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Exact solutions for the unsteady rotational flow of a generalized second grade fluid through a circular cylinder

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Abstract. Here the velocity field and the associated tangential stress corresponding to the rotational flow of a generalized second grade fluid within an infinite circular cylinder are determined by means of the Laplace and finite Hankel transforms. At time $t = 0$ the fluid is at rest and the motion is produced by the rotation of the cylinder around its axis with a time dependent angular velocity Ωt. The solutions that have been obtained are presented under series form in terms of the generalized G-functions. The similar solutions for the ordinary second grade and Newtonian fluids, performing the same motion, are obtained as special cases of our general solution.

Keywords: generalized second grade fluid, velocity field, shear stress, exact solutions.

1 Introduction

Many materials such as drilling mud, certain oils and greases, blood and many emulsions have been used as non-Newtonian fluids. Amongst the many models which have been treated as non-Newtonian behavior, the fluids of differential type have received special attention [1–3]. The second-grade fluids, which are a subclass of the differential type fluids, have been successfully studied in various kinds of flows by different researchers [4–9]. One of the recent advances in the theoretical studies in rheology is the development of one-dimensional fractional derivative models. The simplicity of their form and the fact that they can be used to study shear-thinning, have opened the way for the solution to a series of engineering problems. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of polymer solutions and melts. Generally speaking, the fractional calculus has encountered much success in the description of viscoelasticity. Bagley [10], Friedrich [11], He et al. [12], Huang et al. [13], Xu and Tan [14, 15], Xu [16], Tan et al. [17–19] and Haitao and Hui [20] have sequentially introduced the fractional calculus approach. Fractional derivatives are quite flexible in describing viscoelastic behavior.

The aim of this note is to establish exact solutions for the velocity field and the shear stress corresponding to the unsteady rotational flow of a generalized second grade fluid due to an infinite straight circular cylinder about its axis with an angular velocity Ωt. Using the fractional calculus approach, the governing equations of motion are fractional order partial differential equations. The velocity and adequate shear stress, obtained by means of the finite Hankel and Laplace transforms, are presented under series form in terms of the generalized G-functions. The similar solutions for the ordinary second grade and Newtonian fluids, performing the same motion, are obtained as special cases of our general solution.

2 Governing equations

The flows to be here considered have the velocity $\mathbf{v}$ and the extra-stress $\mathbf{S}$ of the form [20]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{1}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of the cylindrical coordinates system r, θ and z. For such flows, the constraint of incompressibility is automatically satisfied. Furthermore, if the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{v}(r,0) = \mathbf{0}. \tag{2}$$

The governing equations, corresponding to such motions for second grade fluid, are [21,22]

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{3}$$

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{4}$$

where μ is the dynamic viscosity of the fluid, $\alpha_1 = \alpha\rho$ is a material constant (one of the two material moduli which define a second grade fluid), $\nu = \mu/\rho$ is the kinematic viscosity of the fluid (ρ being its constant density), and $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress which is different of zero.

The governing equations corresponding to an incompressible generalized second grade fluid, performing the same motion, are obtained by replacing the inner time derivatives with respect to t from Eqs. (3) and (4), by the fractional differential operator [23]

$$D_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} \, \mathrm{d}\tau, & 0 \le \beta < 1, \\ \frac{\mathrm{d}}{\mathrm{d}t} f(t), & \beta = 1, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. Consequently, the governing equations to be used here are

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^{\beta}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{5}$$

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{6}$$

where the new material constants α and α_1 (although we keep the same notation) reduce to the previous ones for $\beta \to 1$. In the following the fractional differential equations with appropriate initial and boundary conditions will be solved by means of finite Hankel and Laplace transforms.

3 Flow through a circular cylinder

Suppose that an incompressible generalized second grade fluid (GSGF) is situated at rest in an infinite circular cylinder of radius R (> 0). At time $t = 0^+$ the cylinder suddenly begins to rotate about its axis with an angular velocity Ωt. Owing to the shear the inner fluid is gradually moved, its velocity being of the form (1). The governing equations are given by Eqs. (5) and (6), while the appropriate initial and boundary conditions are

$$w(r,0) = 0, \quad r \in [0, R], \tag{7}$$

$$w(R,t) = R\Omega t, \quad t \geq 0, \tag{8}$$

where Ω is a constant.

The partial differential equation (6), also containing fractional derivatives, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. In the following we shall use the Laplace transform to eliminate the time variable and the finite Hankel transform for the spatial variable. However, in order to avoid the burdensome calculations of residues and contour integrals, we shall apply the discrete inverse Laplace transform method.

3.1 Calculation of the velocity field

Applying the Laplace transform to the Eqs. (6) and (8), we get

$$q\overline{w}(r,q) = \left(\nu + \alpha q^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q), \tag{9}$$

$$\overline{w}(R,q) = \frac{\Omega R}{q^2}, \tag{10}$$

where $\overline{w}(r,q)$ and $\overline{w}(R,q)$ are the Laplace transforms of the functions $w(r,t)$ and $w(R,t)$, respectively.

We denote by [23]

$$\overline{w}_H(r_{1n},q) = \int_0^R r\overline{w}(r,q) J_1(rr_{1n})\,\mathrm{d}r, \tag{11}$$

the finite Hankel transform of the function $\overline{w}(r,q)$, and the inverse Hankel transform of $\overline{w}_H(r_{1n},q)$ is given by

$$\overline{w}(r,q) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{J_2^2(Rr_{1n})}\overline{w}_H(r_{1n},q).$$

In view of [24, Eq. (59)], we have

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{w}(r,q)J_1(rr_{1n})\,\mathrm{d}r$$
$$= Rr_{1n}J_2(Rr_{1n})\overline{w}(R,q) - r_{1n}^2\overline{w}_H(r_{1n},q), \tag{12}$$

r_{1n} being the positive roots of the equation $J_1(Rr)=0$ and $J_p(\cdot)$ is the Bessel function of the first kind of order p.

From Eqs. (9), (10) and (12), we find that

$$\overline{w}_H(r_{1n},q) = \left(\nu+\alpha q^\beta\right)\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{1}{q^2(q+\nu r_{1n}^2+\alpha q^\beta r_{1n}^2)}. \tag{13}$$

It can be also written in the suitable form

$$\overline{w}_H(r_{1n},q) = \overline{w}_{1H}(r_{1n},q)+\overline{w}_{2H}(r_{1n},q), \tag{14}$$

where

$$\overline{w}_{1H}(r_{1n},q) = \frac{\Omega R^2 J_2(Rr_{1n})}{r_{1n}}\frac{1}{q^2}, \tag{15}$$

$$\overline{w}_{2H}(r_{1n},q) = -\frac{\Omega R^2 J_2(Rr_{1n})}{r_{1n}}\frac{1}{q(q+\nu r_{1n}^2+\alpha q^\beta r_{1n}^2)}. \tag{16}$$

Applying the inverse Hankel transform to Eqs. (15) and (16), and using the known formula

$$\int_0^R r^2 J_1(rr_{1n})\,\mathrm{d}r = \frac{R^2}{r_{1n}}J_2(Rr_{1n})$$

we get

$$\overline{w}_1(r,q) = \frac{\Omega r}{q^2}, \tag{17}$$

$$\overline{w}_2(r,q) = -2\Omega\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}J_2(Rr_{1n})}\frac{1}{q(q+\nu r_{1n}^2+\alpha q^\beta r_{1n}^2)}. \tag{18}$$

Using the identity

$$\frac{1}{q(q+\nu r_{1n}^2+\alpha q^\beta r_{1n}^2)} = \sum_{k=0}^{\infty}\frac{(-\nu r_{1n}^2)^k q^{-\beta k-\beta-1}}{(q^{1-\beta}+\alpha r_{1n}^2)^{k+1}},$$

Eq. (18) can be written as

$$\overline{w}_2(r,q) = -2\Omega \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}J_2(Rr_{1n})} \sum_{k=0}^{\infty} \frac{(-\nu r_{1n}^2)^k q^{-\beta k-\beta-1}}{(q^{1-\beta}+\alpha r_{1n}^2)^{k+1}}. \tag{19}$$

After taking the inverse Hankel transform of Eq. (14), and using Eqs. (17) and (18), it leads to

$$\overline{w}(r,q) = \frac{\Omega r}{q^2} - 2\Omega \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}J_2(Rr_{1n})} \sum_{k=0}^{\infty} \frac{(-\nu r_{1n}^2)^k q^{-\beta k-\beta-1}}{(q^{1-\beta}+\alpha r_{1n}^2)^{k+1}}. \tag{20}$$

Now taking the inverse Laplace transform of Eq. (20), the velocity field $w(r,t)$ is given by

$$w(r,t) = \Omega rt - 2\Omega \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}J_2(Rr_{1n})} \times \sum_{k=0}^{\infty} \left(-\nu r_{1n}^2\right)^k G_{1-\beta,-\beta k-\beta-1,k+1}\left(-\alpha r_{1n}^2, t\right), \tag{21}$$

where the generalized function $G_{a,b,c}(\cdot\,,\cdot)$ is defined by [25, Eqs. (97) and (101)]

$$G_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = \sum_{k=0}^{\infty} \frac{d^k\Gamma(c+k)}{\Gamma(c)\Gamma(k+1)} \frac{t^{(c+k)a-b-1}}{\Gamma[(c+k)a-b]},$$
$$\mathrm{Re}(ac-b) > 0, \quad \left|\frac{d}{q^a}\right| < 1. \tag{22}$$

3.2 Calculation of the shear stress

Introducing Eq. (21) into Eq. (5), and using the relation

$$D_t^{\beta} G_{a,b,c}(d,t) = G_{a,b+\beta,c}(d,t), \tag{23}$$

we find the shear stress under the form

$$\tau(r,t) = 2\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_{1n})}{J_2(Rr_{1n})} \sum_{k=0}^{\infty} \left(-\nu r_{1n}^2\right)^k \times \left[\mu G_{1-\beta,-\beta k-\beta-1,k+1}\left(-\alpha r_{1n}^2, t\right) + \alpha_1 G_{1-\beta,-\beta k-1,k+1}\left(-\alpha r_{1n}^2, t\right)\right]. \tag{24}$$

4 The special case $\beta \to 1$

Making $\beta \to 1$ into Eqs. (21) and (24), we obtain the similar solutions

$$w(r,t) = \Omega rt - 2\Omega \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}J_2(Rr_{1n})} \times \sum_{k=0}^{\infty} (-\nu r_{1n}^2)^k G_{0,-k-2,k+1}(-\alpha r_{1n}^2, t), \tag{25}$$

and

$$\tau(r,t) = 2\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_{1n})}{J_2(Rr_{1n})} \sum_{k=0}^{\infty} \big(-\nu r_{1n}^2\big)^k \times \big[\mu G_{0,-k-2,k+1}\big(-\alpha r_{1n}^2, t\big) + \alpha_1 G_{0,-k-1,k+1}\big(-\alpha r_{1n}^2, t\big)\big], \tag{26}$$

for a second grade fluid performing the same motion.

These solutions can be also simplified to give the simple expressions (see also Eqs. (A1)–(A2) from Appendix)

$$w(r,t) = \Omega rt - \frac{2\Omega}{\nu} \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})} \left(1 - \exp\left(\frac{-\nu r_{1n}^2 t}{1+\alpha r_{1n}^2}\right)\right), \tag{27}$$

$$\tau(r,t) = 2\rho\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})} \left[1 - \frac{1}{1+\alpha r_{1n}^2} \exp\left(\frac{-\nu r_{1n}^2 t}{1+\alpha r_{1n}^2}\right)\right], \tag{28}$$

obtained in [26, Eqs.(5.1) and (5.3)] by a different technique.

Making $\alpha_1 \to 0$ and then $\alpha \to 0$ into Eqs. (27) and (28), we obtain the velocity field

$$w(r,t) = \Omega rt - \frac{2\Omega}{\nu} \sum_{n=1}^{\infty} \frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})} \big(1 - \exp\big(-\nu r_{1n}^2 t\big)\big), \tag{29}$$

and the associated shear stress

$$\tau(r,t) = 2\rho\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})} \big[1 - \exp\big(-\nu r_{1n}^2 t\big)\big]. \tag{30}$$

corresponding to a Newtonian fluid performing the same motion.

5 Conclusion

In this paper, the velocity field and the adequate shear stress corresponding to the rotational flow of an incompressible generalized second grade fluid induced by an infinite circular cylinder have been determined by using finite Hankel and Laplace transforms. The motion is produced by the circular cylinder that at the moment $t = 0^+$ begins to rotate around its axis with an angular velocity Ωt. The solutions that have been obtained, written under series form in terms of the generalized G-functions, satisfy both the governing equation and all the imposed initial and boundary conditions. The similar solutions for the ordinary second grade and Newtonian fluids, performing the same motion, are obtained as special cases when $\beta \to 1$, respectively $\beta \to 1$ and $\alpha_1 \to 0$.

Appendix

$$\sum_{k=0}^{\infty}\big(-\nu r_{1n}^2\big)^k G_{0,-k-1,k+1}\big(\alpha r_{1n}^2,t\big)=\frac{1}{1+\alpha r_{1n}^2}\exp\biggl(-\frac{\nu r_{1n}^2 t}{1+\alpha r_{1n}^2}\biggr), \tag{A1}$$

$$\sum_{k=o}^{\infty}(-\nu r_{1n}^2)^k G_{0,-k-2,k+1}\big(-\alpha r_{1n}^2,t\big)=\frac{1}{\nu r_{1n}^2}\biggl[1-\exp\biggl(-\frac{\nu r_{1n}^2 t}{1+\alpha r_{1n}^2}\biggr)\biggr]. \tag{A2}$$

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