Taylor–Couette flow of a fractional second grade fluid in an annulus due to a time-dependent couple

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Received: 8 October 2010 / **Revised:** 15 November 2010 / **Published online:** 25 February 2011

Abstract. Exact solutions for the velocity field and the associated shear stress, corresponding to the flow of a fractional second grade fluid between two infinite coaxial cylinders, are determined by means of Laplace and finite Hankel transforms. The motion is produced by the inner cylinder which is rotating about its axis due to a time-dependent torque per unit length $2\pi R_1 f t^2$. The solutions that have been obtained satisfy all imposed initial and boundary conditions. For $\beta \to 1$, respectively $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for ordinary second grade fluids and Newtonian fluids, performing the same motion, are obtained as limiting cases.

Keywords: fractional second grade fluid, velocity field, shear stress, exact solutions.

1 Introduction

The governing equation that describes the flow of a Newtonian fluid is the Navier–Stokes equation. However, some materials such as clay coatings, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers, many emulsions have been treated as non-Newtonian fluids and they cannot be described by the Navier–Stokes equation. For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi-empirical. One of the most popular models for non-Newtonian fluids is the model that is called the second grade fluid or fluid of second grade. It is reasonable to use the second grade fluid model to do numerical calculations. This is particularly so due to the fact that the calculations will generally be simpler. The constitutive equation of a second grade fluid is a linear relation between the stress and the square of the first Rivlin–Ericksen tensor and the second Rivlin–Ericksen tensor [1]. This constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius–Duhem inequality and also due to the assumption that Helmoltz free energy is minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given by Dunn and Fosdick [2] and Dunn and Rajagopal [3]. The restrictions on the two coefficients have not been confirmed by experiments and the sign of the moduli is the subject of much controversy.

During the last years, the fractional calculus has achieved a great success in the description of the complex dynamics. In particular it has been found to be quite flexible in describing the viscoelastic behavior [4, 5]. A very good fit of the experimental data was achieved when the Maxwell model was used with its first-order derivatives replaced by fractional-order derivatives [6]. Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of polymer solutions and melts. In other cases, it has been shown that the constitutive equations employing fractional derivatives are linked to molecular theories [7]. The list of their applications is quite long, it including fractal media, fractional wave diffusion, fractional Hamiltonian dynamics as well as many other topics in physics [8]. In the last time, a lot of papers regarding these fluids have been published but we remember here only a part of those concerning generalized second grade fluids [9–19].

Here, the velocity field and the adequate shear stress, corresponding to the flow of a second grade fluid with fractional model in an annular region between two infinite coaxial cylinders, are determined by means of Laplace and the finite Hankel transforms. The motion is produced by the inner cylinder which is moving about its axis due to a time-dependent torque. The solutions that have been obtained satisfy all imposed initial and boundary conditions. For $\beta \to 1$, respectively $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for ordinary second grade and Newtonian fluids, performing the same motion, are obtained as limiting cases

2 Governing equations

The flows to be here considered have the velocity field of the form [20–22]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\,\mathbf{e}_\theta, \tag{1}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ-direction of the cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. The non-trivial shear stress $\tau(r,t) = S_{r\theta}(r,t)$ corresponding to such a motion of a second grade fluid is given by [23]

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left[\frac{\partial w(r,t)}{\partial r} - \frac{w(r,t)}{r}\right], \tag{2}$$

where μ is the viscosity and α_1 ia a material modulus. In the absence of a pressure gradient in the flow direction and neglecting body forces, the balance of the linear momentum leads to the relevant equation [24, 25]

$$\rho \frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t). \tag{3}$$

Eliminating $\tau(r,t)$ between Eqs. (2) and (3), we get the governing equation

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)w(r,t), \tag{4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, ρ is its constant density and $\alpha = \alpha_1/\rho$.

Generally, the governing equations for a fractional second grade fluid (FSGF) are derived from those of the ordinary fluids by replacing the inner time derivatives of an integer order with the so called Riemann–Liouville operator [26]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\,\mathrm{d}\tau, \quad 0 \le \beta < 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

Consequently, the governing equations corresponding to the motion (1) of a FSGF are (cf. [21, Eqs. (2) and (4)] with $\lambda = 0$)

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha D_t^\beta\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t); \tag{5}$$

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^\beta\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{6}$$

where the new material constant α_1 (for simplicity, we are keeping the same notation) goes to the initial α_1 for $\beta \to 1$.

In this paper, we are interested into the motion of a FSGF whose governing equations are given by Eqs. (5) and (6). The fractional partial differential equations (6), with adequate initial and boundary conditions, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. The Laplace transform will be used to eliminate the time variable and the finite Hankel transform to remove the spatial variable. However, in order to avoid the lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform will be used.

3 Rotational flow between two infinite cylinders

Consider an incompressible FSGF at rest in the annular region between two infinitely coaxial cylinders. At time $t = 0^+$ let the inner cylinder of radius R_1 be set in rotation about its axis by a time dependent torque per unit length $2\pi R_1 f t^2$, while the outer cylinder of radius R_2 is held stationary. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form (1). The governing equations are given by Eqs. (5) and (6) and the appropriate initial and boundary conditions are (see [20, Eqs. (5.2), (5.3)])

$$w(r,0) = 0; \quad r \in \left[R_1, R_2\right], \tag{7}$$

$$\begin{aligned} &\tau(R_1,t) = \left(\mu + \alpha_1 D_t^\beta\right)\left(\frac{\partial w(r,t)}{\partial r} - \frac{w(r,t)}{r}\right)\bigg|_{r=R_1} = f t^2; \\ &w(R_2,t) = 0; \quad t \ge 0, \end{aligned} \tag{8}$$

where f is a constant.

3.1 Calculation of the velocity field

Applying the Laplace transform to the equations (5) and (8) and using the initial condition (7), we get

$$q\overline{w}(r,q) = \left(\nu + \alpha q^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q); \qquad r \in (R_1, R_2), \tag{9}$$

$$\overline{\tau}(R_1,q) = \left(\mu + \alpha_1 q^{\beta}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\overline{w}(r,q)|_{r=R_1} = \frac{2f}{q^3}; \quad \overline{w}(R_2,q) = 0, \tag{10}$$

where $\overline{w}(r,q)$ and $\overline{\tau}(R_1,q)$ are the Laplace transforms of the functions $w(r,t)$ and $\tau(R_1,t)$, respectively. We denote by [21, Eq. (32)]

$$\overline{w}_H(r_n,q) = \int\limits_{R_1}^{R_2} r\overline{w}(r,q)B(r,r_n)\,\mathrm{d}r, \tag{11}$$

the finite Hankel transform of the function $\overline{w}(r,q)$, where

$$B(r,r_n) = J_1(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_1(rr_n), \tag{12}$$

r_n being the positive roots of the equation $B(R_2,r) = 0$ and $J_p(\cdot)$, $Y_p(\cdot)$ are the Bessel functions of the first and second kind of order p.

The inverse Hankel transform of $\overline{w}_H(r_n,q)$ is given by [21, Eq. (35)]

$$\overline{w}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_1^2(R_2r_n)B(r,r_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)}\overline{w}_H(r_n,q). \tag{13}$$

By means of Eq. $(10)_2$ and of the identity

$$J_1(z)Y_2(z) - J_2(z)Y_1(z) = -\frac{2}{\pi z},$$

we can easily prove that

$$\int\limits_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q)B(r,r_n)\,\mathrm{d}r$$

$$= -r_n^2\overline{w}_H(r_n,q) + \frac{2}{\pi r_n}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\overline{w}(r,q)\Big|_{r=R_1}. \tag{14}$$

Multiplying Eq. (9) by $rB(r,r_n)$ and integrating the result with respect to r from R_1 to R_2 and using Eqs. (10) and (14), we find that

$$\overline{w}_H(r_n,q) = \frac{4f}{\pi r_n}\frac{1}{q^3}\frac{1}{\rho q + \alpha_1 q^{\beta} r_n^2 + \mu r_n^2}. \tag{15}$$

Writing $\overline{w}_H(r_n,q)$ under the equivalent forms

$$\overline{w}_H(r_n,q) = \frac{4f}{\mu\pi r_n^3}\left[\frac{1}{q^3} - \frac{1+\alpha r_n^2 q^{\beta-1}}{q^2\{q+(\nu+\alpha q^\beta)r_n^2\}}\right]$$
$$= \frac{4f}{\mu\pi r_n^3}\left[\frac{1}{q^3} - \frac{q^{-\beta-2}+\alpha r_n^2 q^{-3}}{(q^{1-\beta}+\alpha r_n^2)+\nu r_n^2 q^{-\beta}}\right], \tag{16}$$

and applying the Hankel transform to Eq. (16) and using the identities

$$\frac{1}{(q^{1-\beta}+\alpha r_n^2)+\nu r_n^2 q^{-\beta}} = \sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k q^{-\beta k}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}, \tag{17}$$

$$\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n)-J_1^2(R_2 r_n)]} = \frac{1}{2}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right), \tag{18}$$

we find that

$$\overline{w}(r,q) = \frac{f}{\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right)\frac{1}{q^3} - \frac{2\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n)-J_1^2(R_2 r_n)]}$$
$$\times\sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}\left[q^{-\beta k-\beta-2}+\alpha r_n^2 q^{-\beta k-3}\right]. \tag{19}$$

Now applying the inverse Laplace transform to Eq. (19), we find for the velocity field the suitable expression [18]

$$w(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right)t^2 - \frac{2\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n)-J_1^2(R_2 r_n)]}$$
$$\times\sum_{k=o}^{\infty}\left(-\nu r_n^2\right)^k\left[G_{1-\beta,-\beta k-\beta-2,k+1}\left(-\alpha r_n^2,t\right)\right.$$
$$\left.+\alpha r_n^2 G_{1-\beta,-\beta k-3,k+1}\left(-\alpha r_n^2,t\right)\right], \tag{20}$$

where the generalized function $G_{a,b,c}(d,t)$ is defined by [27, Eqs. (97) and (101)]

$$G_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = \sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}; \tag{21}$$
$$\operatorname{Re}(ac-b)>0,\quad \left|\frac{d}{q^a}\right|<1.$$

3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (6), we find that

$$\overline{\tau}(r,q) = \left(\mu+\alpha_1 q^\beta\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{w}(r,q). \tag{22}$$

In order to get a suitable form for $\tau(r,t)$, we rewrite Eq. (15) under the equivalent form

$$\overline{w}_H(r_n,q)=\frac{4f}{\pi r_n^3}\frac{1}{q^3(\mu+\alpha_1 q^\beta)}-\frac{4f}{\pi r_n^3}\frac{1}{q^2(\mu+\alpha_1 q^\beta)(q+\alpha q^\beta r_n^2+\nu r_n^2)}. \tag{23}$$

Applying the inverse Hankel transform to Eq. (23) and using Eq. (13) and the identity (18), we find that

$$\overline{w}(r,q)=f\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right)\frac{1}{q^3(\mu+\alpha_1 q^\beta)}-2\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\times\frac{1}{q^2(\mu+\alpha_1 q^\beta)(q+\alpha q^\beta r_n^2+\nu r_n^2)}. \tag{24}$$

Introducing Eq. (24) into Eq. (22), it results that

$$\overline{\tau}(r,q)=\left(\frac{R_1}{r}\right)^2 f\frac{2}{q^3}+2\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(r,r_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\frac{1}{q^2(q+\alpha q^\beta r_n^2+\nu r_n^2)}, \tag{25}$$

or equivalently (see also Eq. (17))

$$\overline{\tau}(r,q)=\left(\frac{R_1}{r}\right)^2 f\frac{2}{q^3}+2\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(r,r_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\times\sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k q^{-\beta k-\beta-2}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}, \tag{26}$$

where $B_1(r,r_n)=J_2(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_2(rr_n)$.

Now taking the inverse Laplace transform of both sides of Eq. (26), we get for the shear stress $\tau(r,t)$ the expression

$$\tau(r,t)=\left(\frac{R_1}{r}\right)^2 ft^2+2\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(r,r_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\times\sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta-2,k+1}\left(-\alpha r_n^2,t\right). \tag{27}$$

4 The special case $\beta\to 1$ (second grade fluid)

Making $\beta\to 1$ into Eqs. (20) and (27), we obtain the similar solutions

$$w_{SG}(r,t)=\frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right)t^2-\frac{2\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\times\left(1+\alpha r_n^2\right)\sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-k-3,k+1}\left(-\alpha r_n^2,t\right), \tag{28}$$

and

$$\tau_{SG}(r,t) = \left(\frac{R_1}{r}\right)^2 ft^2 + 2\pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r, r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty} \big(-\nu r_n^2 \big)^k G_{0,-k-3,k+1}\big(-\alpha r_n^2, t \big), \tag{29}$$

corresponding to a second grade fluid.

Now, in view of the identity

$$\sum_{k=0}^{\infty} \big(-\nu r_n^2\big)^k G_{0,-(k+3),k+1}\big(-\alpha r_n^2, t\big) = \frac{1+\alpha r_n^2}{(\nu r_n^2)^2} \left[\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2} \right) + \frac{\nu r_n^2 t}{1+\alpha r_n^2} - 1 \right],$$

Eqs. (28) and (29) can be written under the simplest forms

$$w_{SG}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2 \left(r - \frac{R_2^2}{r}\right)\left(t^2 - \frac{2\alpha_1}{\mu} t\right) - \frac{2\pi f}{\mu\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r, r_n)}{r_n^3 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \left\{ t - \frac{(1+\alpha r_n^2)^2}{\nu r_n^2} \left(1 - \exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2} \right) \right) \right\}, \tag{30}$$

and

$$\tau_{SG}(r,t) = \left(\frac{R_1}{r}\right)^2 ft^2 + \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r, r_n)}{r_n^2 [J_2^2\big(R_1 r_n\big) - J_1^2(R_2 r_n)]} \times \left\{ t - \frac{1+\alpha r_n^2}{\nu r_n^2} \left(1 - \exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2} \right) \right) \right\}. \tag{31}$$

5 Newtonian case

Making α_1 and then $\alpha \to 0$ into Eqs. (30) and (31), the velocity field

$$w_N(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2 \left(r - \frac{R_2^2}{r}\right) t^2 - \frac{2\pi f}{\mu\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r, r_n)}{r_n^3 [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \left\{ t - \frac{1}{\nu r_n^2} \big(1 - e^{-\nu r_n^2 t} \big) \right\}, \tag{32}$$

and the associated shear stress

$$\tau_N(r,t) = \left(\frac{R_1}{r}\right)^2 ft^2 + \frac{2\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(r,r_n)}{r_n^2[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \left\{t - \frac{1}{\nu r_n^2}\left(1 - \mathrm{e}^{-\nu r_n^2 t}\right)\right\}, \tag{33}$$

corresponding to a Newtonian fluid are obtained.

6 Conclusions

The purpose of this note is to provide exact analytic solutions for the velocity field $w(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the unsteady rotational flow of a fractional second grade fluid between two infinite coaxial cylinders, the inner cylinder being set in rotation about its axis by a time-dependent shear. The solutions that have been obtained, presented under series form in terms of usual Bessel ($J_1(\cdot)$ and $J_2(\cdot)$) and generalized $G_{a,b,c}(\cdot,t)$ functions, satisfy all imposed initial and boundary conditions. They can be easily specialized to give the similar solutions for ordinary second grade and Newtonian fluids.

The large time solutions corresponding to second grade fluids (see Eqs. (30) and (31))

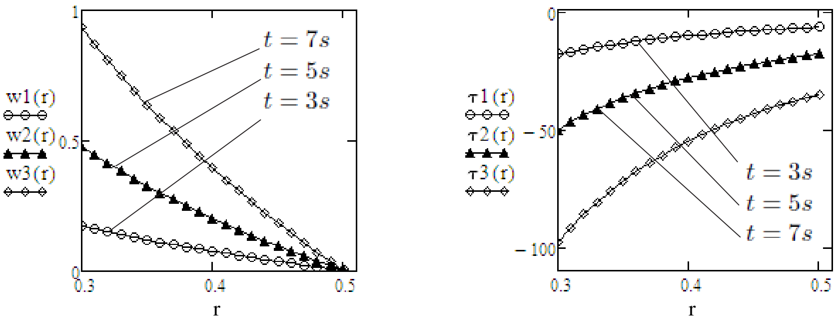
$$w_{LSG}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)\left(t^2 - \frac{2\alpha_1}{\mu}t\right) - \frac{2\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n^3[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\left\{t - \frac{(1+\alpha r_n^2)^2}{\nu r_n^2}\right\}, \tag{34}$$

$$\tau_{LSG}(r,t) = \left(\frac{R_1}{r}\right)^2 ft^2 + \frac{2\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(r,r_n)}{r_n^2[J_2^2(R_1r_n)-J_1^2(R_2r_n)]} \times \left\{t - \frac{1+\alpha r_n^2}{\nu r_n^2}\right\}, \tag{35}$$

are different of those corresponding to Newtonian fluids.

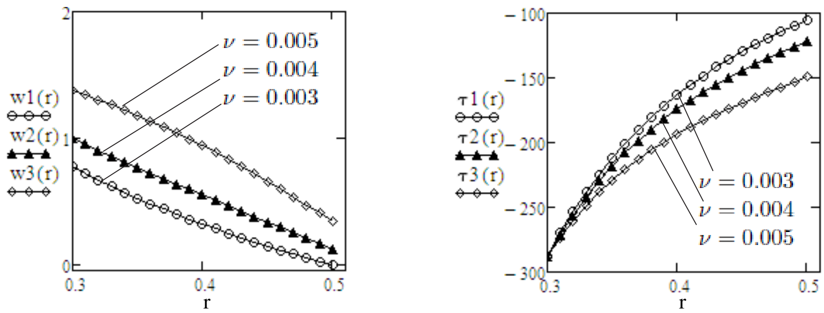
In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity $w(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (20) and (27), have been drawn against r for different values of the time t and of the material parameters. Figs 1(a) and 1(b). show the influence of time on the fluid motion. From these figures it is clearly seen that the velocity as well as the shear stress in absolute value are increasing functions of t. In Figs. 2(a) and 2(b), it is shown the influence of the kinematic viscosity ν on the fluid motion. It is clearly seen from these figures that the velocity and shear stress (in absolute value) are increasing functions of ν. The influence of the material parameter α on the fluid motion is shown by Figs 3. It shows that the velocity is an increasing function, while the shear stress (in absolute value) is a decreasing function with respect to α. In Figs. 4(a) and 4(b) it is shown the influence of the fractional parameter β on the fluid motion. It is clearly seen from these figures that both the velocity and the shear stress (in absolute value) are increasing functions of β.

Finally, for comparison, the diagrams of $w(r,t)$ and $\tau(r,t)$ corresponding to the fractional second grade, ordinary second grade and Newtonian fluids are presented in Figs. 5 for the same values of the common material constants and the time t. In all cases the velocity of the fluid is a decreasing function with respect to r. From these figures, it is clearly seen that, as expected, the Newtonian fluid is the swiftest while the fractional second grade fluid is the slowest. One thing is of worth mentioning that units of the material constants are IS units in all figures and the roots r_n have been approximated by $(2n-1)\pi/[2(R_2-R_1)]$.



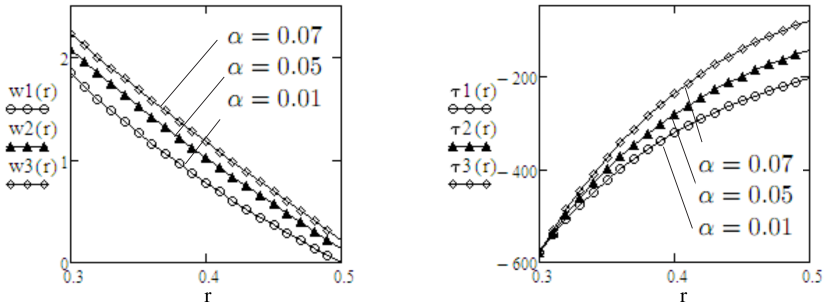
(a) (b)

Fig. 1. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $R_1=0.3$, $R_2=0.5$, $f=-2$, $\nu=0.003$, $\mu=10$, $\alpha=0.003$, $\beta=0.6$ and different values of t.



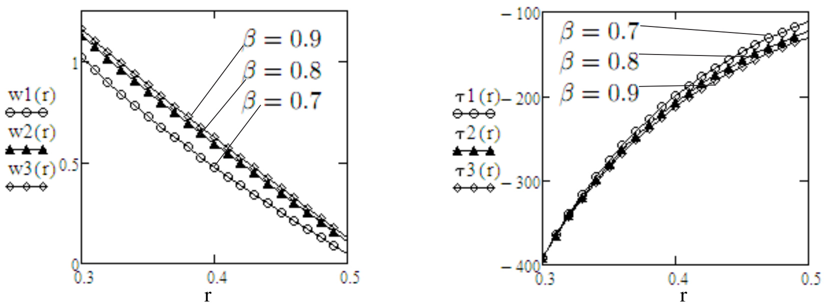
(a) (b)

Fig. 2. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $t=12$ s, $R_1=0.3$, $R_2=0.5$, $f=-2$, $\mu=35$, $\alpha=0.03$, $\beta=0.4$ and different values of ν.



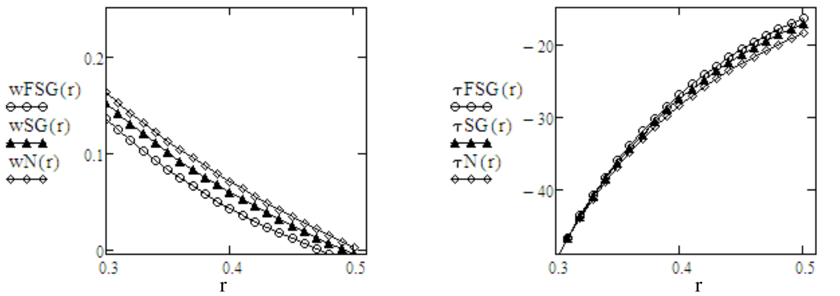
(a) (b)

Fig. 3. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $t = 17$ s, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.0015$, $\mu = 30$, $\beta = 0.4$ and different values of α.



(a) (b)

Fig. 4. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $t = 14$ s, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0,0015$, $\mu = 40$, $\alpha = 0.07$ and different values of β.



(a) (b)

Fig. 5. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, ordinary second grade, fractional second grade fluids, for $t = 5$ s, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.002$, $\mu = 30$, $\alpha = 0.04$ and $\beta = 0.5$.

Acknowledgement

The authors would like to thank reviewers for their careful assessment and very useful suggestions that helped us to improve the paper.

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