(T) + I(T)$.
3. The disease is not genetically inherited. The infected populations do not recover or become immune. We assume that the disease transmission follows the simple law of mass action $\beta S(T)I(T)$ with β as the transmission rate.
4. The infected prey $I(T)$ is removed by death (say, its death rate is positive constant c) or by predation before having the possibility of reproducing. However, the infected prey population $I(T)$ still contribute with $S(T)$ towards the carrying capacity of the system.
5. The infected prey is more vulnerable than susceptible prey. We assume that the predator population consumes only infected prey with ratio-dependent Michaelis-Menten functional response function

$$\eta(I, Y) = \frac{IY}{aY + I} \quad (a > 0).$$

It is assumed that the predator has the death rate constant d ($d > 0$), and the predation coefficient b ($b > 0$). The coefficient in converting prey into predator is p ($0 < p \leq 1$).

This paper extends the above model by incorporating a refuge protecting mI of the infected prey, where $m \in [0, 1)$ is constant. This leaves $(1 - m)I$ of the infected prey available to the predator, and modifying system (1) accordingly yields the system:

$$\frac{dS}{dT} = r_1 S \left(1 - \frac{S + I}{K}\right) - \beta SI, \quad (2a)$$

$$\frac{dI}{dT} = \beta SI - cI - \frac{b(1 - m)IY}{aY + (1 - m)I}, \quad (2b)$$

$$\frac{dY}{dT} = -dY + \frac{pb(1 - m)IY}{aY + (1 - m)I} \quad (2c)$$

with initial data $S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0$.

The model we have just specified has nine parameters, which makes the analysis difficult. To reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system, we nondimensionalize system (2). We choose

$$s = \frac{S}{K}, \quad i = \frac{I}{K}, \quad y = \frac{aY}{K}, \quad \text{and} \quad t = \beta KT.$$

Then system (2) takes the form (after some simplification)

$$\frac{ds}{dt} = rs\{1 - (s + i)\} - si, \tag{3a}$$

$$\frac{di}{dt} = si - b_2i - \frac{l(1 - m)iy}{y + (1 - m)i}, \tag{3b}$$

$$\frac{dy}{dt} = -b_1y + \frac{pl(1 - m)iy}{y + (1 - m)i} \tag{3c}$$

with initial data $s(0) \geq 0, i(0) \geq 0, y(0) \geq 0$, where $r = \frac{r_1}{\beta K}, b_2 = \frac{c}{\beta K}, l = \frac{b}{a\beta K}, b_1 = \frac{d}{a\beta K}$.

3 Boundedness

In theoretical eco-epidemiology, boundedness of a system implies that the system is biologically well behaved. The following theorem ensures the boundedness of system (3):

Theorem 1. *All solutions of system (3) that start in \mathbb{R}_+^3 are uniformly bounded.*

Proof. Let, $(s(t), i(t), y(t))$ be any solution of the system (3). Since, $\frac{ds}{dt} \leq rs(1 - s)$. We have,

$$\limsup_{t \rightarrow \infty} s(t) \leq r.$$

Let, $W = \frac{s}{1+r} + i + \frac{y}{p}$. Then,

$$\begin{aligned} \frac{dW}{dt} &= \frac{r}{1+r}s(1-s) - b_2i - \frac{b_1}{p}y \leq \frac{r}{1+r}s - b_2i - \frac{b_1}{p}y \\ &\leq \frac{2r}{1+r} - \delta W, \quad \text{where } \delta = \min\{1, b_2, b_1\}. \end{aligned}$$

Therefore

$$\frac{dW}{dt} + \delta W \leq \frac{2r}{1+r}.$$

Applying a theorem on differential inequalities [15], we obtain

$$0 \leq W(s, i, y) \leq \frac{2r}{(1+r)\delta} + \frac{W(s(0), i(0), y(0))}{e^{\delta t}},$$

and for $t \rightarrow \infty$

$$0 \leq W \leq \frac{2r}{(1+r)\delta}.$$

Thus, all the solutions of (3) enter into the region

$$B = \left\{ (s, i, y): 0 \leq W \leq \frac{2r}{(1+r)\delta} + \epsilon \text{ for any } \epsilon > 0 \right\}.$$

Hence the theorem. □

4 Boundary equilibria and their stability

In this section, we study the stability of the boundary equilibrium points of the system (3). In the following lemma we have mentioned the boundary equilibria of the system (3) and the condition of their existence.

Lemma 1. *System (3) always has two boundary equilibrium points, namely the trivial equilibrium $E_0(0, 0, 0)$ and the axial equilibrium $E_1(1, 0, 0)$. The predator-free equilibrium point $E_2(\hat{s}, \hat{i}, 0)$ exists if and only if $b_2 < 1$. When this condition is satisfied, \hat{s}, \hat{i} are given by*

$$\hat{s} = b_2, \quad \hat{i} = \frac{r(1 - b_2)}{1 + r}.$$

In terms of original parameters of the system, the condition $b_2 < 1$ becomes $c < \beta K$. This implies that if the ratio of the death rate of the infected prey to the carrying capacity (c/K) is less than the transmission rate (β), then the predator becomes extinct and conversely.

The system (3) cannot be linearized at $E_0(0, 0, 0)$ and $E_1(1, 0, 0)$ and therefore local stability of E_0 and E_1 cannot be studied [16]. Therefore, we are only interested in the stability in the predator-free equilibrium point $E_2(\hat{s}, \hat{i}, 0)$.

The variational matrix $V(E_2)$ at the equilibrium point E_2 is given by

$$V(E_2) = \begin{bmatrix} -r\hat{s} & -(1+r)\hat{s} & 0 \\ \hat{i} & 0 & -l \\ 0 & 0 & -b_1 + pl \end{bmatrix}.$$

The characteristic equation of $V(E_2)$ is

$$(\lambda^2 + B\lambda + C)(\lambda - pl + b_1) = 0,$$

where $B = r\hat{s} > 0$ and $C = (1+r)\hat{s}\hat{i} > 0$.

The eigen values are

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2} \quad \text{and} \quad \lambda_3 = pl - b_1.$$

Since, $B > 0$ and $C > 0$, therefore, the signs of the real parts of λ_1, λ_2 are negative. This implies that E_2 is locally asymptotically stable in the si -plane. Now E_2 is asymptotically stable in the y -direction if and only if $pl - b_1 < 0$, i.e., $b_1 > pl$.

5 Permanence of the system

To prove the permanence of the system (3), we shall use the ‘‘Average Liapunov’’ functions [17].

Theorem 2. *System (3) is permanent if $b_1 > pl + \frac{\alpha_1 r^2}{\alpha_3(1+r)}$.*

Proof. We consider the average Liapunov function of the form $V(s, i, y) = s^{\alpha_1} i^{\alpha_2} y^{\alpha_3}$ where each $\alpha_i (i = 1, 2, 3)$ is assumed positive.

In the interior of \mathfrak{R}_+^3 , we have

$$\begin{aligned} \frac{\dot{V}}{V} = \psi(s, i, y) &= \alpha_1 [r\{1 - (s + i)\} - i] \\ &+ \alpha_2 \left[s - b_2 - \frac{l(1-m)y}{y + (1-m)i} \right] + \alpha_3 \left[-b_1 + \frac{pl(1-m)i}{y + (1-m)i} \right]. \end{aligned}$$

To prove the permanence of the system we shall have to show that $\psi(s, i, y) > 0$ for all boundary equilibria of the system. The following condition should be satisfied for the equilibrium point E_2 ,

$$\begin{aligned} \alpha_1 r\{1 - (\hat{s} + \hat{i})\} - \alpha_1 \hat{i} + \alpha_3(-b_1 + pl) &> 0 \\ \implies \alpha_1 \left[r\left\{1 - \left(b_2 + \frac{r(1-b_2)}{1+r}\right)\right\} - \frac{r(1-b_2)}{1+r} \right] &+ \alpha_3(pl - b_1) > 0. \end{aligned}$$

After some simple calculation, we can get

$$\frac{\alpha_1 r^2}{1+r} < \alpha_3(b_1 - pl).$$

Hence the theorem. □

6 The interior equilibrium point: its existence and stability

First we consider the existence and uniqueness of the interior equilibrium point $E^*(s^*, i^*, y^*)$.

Lemma 2. *The interior equilibrium point $E^*(s^*, i^*, y^*)$ of the system (3) exists if and only if the following two conditions are satisfied:*

- (i) $pl > b_1$;
- (ii) $p(1 - b_2) - (1 - m)(pl - b_1) > 0$.

Furthermore, s^*, i^*, y^* are given by

$$\begin{aligned} s^* &= \frac{pb_2 + (1-m)(pl - b_1)}{p}, \quad i^* = \frac{r}{p(1+r)} \{p(1 - b_2) - (1 - m)(pl - b_1)\}, \\ y^* &= \frac{r(1-m)(pl - b_1)}{b_1 p(1+r)} \{p(1 - b_2) - (1 - m)(pl - b_2)\}. \end{aligned}$$

In terms of original parameters of the system, the conditions (i) and (ii) respectively become $pb > d$ and $ap(\beta K - c) > (1 - m)(pb - d)$, which are the necessary and sufficient conditions for the co-existence of the susceptible prey, infected prey and the predator.

From Lemma 2, we can observe that the interior equilibrium point $E^*(s^*, i^*, y^*)$ exists if and only if both the conditions (i) and (ii) are satisfied. If any one of the condition is violated then $E^*(s^*, i^*, y^*)$ does not exist. Now from condition (ii) of the Lemma 2, we have,

$$p(1 - b_2) - (1 - m)(pl - b_1) > 0 \implies m > 1 - \frac{p(1 - b_2)}{(pl - b_1)}.$$

Hence, to exist the interior equilibrium point $E^*(s^*, i^*, y^*)$ the refuge constant m should lie in the interval

$$1 - \frac{p(1 - b_2)}{(pl - b_1)} < m < 1. \quad (4)$$

Remark 1. It is to be noted that the existence of E^* destabilizes E_2 .

6.1 Local stability analysis of E^*

The variational matrix of (3) at E^* is given by

$$V(E^*) = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & v_{22} & v_{23} \\ 0 & v_{32} & v_{33} \end{bmatrix},$$

where

$$\begin{aligned} v_{11} &= -rs^*, & v_{12} &= -(1 + r)s^*, \\ v_{21} &= i^*, & v_{22} &= \frac{l(1 - m)^2 i^* y^*}{\{y^* + (1 - m)i^*\}^2}, & v_{23} &= -\frac{l(1 - m)^2 i^{*2}}{\{y^* + (1 - m)i^*\}^2}, \\ v_{32} &= \frac{pl(1 - m)y^{*2}}{\{y^* + (1 - m)i^*\}^2}, & v_{33} &= -\frac{pl(1 - m)i^* y^*}{\{y^* + (1 - m)i^*\}^2}. \end{aligned}$$

The characteristic equation is

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0,$$

where

$$\begin{aligned} A_1 &= -\text{tr} [V(E^*)] = -v_{11} - v_{22} - v_{33} = rs^* - \frac{l(1 - m)(1 - m - p)i^* y^*}{\{y^* + (1 - m)i^*\}^2} \\ &= \frac{D}{lp^2}, \\ D &= rlp^2 b_2 + pb_1(pl - b_1) + (1 - m)(pl - b_1)(rpl - b_1), \end{aligned}$$

$$\begin{aligned}
A_2 &= v_{11}v_{22} + v_{11}v_{33} + v_{22}v_{33} - v_{23}v_{32} - v_{12}v_{21} \\
&= i^*s^* \left\{ (1+r) - \frac{rl(1-m)(1-m-p)y^*}{\{y^* + (1-m)i^*\}^2} \right\}, \\
A_3 &= -\det [V(E^*)] = v_{11}v_{23}v_{32} + v_{12}v_{21}v_{33} - v_{11}v_{22}v_{33} \\
&= \frac{pl(1-m)(1+r)s^*i^{*2}y^*}{\{y^* + (1-m)i^*\}^2}.
\end{aligned}$$

Now

$$\begin{aligned}
\Delta &= A_1A_2 - A_3, \\
&= -(v_{11})^2v_{22} - (v_{11})^2v_{33} + v_{11}v_{12}v_{21} - (v_{22})^2v_{33} - v_{11}(v_{22})^2 - 2v_{11}v_{22}v_{33} \\
&\quad + v_{22}v_{12}v_{21} + v_{23}v_{32}v_{22} - v_{22}(v_{33})^2 - v_{11}(v_{33})^2 + v_{23}v_{32}v_{33} \\
&= s^*i^* \left[r(1+r)s^* - \frac{r^2l(1-m)(1-m-p)s^*y^*}{\{y^* + (1-m)i^*\}^2} - \frac{l(1+r)(1-m)^2i^*y^*}{\{y^* + (1-m)i^*\}^2} \right. \\
&\quad \left. + \frac{rl^2(1-m)^2(1-m-p)^2i^*y^{*2}}{\{y^* + (1-m)i^*\}^4} \right].
\end{aligned}$$

Theorem 3. E^* is locally asymptotically stable if and only if $D > 0$ and $\Delta > 0$.

Proof. We notice that

- (i) $D > 0 \Leftrightarrow A_1 > 0$,
 - (ii) $A_3 > 0$ for all values of the parameters,
- and
- (iii) $\Delta = A_1A_2 - A_3 > 0$.

Hence, the theorem follows from Routh Hurwitz criterion. □

6.2 Global stability analysis of E^*

Now, we shall study the global dynamics of the system (3) around the positive equilibrium $E^*(s^*, i^*, y^*)$. We use Liapunov function to prove the global result.

Theorem 4. *Existence of positive interior equilibrium of the system of equations (3) implies its global stability around the positive interior equilibrium if the following two conditions:*

- (i) $i^* < i < \frac{i^*y}{y^*}$ or $\frac{i^*y}{y^*} < i < i^*$ for all i ;
- (ii) $y > \max\{y^*, \frac{iy^*}{i^*}\}$ or $y < \min\{y^*, \frac{iy^*}{i^*}\}$ holds true.

Proof. Let us consider the following positive definite function about E^* :

$$J(s, i, y) = \left(s - s^* - s^* \ln \frac{s}{s^*} \right) + \left(i - i^* - i^* \ln \frac{i}{i^*} \right) + \left(y - y^* - y^* \ln \frac{y}{y^*} \right).$$

Differentiating J with respect to t along the solution of (3), we have (after some calculation)

$$\begin{aligned} \frac{dJ}{dt} &= (s - s^*) [r\{1 - (s + i)\} - i] \\ &\quad + (i - i^*) \left[s - b_2 - \frac{l(1 - m)y}{y + (1 - m)i} \right] + (y - y^*) \left[-b_1 + \frac{pl(1 - m)i}{y + (1 - m)i} \right] \\ &= -r(s - s^*)^2 - r(s - s^*)(i - i^*) \\ &\quad - \frac{l(1 - m)^2(i - i^*)(yi^* - iy^*)}{\{y + (1 - m)i\}\{y^* + (1 - m)i^*\}} - \frac{pl(1 - m)(y - y^*)(yi^* - iy^*)}{\{y + (1 - m)i\}\{y^* + (1 - m)i^*\}}. \end{aligned}$$

Therefore, $\frac{dJ}{dt}$ is negative definite if the above conditions of the theorem are satisfied and consequently, J is a Liapunov function with respect to all solutions in the interior of the positive orthant, proving the theorem. \square

7 Numerical simulation

Analytical studies can never be completed without numerical verification of the results. In this section we present computer simulation of some solutions of the system (3). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view.

We choose the parameters of system (3) as $r = 10.0$, $b_2 = 0.48$, $l = 0.95$, $b_1 = 0.02$, $p = 0.1$ and $(s(0), i(0), y(0)) = (0.5, 0.2, 0.2)$. Also we choose $m = 0.42$ which satisfy the inequality (4). Then the conditions of Theorem 3 is satisfied as $D = 0.0862 > 0$, $\Delta = 0.7523 > 0$ and consequently $E^*(s^*, i^*, y^*) = (0.9150, 0.0773, 0.1681)$ is locally asymptotically stable. The phase diagram is shown in Fig. 1.

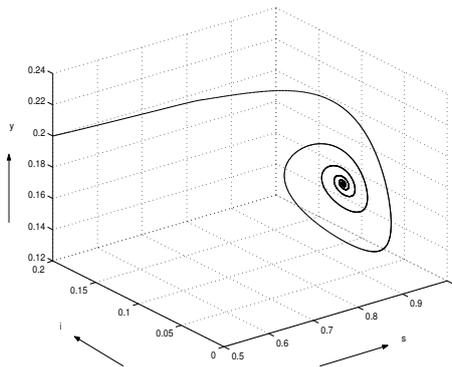


Fig. 1. Phase portrait of the system (3) showing that E^* is locally asymptotically stable. Here $s(0) = 0.5$, $i(0) = 0.2$, $y(0) = 0.2$ and $r = 10.0$, $b_2 = 0.48$, $l = 0.95$, $b_1 = 0.02$, $p = 0.1$, $m = 0.42$.

The sy -plane and iy -plane projections of the solution are shown in Fig. 2(a) and 2(b) respectively. Clearly the solution is a stable spiral converging to E^* . Fig. 2(c) shows that s, i and y populations approach their steady-state values s^*, i^* , and y^* respectively, in finite time.

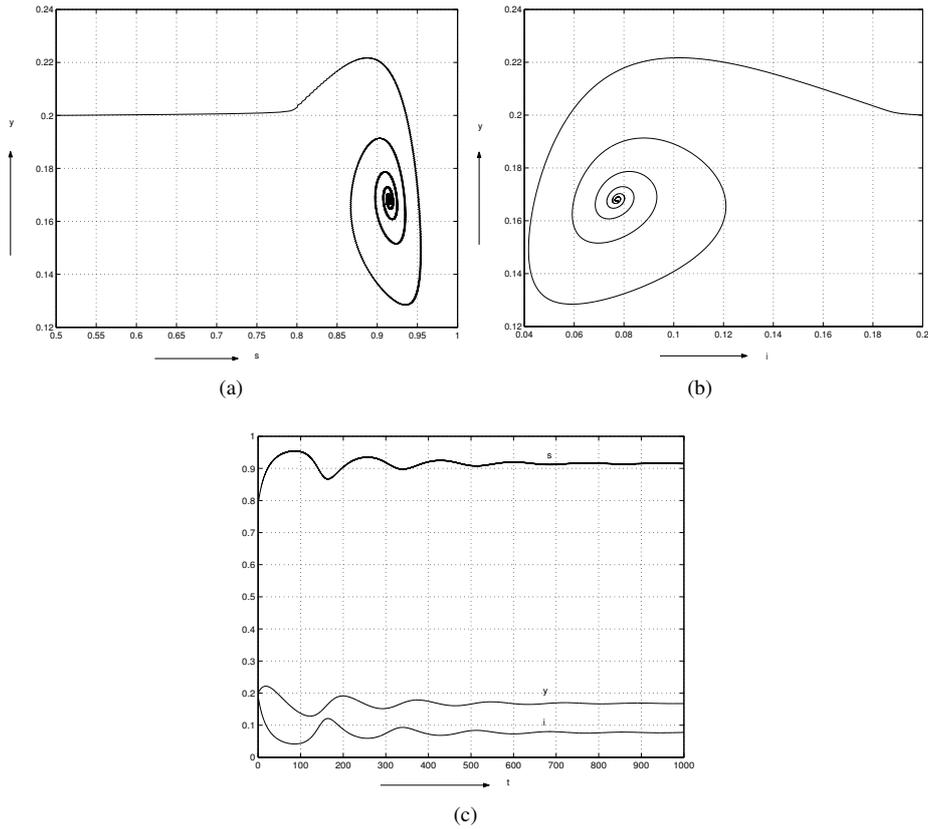


Fig. 2. Here $s(0) = 0.5, i(0) = 0.2, y(0) = 0.2$ and $r = 10.0, b_2 = 0.48, l = 0.95, b_1 = 0.02, p = 0.1, m = 0.42$. (a) sy -plane projection of the solution; (b) iy -plane projection of the solution; (c) Top curve depicts $s(t)$, middle one depicts $y(t)$ and the bottom one depicts $i(t)$. Clearly the s, i, y populations approach to their equilibrium values in finite time.

8 Model with discrete delay

It is already mentioned that time-delay is an important factor in biological system. It is also reasonable to assume that the effect of the infected prey on the predator population will not be instantaneous, but mediated by some discrete time lag τ required for incubation. As a starting point of this section, we consider the following generalization of the

model (3) involving discrete time-delay:

$$\frac{ds}{dt} = rs\{1 - (s + i)\} - si, \quad s(0) \geq 0, \quad (5a)$$

$$\frac{di}{dt} = si - b_2i - \frac{l(1-m)iy}{y + (1-m)i}, \quad i(0) \geq 0, \quad (5b)$$

$$\frac{dy}{dt} = \frac{pl(1-m)i(t-\tau)y(t-\tau)}{y(t-\tau) + (1-m)i(t-\tau)} - b_1y, \quad y(0) \geq 0. \quad (5c)$$

All parameters are the same as in system (3) except that the positive constant τ represents the reaction time or gestation period of the predator y .

The system (5) has the same equilibria as in the previous case. The main purpose of this section is to study the stability behaviour of $E^*(s^*, i^*, y^*)$ in the presence of discrete delay ($\tau \neq 0$). We linearize system (5) by using the following transformation:

$$s = s^* + s_1, \quad i = i^* + i_1, \quad y = y^* + y_1.$$

Then linear system is given by

$$\frac{dU}{dt} = AU(t) + BU(t - \tau), \quad (6)$$

where

$$U(t) = [s_1 \ i_1 \ y_1]^T, \quad A = (a_{ij})_{3 \times 3}, \quad B = (b_{ij})_{3 \times 3}$$

and

$$\begin{aligned} a_{11} &= -rs^*, \quad a_{12} = -(1+r)s^*, \\ a_{21} &= i^*, \quad a_{22} = \frac{l(1-m)^2 i^* y^*}{\{y^* + (1-m)i^*\}^2}, \quad a_{23} = -\frac{l(1-m)^2 i^{*2}}{\{y^* + (1-m)i^*\}^2}, \\ a_{33} &= -b_1 \quad \text{and all other } a_{ij} = 0; \\ b_{32} &= \frac{pl(1-m)y^{*2}}{\{y^* + (1-m)i^*\}^2}, \\ b_{33} &= -\frac{pl(1-m)^2 i^{*2}}{\{y^* + (1-m)i^*\}^2} \quad \text{and other } b_{ij} = 0. \end{aligned}$$

We look for the solution of the model (6) of the form $U(t) = \rho e^{\lambda t}$, $0 \neq \rho \in \mathfrak{R}$. This leads to the following characteristic equation:

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 + (a_4 \lambda^2 + a_5 \lambda + a_6) e^{-\lambda \tau} = 0, \quad (7)$$

where

$$\begin{aligned} a_1 &= -a_{11} - a_{22} - a_{33}, \quad a_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21}, \\ a_3 &= a_{33}(a_{12}a_{21} - a_{11}a_{22}), \quad a_4 = -b_{33}, \\ a_5 &= (a_{11} + a_{22})b_{33} - a_{23}b_{32}, \quad a_6 = a_{12}a_{21}b_{33} - a_{11}(a_{22}b_{33} - a_{23}b_{32}). \end{aligned}$$

It is well known that the signs of the real parts of the solutions of (7) characterize the stability behaviour of E^* . Therefore, substituting $\lambda = \xi + i\eta$ in (7) we obtain real and imaginary parts, respectively as

$$\xi^3 - 3\xi\eta^2 + a_1\xi^2 - a_1\eta^2 + a_2\xi + a_3 + e^{-\xi\tau} [\{a_4(\xi^2 - \eta^2) + a_5\xi + a_6\} \cos \eta\tau + (2a_4\xi + a_5)\eta \sin \eta\tau] = 0 \quad (8)$$

and

$$3\xi^2\eta - \eta^3 + 2a_1\xi\eta + a_2\eta + e^{-\xi\tau} [(2a_4\xi + a_5)\eta \cos \eta\tau - \{a_4(\xi^2 - \eta^2) + a_5\xi + a_6\} \sin \eta\tau] = 0. \quad (9)$$

A necessary condition for a stability change of E^* is that the characteristic equation (7) should have purely imaginary solutions. Hence to obtain the stability criterion, we set $\xi = 0$ in (8) and (9). Then we have,

$$a_1\eta^2 - a_3 = (a_6 - a_4\eta^2) \cos \eta\tau + a_5\eta \sin \eta\tau, \quad (10)$$

$$-\eta^3 + a_2\eta = (a_6 - a_4\eta^2) \sin \eta\tau - a_5\eta \cos \eta\tau. \quad (11)$$

Eliminating τ by squaring and adding (10) and (11), we get the equation for determining η as

$$\eta^6 + d_1\eta^4 + d_2\eta^2 + d_3 = 0, \quad (12)$$

where

$$d_1 = a_1^2 - 2a_2 - a_4^2, \quad d_2 = a_2^2 - 2a_1a_3 + 2a_4a_6 - a_5^2, \quad d_3 = a_3^2 - a_6^2.$$

Substituting $\eta^2 = \sigma$ in (12), we get a cubic equation given by

$$h(\sigma) = \sigma^3 + d_1\sigma^2 + d_2\sigma + d_3 = 0. \quad (13)$$

Since $d_3 = a_3^2 - a_6^2 > 0$ for the parameter values given in previous case, we assume that $d_3 \geq 0$ and have the following claim.

Claim 1. If

$$d_3 \geq 0 \quad \text{and} \quad d_2 > 0, \quad (14)$$

then equation (13) has no positive real roots.

In fact, notice that

$$\frac{dh(\sigma)}{d\sigma} = 3\sigma^2 + 2d_1\sigma + d_2.$$

Set,

$$3\sigma^2 + 2d_1\sigma + d_2. \quad (15)$$

Then the roots of equation (15) can be expressed as

$$\sigma_{1,2} = \frac{-d_1 \pm \sqrt{d_1^2 - 3d_2}}{3}. \tag{16}$$

If $d_2 > 0$, then $d_1^2 - 3d_2 < d_1^2$; that is $\sqrt{d_1^2 - 3d_2} < d_1$. Hence neither σ_1 nor σ_2 is positive. Thus equation (8.12) does not have positive roots. Since $h(0) = d_3 \geq 0$, it follows that the equation (8.9) has no positive roots.

Claim 1 thus implies that there is no η such that $i\eta$ is an eigen value of the characteristic equation (7). Therefore, the real parts of all the eigen values of (7) are negative for all delay $\tau \geq 0$. Summarizing the above analysis, we have the following proposition:

Proposition 1. *Suppose that*

- (i) $a_1 + a_4 > 0, a_3 + a_6 > 0, (a_1 + a_4)(a_2 + a_5) - (a_3 + a_6) > 0;$
- (ii) $d_3 \geq 0$ and $d_2 > 0$.

Then the equilibrium point E^ of the delay model (5) is absolutely stable; that is E^* is asymptotically stable for all $\tau \geq 0$.*

Remark. Proposition 1 indicates that if the parameters satisfy the conditions (i) and (ii), then the steady state of the delay model (5) is asymptotically stable for all delay values; that is, independent of the delay. However, we should point out that if the conditions (condition (ii)) in Proposition 1 are not satisfied, then the stability of the steady state depends on the delay value and the delay could even induce oscillation.

For example, if (a) $d_3 < 0$, then from equation (13) we have $h(0) < 0$ and $\lim_{\sigma \rightarrow \infty} h(\sigma) = \infty$. Thus equation (13) has at least one positive root, say σ_0 . Consequently, equation (12) has at least one positive root, denoted by η_0 .

If (b) $d_2 < 0$, then $\sqrt{d_1^2 - 3d_2} > d_1$. By (16), $\sigma_1 = \frac{1}{3}(-d_1 + \sqrt{d_1^2 - 3d_2}) > 0$. It follows that equation (13), hence equation (12), has a positive root η_0 . This implies that the characteristic equation (7) has a pair of purely imaginary roots $\pm i\eta_0$.

Let $\lambda(\tau) = \xi(\tau) + i\eta(\tau)$ be the eigenvalues of equation (7) such that $\xi(\tau_0) = 0, \eta(\tau_0) = \eta_0$. From (10) and (11) we have,

$$\tau_j = \frac{1}{\eta_0} \arccos \left(\frac{(a_5 - a_1 a_4)\eta^4 + (a_3 a_4 + a_1 a_6 - a_2 a_5)\eta^2 - a_3 a_6}{(a_4 \eta^2 - a_6)^2 + a_5^2 \eta^2} \right) + \frac{2j\pi}{\eta_0},$$

$$j = 0, 1, 2, \dots$$

Also, we can verify that the following transversality conditions:

$$\frac{d}{d\tau} \operatorname{Re}_{\tau=\tau_0} = \frac{d}{d\tau} \xi(\tau)|_{\tau=\tau_0} > 0$$

that is

$$\eta \left[\{ (a_2 - 3\eta^2)(a_6 - a_4\eta^2) + 2a_1 a_5 \eta^2 \} \sin \eta\tau + \eta \{ 2a_1(a_6 - a_4\eta^2) - a_5(a_2 - 3\eta^2) \} \cos \eta\tau + 2a_4 \eta(a_6 - a_4\eta^2) - a_5 \eta \right] > 0 \tag{17}$$

holds.

By continuity, the real part of $\lambda(\tau)$ becomes positive when $\tau > \tau_0$ and the steady state becomes unstable. Moreover, a Hopf bifurcation occurs when τ passes through the critical value τ_0 . The above analysis can be summarized into the following proposition.

Proposition 2. *Suppose that*

$$(i) \ a_1 + a_4 > 0, \ a_3 + a_6 > 0, \ (a_1 + a_3)(a_2 + a_5) - (a_3 + a_6) > 0.$$

If either

$$(ii) \ d_3 < 0 \ \text{or} \ (iii) \ d_3 \geq 0 \ \text{and} \ d_2 < 0$$

is satisfied, then the steady state E^ of the delay model (5) is asymptotically stable when $0 \leq \tau < \tau_0$ and unstable when $\tau > \tau_0$, where*

$$\tau_0 = \frac{1}{\eta_0} \arccos \left(\frac{(a_5 - a_1 a_4) \eta^4 + (a_3 a_4 + a_1 a_6 - a_2 a_5) \eta^2 - a_3 a_6}{(a_4 \eta^2 - a_6)^2 + a_5^2 \eta^2} \right)$$

when $\tau = \tau_0$, a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcates from E^ as τ passes through the critical value τ_0 .*

Proposition 2 indicates that the delay model could exhibit Hopf bifurcation at certain value of the delay if the parameters satisfy the conditions in (ii) and (iii). However, for the parameter values given in Section 7, neither (ii) nor (iii) holds.

9 Estimation of the length of delay to preserve stability

We consider the system (3) and the space of all real valued continuous functions defined on $[-\tau, \infty]$ satisfying the initial conditions on $[-\tau, 0]$. We linearize the system (3) about its interior equilibrium $E^*(s^*, i^*, y^*)$ and get

$$\frac{ds_1}{dt} = -rs^* s_1 - (1+r)s^* i_1, \tag{18a}$$

$$\frac{di_1}{dt} = i^* s_1 + \frac{l(1-m)^2 i^* y^*}{\{y^* + (1-m)i^*\}^2} i_1 - \frac{l(1-m)^2 i^{*2}}{\{y^* + (1-m)i^*\}^2} y_1, \tag{18b}$$

$$\frac{dy_1}{dt} = \frac{pl(1-m)y^{*2}}{\{y^* + (1-m)i^*\}^2} i_1(t-\tau) - b_1 y_1 - \frac{pl(1-m)i^* y^*}{\{y^* + (1-m)i^*\}^2} y_1(t-\tau), \tag{18c}$$

where

$$s(t) = s^* + s_1(t), \quad i(t) = i^* + i_1(t) \quad \text{and} \quad y(t) = y^* + y_1(t).$$

Taking Laplace transform of the system given by (18), we get,

$$\begin{aligned}
 (\alpha + rs^*)\bar{s}_1(\alpha) &= -(r + 1)s^*\bar{i}_1(\alpha) + s_1(0), \\
 \left(\alpha - \frac{l(1-m)^2i^*y^*}{\{y^* + (1-m)i^*\}^2}\right)\bar{i}_1(\alpha) &= i^*\bar{s}_1(\alpha) - \frac{l(1-m)^2i^{*2}}{\{y^* + (1-m)i^*\}^2}\overline{y_1(\alpha)} + i_1(0), \\
 (\alpha + b_1)\bar{y}_1(\alpha) &= \frac{pl(1-m)y^{*2}}{\{y^* + (1-m)i^*\}^2}e^{-\alpha\tau}\overline{i_1(\alpha)} + \frac{pl(1-m)y^{*2}}{\{y^* + (1-m)i^*\}^2}e^{-\alpha\tau}K_1(\alpha) \\
 &\quad - \frac{pl(1-m)i^*y^*}{\{y^* + (1-m)i^*\}^2}e^{-\alpha\tau}\overline{y_1(\alpha)} - \frac{pl(1-m)i^*y^*}{\{y^* + (1-m)i^*\}^2}e^{-\alpha\tau}K_2(\alpha) + y_1(0),
 \end{aligned}$$

where

$$K_1(\alpha) = \int_{-\tau}^0 e^{-\alpha t} i_1(t) dt \quad \text{and} \quad K_2(\alpha) = \int_{-\tau}^0 e^{-\alpha t} y_1(t) dt,$$

and $\bar{s}_1(\alpha)$, $\bar{i}_1(\alpha)$ and $\bar{y}_1(\alpha)$ are the laplace transform of $s_1(t)$, $i_1(t)$ and $y_1(t)$ respectively.

From [7] and using Nyquist criterion (see Appendix), it can be shown that the conditions for local asymptotic stability of $E^*(s^*, i^*, y^*)$ are given by [19]

$$\begin{aligned}
 \text{Im } H(i\eta_0) &> 0, & (19) \\
 \text{Re } H(i\eta_0) &= 0, & (20)
 \end{aligned}$$

where $H(\alpha) = \alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 + e^{-\alpha\tau}(a_4\alpha^2 + a_5\alpha + a_6)$ and η_0 is the smallest positive root of equation (20).

We have already shown that $E^*(s^*, i^*, y^*)$ is stable in absence of delay. Hence, by continuity, all eigenvalues will continue to have negative real parts for sufficiently small $\tau > 0$ provided one can guarantee that no eigenvalues with positive real parts bifurcates from infinity as τ increases from zero. This can be proved by using Butler's lemma [19].

In this case, (19) and (20) gives

$$a_2\eta_0 - \eta_0^3 > (a_6 - a_4\eta_0^2) \sin(\eta_0\tau) - a_5\eta_0 \cos(\eta_0\tau), \tag{21}$$

$$a_3 - a_1\eta_0^2 = -a_5\eta_0 \sin(\eta_0\tau) - (a_6 - a_4\eta_0^2) \cos(\eta_0\tau). \tag{22}$$

(21) and (22), if satisfied simultaneously, are sufficient conditions to guarantee stability. We shall utilize them to get an estimate on the length of delay. Our aim is to find an upper bound η_+ on η_0 , independent of τ so that (9.5) holds for all values of η , $0 \leq \eta \leq \eta_+$ and hence in particular at $\eta = \eta_0$.

We rewrite (22) as

$$a_1\eta_0^2 = a_3 + a_5\eta_0 \sin(\eta_0\tau) + (a_6 - a_4\eta_0^2) \cos(\eta_0\tau). \tag{23}$$

Maximizing $a_3 + a_5\eta_0 \sin(\eta_0\tau) + (a_6 - a_4\eta_0^2) \cos(\eta_0\tau)$ subject to $|\sin(\eta_0\tau)| \leq 1$, $|\cos(\eta_0\tau)| \leq 1$.

We obtain

$$a_1\eta_0^2 \leq a_3 + |a_5|\eta_0 + |a_6| + |a_4|\eta_0^2. \quad (24)$$

Hence, if

$$\eta_+ = \frac{1}{2(a_1 - |a_4|)} \left[|a_5| + \sqrt{a_5^2 + 4(a_1 - |a_4|)(a_3 + |a_6|)} \right] \quad (25)$$

then clearly from (24) we have $\eta_0 \leq \eta_+$

From the inequality (21) we get

$$\eta_0^2 < a_2 + a_5 \cos(\eta_0\tau) - \frac{(a_6 - a_4\eta_0^2)}{\eta_0} \sin(\eta_0\tau). \quad (26)$$

As $E^*(s^*, i^*, y^*)$ is locally asymptotically stable for $\tau = 0$, therefore, for sufficiently small $\tau > 0$, (26) will continue to hold. Substituting (23) in (26) and rearranging we get,

$$\begin{aligned} & (a_6 - a_1a_5 - a_4\eta_0^2) [\cos(\eta_0\tau) - 1] + \left\{ a_5\eta_0 + \frac{a_1(a_6 - a_4\eta_0^2)}{\eta_0} \right\} \sin(\eta_0\tau) \\ & < a_1a_2 + a_1a_5 + a_4\eta_0^2 - a_6 - a_3. \end{aligned} \quad (27)$$

Using the bounds

$$\begin{aligned} & (a_6 - a_1a_5 - a_4\eta_0^2) [1 - \cos(\eta_0\tau)] = (a_6 - a_4\eta_0^2 - a_1a_5) 2 \sin^2 \left(\frac{\eta_0\tau}{2} \right) \\ & \leq \frac{1}{2} |a_6 - a_4\eta_+^2 - a_1a_5| \eta_+^2 \tau^2 \end{aligned}$$

and

$$\left[a_5\eta_0 + \frac{a_1(a_6 - a_4\eta_0^2)}{\eta_0} \right] \sin(\eta_0\tau) \leq [(a_5 - a_1a_4)\eta_+^2 + a_1|a_6|] \tau.$$

So we obtain from (27),

$$k_1\tau^2 + k_2\tau < k_3,$$

where

$$k_1 = \frac{1}{2} |a_6 - a_4\eta_+^2 - a_1a_5| \eta_+^2, \quad (28)$$

$$k_2 = ((a_5 - a_1a_4)\eta_+^2 + a_1|a_6|), \quad (29)$$

$$k_3 = a_1a_2 + a_1a_5 + a_4\eta_0^2 - a_6 - a_3. \quad (30)$$

Hence, if

$$\tau_+ = \frac{1}{2k_2} \left[-k_2 + \sqrt{k_2^2 + 4k_1k_3} \right] \quad (31)$$

then stability is preserved for $0 \leq \tau < \tau_+$.

Appendix

Nyquist criterion. If L be the length of a curve encircling the right half-plane, the curve $\bar{x}_1(L)$ will encircle the origin a number of times equal to the difference between the number of poles and the number of zeroes of $\bar{x}_1(L)$ in the right half-plane.

10 Discussion

In this paper, we have studied an eco-epidemiological model incorporating an prey refuge with disease in the prey population which is governed by modified logistic equation [18]. Incorporating a refuge into system (1) provides a more realistic model. A refuge can be important for the biological control of a pest, however, increasing the amount of refuge can increase prey densities and lead to population outbreaks. It is shown (in Theorem 1) that the non-dimensionalized system (3) is uniformly bounded, which in turn, implies that the system is biologically well behaved. The condition for which all three species will persist are worked out. In deterministic situation, theoretical epidemiologists are usually guided by an implicit assumption that most epidemic models we observe in nature correspond to stable equilibria of the models. From this viewpoint, we have presented the most important equilibrium point $E^*(s^*, i^*, y^*)$. The stability criteria given in Lemma 2 and Theorem 3 are the conditions for stable coexistence of the susceptible prey population, infected prey population and predator population.

Now we observe that the size of the susceptible prey (s) in the absence and presence of infected prey (i) are $\hat{s} = b_2$ and $s^* = \frac{pb_2 + (1-m)(pl - b_1)}{p}$, respectively, so that $s^* - \hat{s} = (1 - m)(pl - b_1) > 0$ whenever $pl > b_1$. This indicates that if the force of infection is high then predator attack causes enhancement of the susceptible preys.

It is mentioned by several researchers that the effect of time-delay must be taken into account to have an epidemiologically useful mathematical model [1, 2, 14]. From this viewpoint, we have formulated (5) where the delay may be looked upon as the gestation period or reaction time of the predator. Then a rigorous analysis leads us to Proposition 1 and Proposition 2 which mentions that the stability criteria in absence of delay is no longer enough to guarantee the stability in the presence of delay, rather there is a value τ_0 of the delay τ such that the system is stable for $\tau < \tau_0$ and become unstable for $\tau > \tau_0$.

All our important mathematical findings without time-delay are numerically verified and graphical representation of a variety of solutions of system (3) are depicted using MATLAB. Though the parameter values in Section 7 gave us a stable state independent of the delay, the delay model (5) itself could exhibit rich dynamics. Under another set of assumptions on the parameters, the stability of the steady state depends on the delay and even delay-induced oscillations could occur via instability.

Finally, our model can be generalized in obvious ways to food chains and competitive systems.

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