

On the convergence rates of Gladyshev's Hurst index estimator

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Abstract. This paper presents the convergence rates for a modified Gladyshev's estimator of the Hurst index of the fractional Brownian motion.

Keywords: fractional Brownian motion, Gladyshev's Hurst index estimator, convergence rate.

1 Introduction

The fractional Brownian motion (fBm) and processes based on it have found many applications in fields as diverse as economics and finance, physics, chemistry, medicine and environmental studies. The fBm is a well-known example of a process with long-range dependence property. Recently much attention has been given to the study of the Hurst parameter, or of the other parameters associated to long range dependence.

By fBm $B^H = \{B_t^H: t \geq 0\}$, $0 < H < 1$, we understand a centered Gaussian process with $B_0^H = 0$ and covariance

$$\text{cov}(B_t^H, B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

The parameter H is called the Hurst index of the process.

In 1961, E. Gladyshev [1] derived a limit theorem for a statistic based on the first order quadratic variations for a class of Gaussian processes. The fBm B^H belongs to this class of processes. Gladyshev proposed an estimator of H which was strongly consistent, but not asymptotically normal. In 1997, another estimator was introduced by J. Istas and G. Lang [2] which again employed the first order quadratic variations and it was asymptotically normal for $H \in (1/2; 3/4)$. In 2005, A. Bégyn [3, 4] considered the

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second order quadratic variations for processes with Gaussian increments. In 2008–2010, K. Kubilius and D. Melichov [7–10] studied the behavior of the first and second order quadratic variations of the pathwise solution of certain stochastic differential equations driven by fBm. It was shown that the quadratic variation based estimators remain strongly consistent in that case as well.

In this note we define the modified Gladyshev’s estimator of fBm parameter H and derive the rate of convergence of it to its real value. To our knowledge, this problem is new and interesting from the practical point of view.

Let $\pi_n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n = 1\}$ be a sequence of subdivisions of the interval $[0, 1]$ such that $t_k^n = \frac{k}{N_n}$ for all $n \in \mathbb{N}$ and all $k \in \{0, \dots, N_n\}$, where (N_n) is an increasing sequence of natural numbers. Such subdivision π_n is called regular.

For a real-valued process $X = \{X_t, t \in [0, 1]\}$ taking values at the points t_k^n , $k = 0, \dots, N_n$, the first order quadratic variation is defined as

$$V_n^{(1)}(X, 2) = \sum_{k=1}^{N_n} (\Delta X_k^n)^2, \quad \Delta X_k^n = X(t_k^n) - X(t_{k-1}^n).$$

Let B^H be the fractional Brownian motion with the Hurst index H . Set $t_k^n = k2^{-n}$, $k = 1, \dots, 2^n$. It is known (see Gladyshev [1]) that

$$2^{n(2H-1)} V_n^{(1)}(B^H, 2) \xrightarrow{\text{a.s.}} 1 \quad \text{as } n \rightarrow \infty.$$

This result yields that

$$\tilde{H}_n = \frac{1}{2} - \frac{\ln V_n^{(1)}(B^H, 2)}{2n \ln 2}$$

is a strongly consistent estimator of H .

Let us define a modified Gladyshev’s estimator of fBm parameter H by

$$\hat{H}_n = \left(\frac{1}{2} - \frac{\ln [V_n^{(1)}(B^H, 2)]}{2 \ln N_n} \right) \mathbf{1}_{C_n},$$

for a regular subdivision π_n , where

$$C_n = \{V_n^{(1)}(B^H, 2) \geq N_n^{-2}\}.$$

The estimate \hat{H}_n is strongly consistent. Moreover, we can derive the rate of convergence of it to H . This follows from the following theorem.

Theorem 1. *Let B^H , $1/2 < H < 1$, be the fractional Brownian motion. \hat{H}_n is a strongly consistent estimator of the Hurst index H and the following rates of convergence hold:*

$$|\hat{H}_n - H| = \mathcal{O}\left(\sqrt{N_n^{-1} \ln N_n}\right) \quad \text{a.s. if } \sum_{n=1}^{\infty} N_n^{-2} < \infty \quad (1)$$

and

$$\mathbf{E}|\hat{H}_n - H| = \mathcal{O}\left(\sqrt{N_n^{-1} \ln N_n}\right). \quad (2)$$

2 Proof of Theorem 1

First we have

$$\widehat{H}_n = H \mathbf{1}_{C_n} - \frac{\ln B_n}{2 \ln N_n} \mathbf{1}_{C_n},$$

where $B_n = N_n^{2H-1} V_n^{(1)}(B^H, 2)$. Thus

$$\begin{aligned} |\widehat{H}_n - H| &\leq H \mathbf{1}_{\overline{C}_n} + \left| \frac{\ln B_n}{2 \ln N_n} \right| \mathbf{1}_{\{B_n \geq N_n^{-2}\}} \\ &\leq H \mathbf{1}_{\{B_n < N_n^{-1}\}} - \frac{\ln B_n}{2 \ln N_n} \mathbf{1}_{\{N_n^{-2} \leq B_n < 1\}} + \frac{\ln B_n}{2 \ln N_n} \mathbf{1}_{\{B_n \geq 1\}}. \end{aligned}$$

Let (δ_n) be a sequence of positive numbers such that $\delta_n < 1$ and $\delta_n \downarrow 0$. The inequality $-\ln(1-x) \leq 20x$, $0 \leq x \leq 19/20$, gives

$$\begin{aligned} (-\ln B_n) \mathbf{1}_{\{1-\delta_n \leq B_n < 1\}} &= (-\ln [1 - (1 - B_n)]) \mathbf{1}_{\{1-\delta_n \leq B_n < 1\}} \\ &\leq 20(1 - B_n) \mathbf{1}_{\{1-\delta_n \leq B_n < 1\}}, \end{aligned}$$

if $\delta_n \leq 19/20$. So, we have

$$\begin{aligned} -\frac{\ln B_n}{2 \ln N_n} \mathbf{1}_{\{N_n^{-2} \leq B_n < 1\}} &\leq \mathbf{1}_{\{N_n^{-2} \leq B_n < 1-\delta_n\}} + 10 \frac{1 - B_n}{\ln N_n} \mathbf{1}_{\{1-\delta_n \leq B_n < 1\}} \\ &\leq \mathbf{1}_{\{N_n^{-2} \leq B_n < 1-\delta_n\}} + \frac{10\delta_n}{\ln N_n} \mathbf{1}_{\{1-\delta_n \leq B_n < 1\}}. \end{aligned} \quad (3)$$

An application of inequality $\ln(1+x) \leq x$, $x \geq 0$, yields

$$(\ln B_n) \mathbf{1}_{\{B_n \geq 1\}} = (\ln [1 + (B_n - 1)]) \mathbf{1}_{\{B_n \geq 1\}} \leq (B_n - 1) \mathbf{1}_{\{B_n \geq 1\}}.$$

Thus

$$\begin{aligned} \frac{\ln B_n}{2 \ln N_n} \mathbf{1}_{\{B_n \geq 1\}} &\leq \frac{B_n - 1}{2 \ln N_n} \mathbf{1}_{\{1 \leq B_n \leq 1+\delta_n\}} + \frac{B_n - 1}{2 \ln N_n} \mathbf{1}_{\{B_n > 1+\delta_n\}} \\ &\leq \frac{\delta_n}{2 \ln N_n} \mathbf{1}_{\{1 \leq B_n \leq 1+\delta_n\}} + \frac{B_n - 1}{2 \ln N_n} \mathbf{1}_{\{B_n > 1+\delta_n\}}. \end{aligned} \quad (4)$$

Inequalities (3), (4) implies that

$$|\widehat{H}_n - H| \leq \left(2 + \frac{B_n - 1}{2 \ln N_n} \right) \mathbf{1}_{\{|B_n - 1| > \delta_n\}} + \frac{10\delta_n}{\ln N_n}. \quad (5)$$

To complete the proof, it suffices to estimate the first term in inequality (5) by using Hanson and Wright inequality [5]. Note that $N_n^{2H-1} V_n^{(1)}(B^H, 2)$ is the square of the Euclidean norm of one N_n -dimensional Gaussian vector X_n with components

$$N_n^{2H-1} \Delta B_k^{H,n}, \quad 1 \leq k \leq N_n.$$

By linear transformation of X_n one can get a new Gaussian vector Y_n with independent components. So there exists nonnegative real numbers $(\lambda_{1,n}, \dots, \lambda_{N_n, N_n})$ and one N_n -dimensional Gaussian vector Y_n , such that its components are independent Gaussian variables $\mathcal{N}(0, 1)$ and

$$N_n^{2H-1} V_n^{(1)}(B^H, 2) = \sum_{j=1}^{N_n} \lambda_{j, N_n} (Y_n^{(j)})^2.$$

Numbers $(\lambda_{1,n}, \dots, \lambda_{N_n, N_n})$ are the eigenvalues of the symmetric $N_n \times N_n$ -matrix

$$(N_n^{2H-1} \mathbf{E}[\Delta B_j^{H,n} \Delta B_k^{H,n}])_{1 \leq j, k \leq N_n}.$$

With the arguments of [6] and [3] one can get inequality

$$\mathbf{P}(N_n^{2H-1} |V_n^{(1)}(B^H, 2) - \mathbf{E}V_n^{(1)}(B^H, 2)| \geq \varepsilon) \leq 2 \exp(-K\varepsilon^2 N_n), \quad (6)$$

which follows directly from Hanson and Wright inequality, where $0 < \varepsilon \leq 1$, K is a positive constant.

Set

$$\delta_n^2 = \frac{2 \ln N_n}{K N_n}.$$

From inequality (6) it follows that

$$\mathbf{P}(|B_n - 1| > \delta_n) \leq \frac{2}{N_n^2}. \quad (7)$$

Obviously,

$$\mathbf{P}\left(\left(2 + \frac{B_n - 1}{2 \ln N_n}\right) \mathbf{1}_{\{|B_n - 1| > \delta_n\}} > 0\right) \leq \mathbf{P}(|B_n - 1| > \delta_n) \leq \frac{2}{N_n^2}.$$

Under condition of the theorem and from the Borel–Cantelli lemma it follows that

$$\mathbf{P}\left(\limsup_{n \rightarrow \infty} \left\{ \left(\frac{1}{2} + \frac{B_n - 1}{2 \ln N_n} \right) \mathbf{1}_{\{|B_n - 1| > \delta_n\}} > 0 \right\}\right) = 0,$$

i.e.,

$$\left(2 + \frac{B_n - 1}{2 \ln N_n}\right) \mathbf{1}_{\{|B_n - 1| > \delta_n\}} = 0$$

for sufficiently large n . From what has been said above and inequality (5) it follows that

$$|\hat{H}_n - H| = \mathcal{O}\left(\sqrt{N_n^{-1} \ln N_n}\right) \quad \text{a.s.}$$

which completes the proof of (1). Note that from the inequalities (5) and (7) we get

$$\mathbf{E}|\widehat{H}_n - H| \leq \frac{2}{N_n^2} + \mathbf{E} \frac{|B_n - 1|}{2 \ln N_n} \mathbf{1}_{\{|B_n - 1| > \delta_n\}} + \frac{10\delta_n}{\ln N_n}.$$

We now estimate the second term on the right side of the previous inequality. Note that

$$\begin{aligned} & \mathbf{E}|B_n - 1| \mathbf{1}_{\{|B_n - 1| > \delta_n\}} \\ & \leq \mathbf{E}^{1/2} |B_n - 1|^2 \sqrt{\mathbf{P}(|B_n - 1| > \delta_n)} \leq \frac{2}{N_n} \mathbf{E}^{1/2} (B_n^2 + 1) \\ & \leq \frac{2}{N_n} \left(N_n^{2H-1/2} \mathbf{E}^{1/2} \sum_{k=1}^{N_n} |\Delta B_k^{H,n}|^4 + 1 \right) \\ & \leq \frac{2}{N_n} (\sqrt{3} N_n^{-1/2} + 1). \end{aligned}$$

Thus

$$\mathbf{E}|\widehat{H}_n - H| \leq \frac{2}{N_n^2} + \frac{\sqrt{3} N_n^{-1/2} + 1}{N_n \ln N_n} + \frac{10\delta_n}{\ln N_n}.$$

The proof of (2) is completed.

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