Limit theorems for a quadratic variation of Gaussian processes

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Abstract. In the paper a weighted quadratic variation based on a sequence of partitions for a class of Gaussian processes is considered. Conditions on the sequence of partitions and the process are established for the quadratic variation to converge almost surely and for a central limit theorem to be true. Also applications to bifractional and sub-fractional Brownian motion and the estimation of their parameters are provided.

Keywords: quadratic variation, bifractional Brownian motion, sub-fractional Brownian motion, Hurst index, almost sure convergence, central limit theorem.

1 Introduction

The phenomenon of long range dependence is observed in fields such as hydrology, finance, chemistry, mathematics, physics and others, therefore many statistical and stochastic models assuming independence or weak dependence of observations are inappropriate. First mathematically defined by Kolmogorov, fractional Brownian motion (fBm), the prototype of self similar and long range dependent processes, has been widely applied as a modelling tool. Since a long memory parameter (or the Hurst index/exponent, usually denoted H) determines the mathematical properties of the model, it's estimation is of great importance. One of the methods for estimation of the Hurst index involves quadratic variation.

The problem of the almost sure convergence of a quadratic variation has been solved for a wide class of processes with Gaussian increments by Baxter [1] and Gladyshev [2]. These authors employed dyadic partitions. Klein and Giné [3] used more general partitions and proved that particular functions of the mesh of the partition must be at most $o(1/\log n)$ for the almost sure convergence to hold.

Using his result of convergence of the quadratic variation Gladyshev [2] constructed a strongly consistent estimator of H. Istas and Lang [4] proposed another estimator, which also involved a quadratic variation and was asymptotically normal only when $H \in (0, 3/4)$. To avoid this drawback second order quadratic variation has been considered by Cohen et al. [5] and Bégyn [6] to name a few. In a recent paper of Kubilius and Melichov [7] a modified Gladyshev's estimator of the fBm parameter H is defined and convergence rate to its real value is derived.

Stationarity of the increments of fBm is a useful feature in certain applications, however there are cases when it is unwanted. In order to enlarge the variety of models to choose from, extensions of fBm have been introduced recently by Houdré and Villa [8] (bifractional Brownian motion) and Bojdecki et al. [9] (sub-fractional Brownian motion). These processes share properties with fBm such as self similarity, gaussianity and others, however they do not have stationary increments and possess some new features.

Norvaiša [10] extended Gladyshev's theorem to a class of Gaussian processes that includes bifractional and sub-fractional Brownian motion. In this paper we extend the main result in Norvaiša [10] by using a general class of partitions that may be irregular. We prove in Theorem 1 that particular functions of the mesh of a partition must be at most $o(1/\log n)$ for the result to hold as in Klein and Giné [3]. We proceed further by showing in Theorem 3 that the central limit theorem holds for quadratic variation for particular values of parameters of the process.

2 Almost sure convergence

In this section we prove the almost sure convergence of a quadratic variation.

We begin with some notation. We denote by $\pi = \{t_k\}_{k=0}^m$ a partition of $[0,T]$ such that $0 = t_0 < \cdots < t_m = T$, by $\mathrm{c}(\pi) = m$ the number of intervals (t_k, t_{k+1}), $k = 0, \ldots, m-1$, by $\mathrm{m}(\pi)$ its mesh, i.e., $\mathrm{m}(\pi) := \max\{t_k - t_{k-1}\colon t_k \in \pi \setminus \{t_0\}\}$ and by $\mathrm{p}(\pi)$ the quantity $\min\{t_k - t_{k-1}\colon t_k \in \pi \setminus \{t_0\}\}$. Throughout this paper $\{\pi_n\}_{n=1}^\infty$ with $\pi_n = \{t_k^{(n)}\}_{k=0}^{\mathrm{c}(\pi_n)}$ will denote a sequence of partitions of $[0,T]$ but we will drop the superindex n in $t_k^{(n)}$ whenever possible. We often assume that there exists $c \geq 1$ such that $\mathrm{m}(\pi_n)/\mathrm{p}(\pi_n) \leq c$ for all n. Then

$$\mathrm{c}(\pi_n) \leq \frac{T}{\mathrm{p}(\pi_n)} \leq \frac{cT}{\mathrm{m}(\pi_n)}. \tag{1}$$

Given a function $F\colon [0,T] \to \mathbb{R}$, a parameter $\gamma \in (0,2)$ and a partition $\pi_n = \{t_k\}_{k=0}^{\mathrm{c}(\pi_n)}$, for each $n = 1, 2, \ldots$, let

$$\begin{aligned} \Delta F_k &:= \frac{F(t_k) - F(t_{k-1})}{(t_k - t_{k-1})^{(1-\gamma)/2}}, \\ \Delta t_k &:= t_k - t_{k-1} \quad \text{and} \quad S_n(F) := \sum_{k=1}^{\mathrm{c}(\pi_n)} (\Delta F_k)^2. \end{aligned} \tag{2}$$

For a Gaussian stochastic process $X = \{X(t),\ t \geq 0\}$ let M be the mean function with values $M(t) := EX(t)$ and r_X be the covariance function with values

$$r_X(t,s) := \mathbf{E}\big[X(t) - M(t)\big]\big[X(s) - M(s)\big] \tag{3}$$

for $(s,t) \in [0,\infty)^2$.

Definition 1. Let $0 < \gamma < 2$, $d > 0$, $b : [0, \infty)^2 \to [0, \infty)$ be a symmetric function and L be a finite constant. A Gaussian process $X = \{X(t),\ t \geq 0\}$ is said to belong to the class $\mathcal{G}(\gamma, d, b, L)$ if its structure function ψ_X with values $\psi_X(s,t) := \mathbf{E}[X(t) - X(s)]^2$ for $(s,t) \in [0, \infty)^2$ satisfies:

(i) $\psi_X(s,t) = d|t - s|^{2-\gamma} + b(s,t)$,

(ii) $\psi_X(t - h, t) \leq Lh^{2-\gamma}$ holds for each $0 < h \leq t \leq T$.

We will denote by M_n the set of all $n \times n$ matrices over the field of scalars $\mathbb{R}$, by $\rho(Q)$ the spectral radius and by $\sigma(Q)$ the set of eigenvalues of a matrix $Q \in M_n$.

Lemma 1. *Let $A = (a_{kl}) \in M_n$ be a symmetric, positive semidefinite matrix. Then*

$$\rho(A) \leq \inf_{m \in \mathbb{N}} (\operatorname{tr} A^m)^{1/m},$$

where for a $B = (b_{kl}) \in M_n$, $\operatorname{tr} B$ denotes the trace of B, i.e., $\operatorname{tr} B := \sum_{k=1}^n b_{kk}$.

Proof. Consider the Frobenius norm on M_n defined for $Q = (q_{kl}) \in M_n$ by

$$\|Q\|_2 := \left(\sum_{k,l=1}^n q_{kl}^2 \right)^{1/2}.$$

Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. It is well known that for any $m \in \mathbb{N}$, $\sigma(A^m) = \{\lambda_1^m, \ldots, \lambda_n^m\}$ and $\operatorname{tr} A^m = \sum_{k=1}^n \lambda_k^m$. Since A is symmetric

$$\|A\|_2 = \sqrt{\operatorname{tr} A^2} = \left(\sum_{k=1}^n \lambda_k^2 \right)^{1/2}.$$

It is easy to prove that

$$\rho(A) \leq \left\| A^m \right\|_2^{1/m}, \quad \forall m = 1, 2, \ldots. \tag{4}$$

Since A is positive semidefinite, $\lambda_k \geq 0$, $k = 1, 2, \ldots, n$. Thus for every $m = 1, 2, \ldots$ we have that

$$\left(\operatorname{tr} A^m\right)^2 = \left(\sum_{k=1}^n \lambda_k^m \right)^2 \geq \sum_{k=1}^n \lambda_k^{2m} = \left\| A^m \right\|_2^2. \tag{5}$$

Then the statement of the lemma follows from (4) and (5). □

Given a Gaussian stochastic process $X = \{X(t),\ t \in [0, T]\}$, $0 < T < \infty$, and π a partition of $[0, T]$ denote $a_{kl} := \mathbf{E}\Delta X_k \Delta X_l$ and a $\mathrm{c}(\pi) \times \mathrm{c}(\pi)$ matrix $A(\pi) := (a_{kl})_{k,l=1}^{\mathrm{c}(\pi)}$.

Lemma 2. *Let γ, d, b, L be as in Definition 1 and $X = \{X(t),\ t \in [0, T]\}$, $0 < T < \infty$, be a mean zero Gaussian stochastic process from $\mathcal{G}(\gamma, d, b, L)$. Let $\{\pi_n\}_{n=1}^\infty$ be a sequence of partitions of $[0, T]$ such that there exists $c \geq 1$ satisfying* $\mathrm{m}(\pi_n)/\mathrm{p}(\pi_n) \leq c$

for every n and there exist $\xi(\gamma) > 0$ depending only on γ and a function $f_\gamma : (0,\infty) \to (0,\infty)$ such that:

(i) *there exists $N_1 \in \mathbb{N}$ such that the bound*

$$\max_{1<l\le \mathrm{c}(\pi_n)} \sum_{k\in D(l)} |\mathbf{E}\Delta X_k \Delta X_l| \le \xi(\gamma) f_\gamma\big(\mathrm{m}(\pi_n)\big) =: H_n(\gamma), \tag{6}$$

where $D(l) = \{2,3,\dots,\mathrm{c}(\pi_n)\} \setminus \{l-1,l,l+1\}$, holds for all $n \ge N_1$;

(ii) *there exists $\tau > 0$ such that $x \le f_\gamma(x)$ when $x < \tau$*;

(iii) $\mathrm{m}(\pi_n) = o(1)$ *and $H_n(\gamma) = o(1)$ as $n \to \infty$.*

Then there exist constants $\theta > 0$ and $N \in \mathbb{N}$, $N \ge N_1$ such that

$$\inf_{m\in\mathbb{N}} \big(\mathrm{tr}\,\big[\big(A(\pi_n)\big)^m\big]\big)^{1/m} \le \theta H_n(\gamma),$$

for all $n \ge N$.

Proof. Throughout the proof we assume $n \ge N_1$.

Using the condition (ii) in Definition 1 and Cauchy–Bunyakowski–Schwarz inequality for $k,l = 1,\dots,\mathrm{c}(\pi_n)$ we obtain upper bounds

$$|a_{kl}| \le \sqrt{\psi_X(t_{k-1},t_k)}\sqrt{\psi_X(t_{l-1},t_l)}(\Delta t_k \Delta t_l)^{(\gamma-1)/2} \le L\mathrm{m}(\pi_n). \tag{7}$$

For every $j = 1,2,\dots$ we set

$$\Lambda_j := \sum_{k_1=1}^{\mathrm{c}(\pi_n)} \dots \sum_{k_{j+1}=1}^{\mathrm{c}(\pi_n)} \prod_{i=1}^{j} |a_{k_i k_{i+1}}|.$$

Then denoting by $(A(\pi_n))^j = (a_{kl}^{(j)})_{k,l=1}^{\mathrm{c}(\pi_n)}$ the jth power of $A(\pi_n)$ it can be seen by induction that

$$\mathrm{tr}\big[\big(A(\pi_n)\big)^j\big] \le \sum_{k,l} \big|a_{kl}^{(j)}\big| \le \Lambda_j, \quad j = 1,2,\dots. \tag{8}$$

Using straightforward algebra one can estimate for $m > 2$

$$\begin{aligned}
\Lambda_m &= \sum_{k_1=1}^{\mathrm{c}(\pi_n)} \dots \sum_{k_{m-1}=1}^{\mathrm{c}(\pi_n)} \prod_{i=1}^{m-2} |a_{k_i k_{i+1}}| \sum_{k_m=1}^{\mathrm{c}(\pi_n)} |a_{k_{m-1}k_m}| \sum_{k_{m+1}=1}^{\mathrm{c}(\pi_n)} |a_{k_m k_{m+1}}| \\
&\le \sum_{k_1=1}^{\mathrm{c}(\pi_n)} \dots \sum_{k_{m-1}=1}^{\mathrm{c}(\pi_n)} \prod_{i=1}^{m-2} |a_{k_i k_{i+1}}| \Bigg[|a_{k_{m-1}1}| \sum_{k_{m+1}=1}^{\mathrm{c}(\pi_n)} |a_{1k_{m+1}}| + \\
&\quad + \sum_{k_m=1}^{\mathrm{c}(\pi_n)} |a_{k_{m-1}k_m}||a_{k_m 1}| + \sum_{k_m=2}^{\mathrm{c}(\pi_n)} |a_{k_{m-1}k_m}| \sum_{k_{m+1}=2}^{\mathrm{c}(\pi_n)} |a_{k_m k_{m+1}}| \Bigg] \\
&=: \Lambda_{m,1} + \Lambda_{m,2} + \Lambda_{m,3}.
\end{aligned} \tag{9}$$

Using (7) and then (1) we get

$$\Lambda_{m,1} \le \Lambda_{m-2} L\mathrm{m}(\pi_n)\mathrm{c}(\pi_n) L\mathrm{m}(\pi_n) \le cL^2 T\mathrm{m}(\pi_n)\Lambda_{m-2} \tag{10}$$

and

$$\Lambda_{m,2} \le L\mathrm{m}(\pi_n)\Lambda_{m-1}. \tag{11}$$

(7) and (6) give

$$\begin{aligned}\Lambda_{m,3} &= \sum_{k_1=1}^{\mathrm{c}(\pi_n)} \dots \sum_{k_{m-1}=1}^{\mathrm{c}(\pi_n)} \prod_{i=1}^{m-2} |a_{k_i k_{i+1}}| \sum_{k_m=2}^{\mathrm{c}(\pi_n)} |a_{k_{m-1}k_m}| \sum_{\substack{k_{m+1}=2 \\ |k_{m+1}-k_m|\le 1}}^{\mathrm{c}(\pi_n)} |a_{k_m k_{m+1}}| \\ &\quad + \sum_{k_1=1}^{\mathrm{c}(\pi_n)} \dots \sum_{k_{m-1}=1}^{\mathrm{c}(\pi_n)} \prod_{i=1}^{m-2} |a_{k_i k_{i+1}}| \sum_{k_m=2}^{\mathrm{c}(\pi_n)} |a_{k_{m-1}k_m}| \sum_{k_{m+1}\in D(k_m)} |a_{k_m k_{m+1}}| \\ &\le \bigl(3L\mathrm{m}(\pi_n) + H_n(\gamma)\bigr)\Lambda_{m-1}. \end{aligned} \tag{12}$$

Let

$$\theta := 2\max\biggl\{\frac{5L}{\xi(\gamma)} + 1,\ \frac{cL^2T}{\xi(\gamma)},\ cLT\biggl(\frac{5L}{\xi(\gamma)} + 1\biggr)\biggr\}. \tag{13}$$

Then by the assumptions (ii) and (iii) there exists $N_2 \in \mathbb{N}$ such that $\mathrm{m}(\pi_n) \le H_n(\gamma)$ for $n \ge N_2$ and so

$$4L\mathrm{m}(\pi_n) + H_n(\gamma) \le \frac{\theta}{2} H_n(\gamma) \quad \text{and} \quad cL^2T\mathrm{m}(\pi_n) \le \frac{\theta}{2} H_n(\gamma).$$

for $n \ge N_2$. Substituting (10), (11) and (12) into (9) yields

$$\begin{aligned}\Lambda_m &\le \bigl(4L\mathrm{m}(\pi_n) + H_n(\gamma)\bigr)\Lambda_{m-1} + cL^2T\mathrm{m}(\pi_n)\Lambda_{m-2} \\ &\le \theta H_n(\gamma)\max\{\Lambda_{m-1}, \Lambda_{m-2}\} \\ &\le \bigl(\theta H_n(\gamma)\bigr)^2 \max\{\Lambda_{m-2}, \Lambda_{m-3}\} \\ &\dots \\ &\le \bigl(\theta H_n(\gamma)\bigr)^{m-2} \max\{\Lambda_2, \Lambda_1\}, \end{aligned} \tag{14}$$

for $n \ge N_2$. Applying (7) and (6) we estimate

$$\begin{aligned}\Lambda_1 &= \sum_{k,l=1}^{\mathrm{c}(\pi_n)} |a_{kl}| \le 2\sum_{l=1}^{\mathrm{c}(\pi_n)} |a_{1l}| + \sum_{k,l=2}^{\mathrm{c}(\pi_n)} |a_{kl}| \\ &= 2\sum_{l=1}^{\mathrm{c}(\pi_n)} |a_{1l}| + \sum_{\substack{k,l=2 \\ |k-l|\le 1}}^{\mathrm{c}(\pi_n)} |a_{kl}| + \sum_{\substack{k,l=2 \\ |k-l|>1}}^{\mathrm{c}(\pi_n)} |a_{kl}| \\ &\le \mathrm{c}(\pi_n)\bigl(5L\mathrm{m}(\pi_n) + H_n(\gamma)\bigr). \end{aligned} \tag{15}$$

By (7) and (1) we have

$$\Lambda_2 = \sum_{j=1}^{\mathrm{c}(\pi_n)} \sum_{k,l=1}^{\mathrm{c}(\pi_n)} |a_{kl}||a_{lj}| \le L\mathrm{m}(\pi_n)\mathrm{c}(\pi_n) \sum_{k,l=1}^{\mathrm{c}(\pi_n)} |a_{kl}| \le cLT\Lambda_1. \tag{16}$$

Therefore by (15), (16) and (13) for $n \ge N_2$ we can bound

$$\max\{\Lambda_2, \Lambda_1\} \le \mathrm{c}(\pi_n)\theta H_n(\gamma). \tag{17}$$

From the assumption (c) it follows that there exists $N_3 \in \mathbb{N}$ such that $\theta H_n(\gamma) < 1$ for all $n \ge N_3$. Let $N := \max\{N_1, N_2, N_3\}$. Then by (8), (14) and (17) we conclude that for $n \ge N$

$$\begin{aligned} &\inf_{m\in\mathbb{N}} \bigl(\mathrm{tr}\bigl[\bigl(A(\pi_n)\bigr)^m\bigr]\bigr)^{1/m} \\ &\quad \le \inf_{m\in\mathbb{N}} \bigl(\theta H_n(\gamma)\bigr)^{1-1/m} \inf_{m\in\mathbb{N}} \bigl(\mathrm{c}(\pi_n)\bigr)^{1/m} = \theta H_n(\gamma). \end{aligned}$$ □

Theorem 1. *Let γ, d, b, L be as in Definition* 1 *and $X = \{X(t),\ t \in [0,T]\}$, $0 < T < \infty$, be a Gaussian stochastic process from $\mathcal{G}(\gamma, d, b, L)$. Let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,T]$ satisfying the assumptions of Lemma* 2*, and $H_n(\gamma) = o(1/\log n)$ as $n \to \infty$ with $H_n(\gamma)$ defined in* (6)*. Suppose that*:

(i) *the mean function M of X is Lipschitz continuous on $[0,T]$*;

(ii) *for each $\varepsilon \in (0,T)$*

$$\lim_{n\to\infty} \max_{\mathcal{K}(\varepsilon,n)} \left\{ \frac{|b(t_{k-1}^{(n)}, t_k^{(n)})|}{(t_k^{(n)} - t_{k-1}^{(n)})^{2-\gamma}} \right\} = 0, \tag{18}$$

where $\mathcal{K}(\varepsilon, n) := \{k \in \{1, \ldots, \mathrm{c}(\pi_n)\}\colon t_k^{(n)} \in (\varepsilon, T]\}$.

Then for X the relation

$$\lim_{n\to\infty} \sum_{k=1}^{\mathrm{c}(\pi_n)} \left[\frac{X(t_k^{(n)}) - X(t_{k-1}^{(n)})}{(t_k^{(n)} - t_{k-1}^{(n)})^{(1-\gamma)/2}} \right]^2 = dT \tag{19}$$

holds with probability 1.

Proof. It is enough to prove the theorem in the case when $M = 0$. Indeed, if $M \ne 0$, let $\overline{X} := X - M$ and suppose that (19) holds with $\overline{X}$ in place of X. For each n

$$S_n(X) = S_n(\overline{X}) + 2 \sum_{k=1}^{\mathrm{c}(\pi_n)} \Delta\overline{X}_k \Delta M_k + S_n(M).$$

Applying Cauchy–Bunyakowsky–Schwarz inequality we have

$$\left| \sum_{k=1}^{\mathrm{c}(\pi_n)} \Delta\overline{X}_k \Delta M_k \right| \le \sqrt{S_n(\overline{X})}\sqrt{S_n(M)}.$$

Since M is a Lipschitz function, there exists $K \geq 0$ such that

$$S_n(M) = \sum_{k=1}^{\mathrm{c}(\pi_n)} \big(M(t_k) - M(t_{k-1})\big)^2 \Delta t_k^{\gamma-1} \\ \leq K^2 \big(\mathrm{m}(\pi_n)\big)^\gamma \sum_{k=1}^{\mathrm{c}(\pi_n)} \Delta t_k = K^2 T \big(\mathrm{m}(\pi_n)\big)^\gamma \to 0, \quad \text{as } n \to \infty.$$

As $S_n(\overline{X})$ is bounded in n, it then follows that $S_n(X) \to dT$ as $n \to \infty$ with probability 1. Therefore for the rest of the proof we can and do assume that X is a mean zero Gaussian process.

Next we prove that

$$\lim_{n\to\infty} \mathbf{E} S_n(X) = dT. \tag{20}$$

For each n since X is from $\mathcal{G}(\gamma, d, b, L)$, we have

$$\mathbf{E} S_n(X) = dT + \sum_{k=1}^{c(\pi_n)} b(t_{k-1}, t_k) \Delta t_k^{\gamma-1}.$$

It is enough to prove that the second term on the right hand side converges to zero as $n \to \infty$. Let $\varepsilon \in (0,1)$. By (18) there exists an $N \in \mathbb{N}$ such that

$$\max_{\mathcal{K}(\varepsilon,n)} \left\{ \frac{|b(t_{k-1}, t_k)|}{\Delta t_k^{2-\gamma}} \right\} < \varepsilon$$

for each $n \geq N$. Then using the condition (ii) in Definition 1 we have

$$\sum_{k=1}^{c(\pi_n)} \big| b(t_{k-1}, t_k) \Delta t_k^{\gamma-1} \big| \\ = \Bigg(\sum_{k\colon\, 0 < t_k \leq \varepsilon} + \sum_{k\colon\, \varepsilon < t_k \leq T} \Bigg) \big| b(t_{k-1}, t_k) \big| \Delta t_k^{\gamma-1} \\ < L \sum_{k\colon\, 0 < t_k \leq \varepsilon} \Delta t_k + \varepsilon \sum_{k\colon\, \varepsilon < t_k \leq T} \Delta t_k \leq \varepsilon (L + T),$$

for each $n \geq N$. Since ε is arbitrary the desired relation (20) holds.

Finally we are left to prove that $S_n(X) - \mathbf{E} S_n(X)$ converges to zero as $n \to \infty$ with probability 1. For this we only need the existence of a sequence $\{\varepsilon_n\}$ satisfying

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbf{P}\big\{ \big| S_n(X) - \mathbf{E} S_n(X) \big| \geq \varepsilon_n \big\} < \infty. \tag{21}$$

Provided such a sequence exists Borel–Cantelli lemma and (20) then yield the desired conclusion (19). In order to find such a sequence we will follow Klein and Giné [3] and use Hanson and Wright's [11] bound.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which X is defined and so $\{\Delta X_k\}_{k=1}^{\mathrm{c}(\pi_n)} \subset L^2(\Omega, \mathcal{F}, \mathbf{P})$ and let $\langle \cdot\,, \cdot \rangle$ denote the inner product in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. If $\{Y_k\}_{k=1}^{r}$, $r \leq \mathrm{c}(\pi_n)$ is an orthonormal basis for the linear span of $\{\Delta X_k\}_{k=1}^{\mathrm{c}(\pi_n)}$, for $k = 1, \dots, \mathrm{c}(\pi_n)$ and $l = 1, \dots, r$ define $b_{kl} := \langle \Delta X_k, Y_l \rangle$, a $\mathrm{c}(\pi_n) \times r$ matrix $B_n := (b_{kl})$, a $\mathrm{c}(\pi_n) \times \mathrm{c}(\pi_n)$ matrix $A(\pi_n) := (a_{kl}) = B_n B_n'$ and an $r \times r$ matrix $C(\pi_n) := (c_{kl}) = B_n' B_n$.

From the definition of a_{kl} and c_{kl} we get

$$a_{kl} = \mathbf{E}\Delta X_k \Delta X_l \quad \text{and} \quad S_n(X) = \sum_{k,l=1}^{r} c_{kl} Y_k Y_l, \tag{22}$$

and so, applying the Hanson and Wright's bound and the argument in [3, p. 718], with more details in [6, pp. 698–699], yields that there exists a constant $M > 0$ such that for every $0 < \varepsilon \leq 1$ and for all $n \in \mathbb{N}$

$$\mathbf{P}\bigl\{\bigl|S_n(X) - \mathbf{E}S_n(X)\bigr| \geq \varepsilon\bigr\} \leq 2\exp\biggl(-\frac{\varepsilon^2 M}{\rho(C(\pi_n))}\biggr). \tag{23}$$

Now we want to obtain an upper bound for $\rho(C(\pi_n))$ from upper bounds for $|a_{kl}|$, $k, l = 1, \dots, \mathrm{c}(\pi_n)$. By (22) we have that the definition of the matrix $A(\pi_n)$ coincides with the one in Lemma 2. Thus, from Lemma 1 and Lemma 2 it follows that there exist constants $\theta > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$

$$\rho\bigl(C(\pi_n)\bigr) = \rho\bigl(A(\pi_n)\bigr) \leq \theta H_n(\gamma). \tag{24}$$

Now, the sequence $\{\varepsilon_n\}$ defined by $\varepsilon_n^2 = 2M^{-1}\theta H_n(\gamma)\log n$ converges to zero by hypothesis. Moreover, $\mathbf{P}\{|S_n(X) - \mathbf{E}S_n(X)| \geq \varepsilon_n\} \leq 2n^{-2}$ for $n \geq N$ by (23) and (24). Therefore, $\{\varepsilon_n\}$ satisfies (21). □

Also an analogous result to Theorem 2 in [2] can be proved:

Theorem 2. *Let γ, d, b, L be as in Definition* 1. *Let $X = \{X(t),\ t \in [0,T]\}$, $0 < T < \infty$, be a Gaussian stochastic process from $\mathcal{G}(\gamma, d, b, L)$ and $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,T]$ both satisfying assumptions of Lemma* 2 *and Theorem* 1. *Let also $\Delta t_k^{(n)} = \mathrm{m}(\pi_n)$, $k = 1, 2, \dots, \mathrm{c}(\pi_n)$. Then*

$$\lim_{n\to\infty}\biggl\{1 - \frac{1}{\log(\mathrm{m}(\pi_n))}\log\sum_{k=1}^{\mathrm{c}(\pi_n)}\bigl[X\bigl(t_k^{(n)}\bigr) - X\bigl(t_{k-1}^{(n)}\bigr)\bigr]^2\biggr\} = \gamma$$

holds with probability 1.

Proof. Since $dT > 0$, one can take logarithms and divide both sides of (19) by $\log(\mathrm{m}(\pi_n))$ before passing to the limit when $n \to \infty$. □

3 Application to some processes

In [10] applications of his result to bifractional and sub-fractional Brownian motion are presented. These processes are shown to satisfythe hypotheses of Theorem 1 in [10] and fail to satisfy some hypotheses of Theorem 1 in [2]. In this section we apply Theorem 1 to these processes.

Let $X = \{X_t,\ t \in [0,T]\}$ be a zero mean Gaussian process as in Lemma 2. Given $\{t_k\}_{k=1}^m$ a partition of $[0,T]$ using the notation (2) and (3) one can rewrite

$$\begin{aligned}\mathbf{E}\Delta X_k \Delta X_l &= \mathbf{E}\big(X(t_k) - X(t_{k-1})\big)\big(X(t_l) - X(t_{l-1})\big)(\Delta t_k \Delta t_l)^{(1/2)(\gamma-1)} \\ &= \big[r_X(t_k,t_l) - r_X(t_k,t_{l-1}) - r_X(t_{k-1},t_l) + r_X(t_{k-1},t_{l-1})\big] \\ &\quad \times (\Delta t_k \Delta t_l)^{(1/2)(\gamma-1)} \\ &= (\Delta t_k \Delta t_l)^{(1/2)(\gamma-1)} \int\limits_{I_{kl}} \frac{\partial^2 r_X}{\partial s \partial t}(s,t)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{25}$$

where the last equality holds with $I_{kl} := [t_{k-1}, t_k] \times [t_{l-1}, t_l]$ being in the region $\{(s,t)\colon 0 < s \neq t \leq T\}$ and $|k-l| > 1$. Representation (25) will be used to prove the corrolaries of the following subsections.

3.1 Bifractional Brownian motion

Let $H \in (0,1)$ and $K \in (0,1]$. The function $R_{H,K}\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ with values

$$R_{H,K}(s,t) := 2^{-K}\big[\big(t^{2H} + s^{2H}\big)^K - |t-s|^{2HK}\big], \quad s,t \geq 0,$$

is positive definite by Proposition 2.1 in [8]. A mean zero Gaussian stochastic process $B_{H,K} = \{B_{H,K}(t)\colon t \geq 0\}$ with the covariance function $R_{H,K}$ is called a *bifractional Brownian motion* with index (H,K). The structure function of $B_{H,K}$ is given by

$$\psi_{B_{H,K}}(s,t) = 2^{1-K}\big[|t-s|^{2HK} - \big(t^{2H} + s^{2H}\big)^K\big] + t^{2HK} + s^{2HK} \tag{26}$$

for each $t,s \geq 0$. It is shown in [8, Proposition 3.1] that $\psi_{B_{H,K}}(s,t) \leq 2^{1-K}|t-s|^{2HK}$. It then follows that a bifractional Brownian motion $B_{H,K}$ is a Gaussian process from the class $\mathcal{G}(\gamma,d,b,L)$ with $\gamma = 2 - 2HK$, $d = L = 2^{1-K}$ and $b(s,t) = \psi_{B_{H,K}}(s,t) - 2^{1-K}|t-s|^{2HK}$. For notational simplicity γ will be used in place of $2 - 2HK$ in this subsection.

Corollary 1. *Let $H \in (0,1)$, $K \in (0,1]$ and $0 < T < \infty$. Let $\{\pi_n\}_{n=1}^\infty$ be a sequence of partitions of $[0,T]$ such that there exists $c \geq 1$ satisfying $\mathrm{m}(\pi_n)/\mathrm{p}(\pi_n) \leq c$ for every n and $\mathrm{m}(\pi_n)^{\gamma/2} = o(1/\log n)$ as $n \to \infty$. Then for a bifractional Brownian motion $B_{H,K}$ the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{\mathrm{c}(\pi_n)} \left[\frac{B_{H,K}(t_k) - B_{H,K}(t_{k-1})}{(t_k - t_{k-1})^{(1-\gamma)/2}}\right]^2 = 2^{1-K}T$$

holds with probability 1.

Proof. This corollary is a consequence of Theorem 1, thus it is enough to check the hypotheses in Lemma 2 and Theorem 1. For a bifractional Brownian motion, Norvaiša verified the hypothesis (i) of Theorem 1. Therefore, we are left to check the hypotheses (i)–(iii) of Lemma 2 and the hypothesis (ii) of Theorem 1.

Let us check the hypotheses of Lemma 2. We start with (6). Norvaiša [10] showed that functions f_1 and f_2 with values

$$f_1(s,t) := \frac{|s-t|^\gamma}{c_1} \quad \text{and} \quad f_2(s,t) := \frac{(st)^{\gamma/2}}{c_2} \tag{27}$$

and some constants c_1 and c_2 can be used to bound

$$\left|\frac{\partial^2 R_{H,K}}{\partial s \partial t}(s,t)\right| \le \frac{1}{f_1(s,t)} + \frac{1}{f_2(s,t)}, \tag{28}$$

when $s,t \in (0,T]$ and $s \ne t$. For f_1 we have

$$d_1(k,l) := \inf\{f_1(s,t)\colon\ (s,t) \in I_{kl}\} = \begin{cases} |t_{k-1}-t_l|^\gamma/c_1 & \text{if } k>l, \\ |t_{l-1}-t_k|^\gamma/c_1 & \text{if } k<l. \end{cases}$$

Therefore

$$\begin{aligned} c_1^{-1} \max_{l>1} \sum_{k\in D(l)} \bigl[d_1(k,l)\bigr]^{-1} &= \max_{l>1} \Biggl[\sum_{k=2}^{l-2} |t_{l-1}-t_k|^{-\gamma} + \sum_{k=l+2}^{\mathrm{c}(\pi_n)} |t_{k-1}-t_l|^{-\gamma}\Biggr] \\ &\le \frac{1}{(\mathrm{p}(\pi_n))^\gamma} \max_{l>1} \Biggl[\sum_{k=2}^{l-2} (l-1-k)^{-\gamma} + \sum_{k=l+2}^{\mathrm{c}(\pi_n)} (k-1-l)^{-\gamma}\Biggr] \\ &= \frac{1}{(\mathrm{p}(\pi_n))^\gamma} \max_{l>1} \Biggl[\sum_{k=1}^{l-3} k^{-\gamma} + \sum_{k=1}^{\mathrm{c}(\pi_n)-l-1} k^{-\gamma}\Biggr] \\ &\le \frac{1}{(\mathrm{p}(\pi_n))^\gamma} \Biggl(\max_{l>1} \Biggl[\sum_{k=1}^{l-3} k^{-\gamma}\Biggr] + \max_{l>1} \Biggl[\sum_{k=1}^{\mathrm{c}(\pi_n)-l-1} k^{-\gamma}\Biggr]\Biggr) \\ &= \frac{1}{(\mathrm{p}(\pi_n))^\gamma} \Biggl(\max_{l>3} \Biggl[\sum_{k=1}^{l-3} k^{-\gamma}\Biggr] + \max_{1<l<\mathrm{c}(\pi_n)-1} \Biggl[\sum_{k=1}^{\mathrm{c}(\pi_n)-l-1} k^{-\gamma}\Biggr]\Biggr) \\ &\le \frac{2}{(\mathrm{p}(\pi_n))^\gamma} \sum_{k=1}^{\mathrm{c}(\pi_n)} k^{-\gamma}. \end{aligned} \tag{29}$$

Bounding the sum in the preceding inequality with integral, we have

$$\sum_{k=2}^{\mathrm{c}(\pi_n)} k^{-\gamma} \le \int\limits_1^{\mathrm{c}(\pi_n)} x^{-\gamma}\,\mathrm{d}x$$
$$= \begin{cases} \frac{1}{1-\gamma}[(\mathrm{c}(\pi_n))^{1-\gamma} - 1] & \text{if } \gamma \in (0,1), \\ \log(\mathrm{c}(\pi_n)) & \text{if } \gamma = 1, \\ \frac{1}{\gamma-1}[1 - (\mathrm{c}(\pi_n))^{1-\gamma}] & \text{if } \gamma \in (1,2), \end{cases}$$
$$\le \begin{cases} \frac{1}{1-\gamma}(\frac{cT}{\mathrm{m}(\pi_n)})^{1-\gamma} & \text{if } \gamma \in (0,1), \\ \log(\frac{cT}{\mathrm{m}(\pi_n)}) & \text{if } \gamma = 1, \\ \frac{1}{\gamma-1} & \text{if } \gamma \in (1,2). \end{cases} \tag{30}$$

Substituting (30) into (29) we get

$$\max_{l>1} \sum_{k\in D(l)} \big[d_1(k,l)\big]^{-1} \le \begin{cases} \frac{2cc_1T^{1-\gamma}}{(1-\gamma)\mathrm{m}(\pi_n)} & \text{if } \gamma \in (0,1), \\ 2cc_1\frac{|\log((cT)^{-1}\mathrm{m}(\pi_n))|}{\mathrm{m}(\pi_n)} & \text{if } \gamma = 1, \\ \frac{1}{\gamma-1}\frac{c^\gamma c_1}{\mathrm{m}(\pi_n)^\gamma} & \text{if } \gamma \in (1,2). \end{cases} \tag{31}$$

For f_2 we have

$$d_2(k,l) := \inf\big\{f_2(s,t)\colon (s,t) \in I_{kl}\big\} = \frac{(t_{k-1}t_{l-1})^{\gamma/2}}{c_2}.$$

Therefore

$$\max_{l>1} \sum_{k\in D(l)} [d_2(k,l)]^{-1}$$
$$= c_2 \max_{l>1} \sum_{k\in D(l)} (t_{k-1}t_{l-1})^{-\gamma/2} = c_2 \max_{l>1} t_{l-1}^{-\gamma/2} \sum_{k\in D(l)} t_{k-1}^{-\gamma/2}$$
$$\le \frac{c_2}{(\mathrm{p}(\pi_n))^{\gamma/2}} \sum_{k=2}^{\mathrm{c}(\pi_n)} t_{k-1}^{-\gamma/2} \frac{c_2c^\gamma}{(\mathrm{m}(\pi_n))^\gamma(1-\gamma/2)}\big(\mathrm{c}(\pi_n)\big)^{1-\gamma/2}$$
$$\le \frac{c_2c^{2-\gamma/2}T^{1-\gamma/2}}{(1-\gamma/2)(\mathrm{m}(\pi_n))^{1+\gamma/2}} =: M_n. \tag{32}$$

It can be seen from (31) and (32) that for n large enough

$$\max_{l>1} \sum_{k\in D(l)} \big[d_1(k,l)\big]^{-1} < M_n. \tag{33}$$

Thus using (25) with $X = B_{H,K}$, (28), (32) and (33) we conclude that

$$\begin{aligned}
&\max_{l>1} \sum_{k\in D(l)} |\mathbf{E}\Delta X_k \Delta X_l| \\
&\quad \le \max_{l>1} \sum_{k\in D(l)} (\Delta t_k \Delta t_l)^{\frac{1}{2}(\gamma-1)} \int\limits_{I_{kl}} \left(\frac{1}{f_1(s,t)} + \frac{1}{f_2(s,t)}\right) \mathrm{d}s\, \mathrm{d}t \\
&\quad \le \mathrm{m}(\pi_n)^{\gamma+1} \max_{l>1} \sum_{k\in D(l)} \left(\left[d_1(k,l)\right]^{-1} + \left[d_2(k,l)\right]^{-1}\right) \\
&\quad < H_1 \mathrm{m}(\pi_n)^{\gamma+1} \mathrm{m}(\pi_n)^{-1-\gamma/2} = H_1 \mathrm{m}(\pi_n)^{\gamma/2},
\end{aligned}$$

with some constant H_1, so (6) holds with $f_\gamma(x) = x^{\gamma/2}$ and $\xi(\gamma) = H_1$.

The hypothesis (ii) holds with $\tau = 1$ since $0 < \gamma < 2$ and the hypothesis (iii) is clearly satisfied by the hypotheses of this corrolary.

The proof of (18) can be given by a slight modification of the proof in [10]. □

3.2 Sub-fractional Brownian motion

Let $H \in (0,1)$. The function $C_H : [0,\infty) \times [0,\infty) \to \mathbb{R}$ with values

$$C_H(s,t) := s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right]$$

is positive definite as shown in [9]. A *sub-fractional Brownian motion* with index H is a mean zero Gaussian stochastic process $G_H = \{G_H(t)\colon t \ge 0\}$ with the covariance function C_H. The structure function of G_H is given by

$$\psi_{G_H}(s,t) = |s-t|^{2H} + (s+t)^{2H} - 2^{2H-1}\left[t^{2H} + s^{2H}\right]$$

for $s,t \ge 0$. It is shown in [9] that $\psi_{G_H}(s,t) \le \max\{1, 2-2^{2H-1}\}(t-s)^{2H}$ for $t \ge s$. It then follows that a sub-fractional Brownian motion G_H is a Gaussian process from the class $\mathcal{G}(\gamma,d,b,L)$ with $\gamma = 2-2H$, $d = 1$, $b(s,t) = \psi_{G_H}(s,t) - |s-t|^{2H}$ and $L = \max\{1, 2-2^{2H-1}\}$. For notational simplicity γ will be used in place of $2-2H$ in this subsection.

Corollary 2. *Let $H \in (0,1)$ and $0 < T < \infty$. Let $\{\pi_n\}_{n=1}^\infty$ be a sequence of partitions of $[0,T]$ such that there exists $c \ge 1$ satisfying $\mathrm{m}(\pi_n)/\mathrm{p}(\pi_n) \le c$ for every n and*

$$\begin{cases}
\mathrm{m}(\pi_n)^\gamma = o(1/\log n) & \text{if } \gamma \in (0,1),\\
\mathrm{m}(\pi_n)|\log(\mathrm{m}(\pi_n))| = o(1/\log n) & \text{if } \gamma = 1,\\
\mathrm{m}(\pi_n) = o(1/\log n) & \text{if } \gamma \in (1,2).
\end{cases}$$

Then for a sub-fractional Brownian motion G_H the relation

$$\lim_{n\to\infty} \sum_{k=1}^{\mathrm{c}(\pi_n)} \left[\frac{G_H(t_k) - G_H(t_{k-1})}{(t_k - t_{k-1})^{(1-\gamma)/2}}\right]^2 = T$$

holds with probability 1.

Proof. It is enough again to check the hypotheses of Lemma 2 and Theorem 1 for $X = G_H$. For a sub-fractional Brownian motion, Norvaiša verified the hypothesis (i) of Theorem 1. Therefore, we are left to check the hypotheses (i)–(iii) of Lemma 2 and the hypothesis (ii) of Theorem 1.

Let us check the hypotheses of Lemma 2. We start with (6). Norvaiša [10] showed that for some constant C

$$\left|\frac{\partial^2 C_H}{\partial s \partial t}(s,t)\right| \le \frac{C}{|s-t|^{\gamma}}, \tag{34}$$

when $s,t \in (0,T]$ and $s \neq t$, therefore using (25) with $X = G_H$, (34) and (31) we conclude that

$$\begin{aligned}
&\max_{l>1} \sum_{k\in D(l)} |\mathbf{E}\Delta X_k \Delta X_l| \\
&\quad\le \max_{l>1} \sum_{k\in D(l)} (\Delta t_k \Delta t_l)^{\frac{1}{2}(\gamma-1)} \int\limits_{I_{kl}} \frac{C}{|s-t|^{\gamma}} \,\mathrm{d}s \,\mathrm{d}t \\
&\quad\le \mathrm{m}(\pi_n)^{\gamma+1} \max_{l>1} \sum_{k\in D(l)} \frac{C}{\inf\{|s-t|^{\gamma}\colon (s,t)\in I_{kl}\}} \\
&\quad\le \zeta(\gamma) f_\gamma\big(\mathrm{m}(\pi_n)\big),
\end{aligned}$$

where $\zeta(\gamma) \in \mathbb{R}$ depends only on γ, so (6) holds with

$$f_\gamma(x) := \begin{cases} x^{\gamma} & \text{if } \gamma \in (0,1), \\ x|\log(x)| & \text{if } \gamma = 1, \\ x & \text{if } \gamma \in (1,2). \end{cases}$$

and $\xi(\gamma) = \zeta(\gamma)$.

Again, the hypotheses (ii) and (iii) are clearly satisfied.

The proof of (18) can be given by a slight modification of the proof in [10]. □

Remark 1. Corrolaries 1 and 2 are valid for the most frequently used partitions, i.e., when $\Delta t_k = T/n$ or $\Delta t_k = T2^{-n}$, $\forall k = 1,2,\dots,\mathrm{c}(\pi_n)$.

3.3 Estimating parameters of Gaussian processes

It follows from Corollary 1, Corollary 2 and Theorem 2 that given a partition π_n from a sequence of partitions $\{\pi_n\}_{n=1}^{\infty}$ satisfying assumptions of Lemma 2, Theorem 1 and Corollary 1 or Corollary 2 respectively

$$\widehat{V}_n = \frac{1}{2} + \frac{1}{2\log(\mathrm{m}(\pi_n))} \log \sum_{k=1}^{\mathrm{c}(\pi_n)} \big[F_V\big(t_k^{(n)}\big) - F_V\big(t_{k-1}^{(n)}\big)\big]^2, \tag{35}$$

where F_V denotes either a bifractional ($V = HK$) or a sub-fractional Brownian motion ($V = H$), is a strongly consistent estimator of V.

4 Central limit theorem

By Theorem 1 we know that $S_n(X) - \mathbf{E}S_n(X) \xrightarrow{\text{a.s.}} 0$. Since the estimator of the parameters of Gaussian processes (35) is based on quadratic variation, in applications we need a central limit theorem for the quadratic variation in order to carry out some tests or construct confidence intervals.

Theorem 3. *Let γ, d, b, L be as in Definition 1. Let $X = \{X(t),\ t \in [0,T]\}$, $0 < T < \infty$, be a zero mean Gaussian stochastic process from $\mathcal{G}(\gamma, d, b, L)$ and $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,T]$ both satisfying assumptions of Lemma 2. Suppose also that $H_n^2(\gamma)/\mathrm{p}(\pi_n) = o(1)$ as $n \to \infty$ and the hypothesis (ii) of Theorem 1 holds. Then*

$$\frac{S_n(X) - \mathbf{E}S_n(X)}{\sqrt{\operatorname{var}(S_n(X))}} \xrightarrow{\mathrm{d}} \zeta, \quad \text{as } n \to \infty,$$

where $\xrightarrow{\mathrm{d}}$ denotes the convergence in distribution and ζ is a standard Gaussian random variable.

Proof. Consider a centered Gaussian vector

$$\mathbb{X}_n = \begin{bmatrix} \Delta X_1 \\ \dots \\ \Delta X_{\mathrm{c}(\pi_n)} \end{bmatrix}$$

and denote it's covariance matrix $\Sigma_{\mathbb{X}_n}$. By the spectral decomposition theorem one can write $\Sigma_{\mathbb{X}_n} = P_n \Lambda_n P_n'$, where P_n is an orthogonal matrix and $\Lambda_n = \operatorname{diag}(\lambda_{1,n}, \dots, \lambda_{\mathrm{c}(\pi_n),n})$ with $\{\lambda_{i,n}\}$ being the eigenvalues of $\Sigma_{\mathbb{X}_n}$. Define

$$\boldsymbol{\eta}_n = \begin{bmatrix} \eta_{1,n} \\ \dots \\ \eta_{\mathrm{c}(\pi_n),n} \end{bmatrix} := \Lambda_n^{-1/2} P_n' \mathbb{X}_n.$$

Then $\boldsymbol{\eta}_n$ is a vector of independent standard Gaussian variables and

$$S_n(X) = \sum_{k=1}^{\mathrm{c}(\pi_n)} (\Delta X_k)^2 = \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n} \eta_{k,n}^2. \tag{36}$$

Indeed,

$$\begin{aligned} \operatorname{cov}(\boldsymbol{\eta}_n) &= \Lambda_n^{-1/2} P_n' \operatorname{cov}(\mathbb{X}_n) P_n \Lambda_n^{-1/2} \\ &= \Lambda_n^{-1/2} P_n' P_n \Lambda_n P_n' P_n \Lambda_n^{-1/2} = I_{\mathrm{c}(\pi_n)}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{\mathrm{c}(\pi_n)} (\Delta X_k)^2 &= \mathbb{X}_n' \mathbb{X}_n = \mathbb{X}_n' P_n \Lambda_n^{-1/2} \Lambda_n \Lambda_n^{-1/2} P_n' \mathbb{X}_n \\ &= \boldsymbol{\eta}_n' \Lambda_n \boldsymbol{\eta}_n = \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n} \eta_{k,n}^2. \end{aligned}$$

Let $a_{kl} = \mathbf{E}\Delta X_k \Delta X_l$ for $k,l = 1,2,\dots,\mathrm{c}(\pi_n)$ and the matrix $A(\pi_n) = (a_{kl}) \in M_{\mathrm{c}(\pi_n)}$. Then $A(\pi_n) = \Sigma_{\mathbb{X}_n}$ and $\{\lambda_{k,n}\}_{k=1}^{\mathrm{c}(\pi_n)}$ are the eigenvalues of $A(\pi_n)$. At first we will check that $\lambda_n^* := \rho(A(\pi_n)) = o(\sqrt{\operatorname{var}(S_n(X))})$. By Lemma 1 and Lemma 2 we have that $\lambda_n^* \le \theta H_n(\gamma)$.

We need a lower bound for $\operatorname{var}(S_n(X))$. We can write

$$\bigl[\mathbf{E}S_n(X)\bigr]^2 = \Biggl(\sum_{k=1}^{\mathrm{c}(\pi_n)} a_{kk}\Biggr)^2 = \sum_{k=1}^{\mathrm{c}(\pi_n)} a_{kk}^2 + 2\sum_{\substack{k,l=1\\k<l}}^{\mathrm{c}(\pi_n)} a_{kk}a_{ll}.$$

We have that

$$S_n^2(X) = \sum_{k=1}^{\mathrm{c}(\pi_n)} (\Delta X_k)^4 + 2\sum_{\substack{k,l=1\\k<l}}^{\mathrm{c}(\pi_n)} (\Delta X_k)^2(\Delta X_l)^2.$$

Since $\mathbb{X}_n$ is a Gaussian vector, applying Isserlis' theorem gives

$$\mathbf{E}(\Delta X_k)^4 = 3\bigl[\mathbf{E}(\Delta X_k)^2\bigr]^2 = 3a_{kk}^2$$

and

$$\begin{aligned}\mathbf{E}(\Delta X_k)^2(\Delta X_l)^2 &= \mathbf{E}(\Delta X_k)^2\mathbf{E}(\Delta X_l)^2 + 2[\mathbf{E}\Delta X_k\Delta X_l]^2\\ &= a_{kk}a_{ll} + 2a_{kl}^2.\end{aligned}$$

Then

$$\mathbf{E}S_n^2(X) = 3\sum_{k=1}^{\mathrm{c}(\pi_n)} a_{kk}^2 + 2\sum_{\substack{k,l=1\\k<l}}^{\mathrm{c}(\pi_n)} \bigl(a_{kk}a_{ll} + 2a_{kl}^2\bigr),$$

which yields

$$\operatorname{var}\bigl(S_n(X)\bigr) = 2\sum_{k=1}^{\mathrm{c}(\pi_n)} a_{kk}^2 + 4\sum_{\substack{k,l=1\\k<l}}^{\mathrm{c}(\pi_n)} a_{kl}^2. \tag{37}$$

Let $\delta \in (0,T)$. Let us notice that by the definition of ψ_X (in Definition 1)

$$a_{kk} = \frac{\psi_X(t_{k-1},t_k)}{\Delta t_k^{1-\gamma}}$$

and that (18) implies

$$\max_{\mathcal{K}(\delta,n)}\left|\frac{|\psi_X(t_{k-1},t_k)|}{\Delta t_k^{2-\gamma}} - d\right| \to 0, \quad \text{as } n\to\infty,$$

with $\mathcal{K}(\delta, n)$ defined in (18), which gives that

$$\max_{\mathcal{K}(\delta,n)} \left| \frac{|a_{kk}|}{\Delta t_k} - d \right| \to 0, \quad \text{as } n \to \infty. \tag{38}$$

Denoting $g_k := |a_{kk}|/\Delta t_k - d$ for $k \in \mathcal{K}(\delta, n)$ gives $|a_{kk}| = \Delta t_k(g_k + d)$. From (37) and (38) we get

$$\begin{aligned}
&\operatorname{var}\bigl(S_n(X)\bigr) \\
&\geq 2 \sum_{k=1}^{\mathrm{c}(\pi_n)} a_{kk}^2 \geq 2 \sum_{k\in\mathcal{K}(\delta,n)} a_{kk}^2 \geq 2d^2 \sum_{k\in\mathcal{K}(\delta,n)} \Delta t_k^2 + 4d \sum_{k\in\mathcal{K}(\delta,n)} \Delta t_k^2 g_k \\
&\geq 2d^2\bigl(T - \delta + \mathrm{p}(\pi_n)\bigr)\mathrm{p}(\pi_n) + 4d\bigl(T - \delta + \mathrm{p}(\pi_n)\bigr)\mathrm{p}(\pi_n) \min_{\mathcal{K}(\delta,n)} g_k \\
&= 2d^2\bigl(T - \delta + \mathrm{p}(\pi_n)\bigr)\mathrm{p}(\pi_n) + o\bigl(\mathrm{p}(\pi_n)\bigr).
\end{aligned}$$

Hence $\operatorname{var}(S_n(X)) > J\mathrm{p}(\pi_n)$ for some positive constant J and

$$\frac{(\lambda_n^*)^2}{\operatorname{var}(S_n(X))} \leq \frac{(\theta H_n(\gamma))^2}{J\mathrm{p}(\pi_n)} \to 0, \quad \text{as } n \to \infty.$$

In order to prove the statement of the theorem we will check the Lindeberg condition for $\{\lambda_{k,n}(\eta_{k,n}^2 - 1)\}_{k=1}^{\mathrm{c}(\pi_n)}$: for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\operatorname{var}(S_n(X))} \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}^2 \mathbf{E}\bigl[\eta_{k,n}^2 - 1\bigr]^2 \mathbf{1}_{\bigl\{|\lambda_{k,n}(\eta_{k,n}^2 - 1)| > \varepsilon\sqrt{\operatorname{var}(S_n(X))}\bigr\}} = 0.$$

For $k = 1, 2, \ldots, \mathrm{c}(\pi_n)$ denote

$$V_{k,n} := \bigl[\eta_{k,n}^2 - 1\bigr]^2 \mathbf{1}_{\bigl\{|\lambda_{k,n}(\eta_{k,n}^2 - 1)| > \varepsilon\sqrt{\operatorname{var}(S_n(X))}\bigr\}}.$$

From $\eta_{k,n} \sim N(0,1)$, $\forall n = 1, 2, \ldots$ and $k = 1, 2, \ldots, \mathrm{c}(\pi_n)$ it follows that

$$\begin{aligned}
\mathbf{E}V_{k,n} &\leq \mathbf{E}\bigl[\eta_{k,n}^2 - 1\bigr]^2 \mathbf{1}_{\bigl\{|\eta_{k,n}^2 - 1| > \varepsilon \frac{\sqrt{\operatorname{var}(S_n(X))}}{\lambda_n^*}\bigr\}} \\
&= \mathbf{E}\bigl[\eta_{1,1}^2 - 1\bigr]^2 \mathbf{1}_{\bigl\{|\eta_{1,1}^2 - 1| > \varepsilon \frac{\sqrt{\operatorname{var}(S_n(X))}}{\lambda_n^*}\bigr\}} =: \mathbf{E}\widetilde{V}_n.
\end{aligned} \tag{39}$$

Since $\widetilde{V}_n \leq [\eta_{1,1}^2 - 1]^2$ and

$$\mathbf{E}\bigl[\eta_{1,1}^2 - 1\bigr]^2 = \mathbf{E}\bigl[\eta_{1,1}^4 - 2\eta_{1,1}^2 + 1\bigr] = 2,$$

we can use the dominated convergence theorem to conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\widetilde{V}_n \\
&\quad = \mathbf{E} \lim_{n\to\infty} \widetilde{V}_n = \mathbf{E}\bigl[\eta_{1,1}^2 - 1\bigr]^2 \lim_{n\to\infty} \mathbf{1}_{\bigl\{|\eta_{1,1}^2 - 1| > \varepsilon \frac{\sqrt{\operatorname{var}(S_n(X))}}{\lambda_n^*}\bigr\}} = 0.
\end{aligned} \tag{40}$$

By (36)

$$\operatorname{var}\bigl(S_n(X)\bigr) = \operatorname{var}\Biggl(\sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}\eta_{k,n}^2\Biggr) = \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}^2 \operatorname{var}(\eta_{k,n}^2) = 2\sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}^2.$$

Let $\varepsilon > 0$. Then by (39) and (40)

$$\lim_{n\to\infty} \frac{1}{\operatorname{var}(S_n(X))} \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}^2 \mathbf{E} V_{k,n} \le \lim_{n\to\infty} \frac{\mathbf{E}\widetilde{V}_n}{\operatorname{var}(S_n(X))} \sum_{k=1}^{\mathrm{c}(\pi_n)} \lambda_{k,n}^2 = \frac{1}{2}\lim_{n\to\infty} \mathbf{E}\widetilde{V}_n = 0,$$

thus the Lindeberg condition is satisfied. The conclusion then follows by the Lindeberg's CLT. □

Remark 2. Guyon and Léon [12] proved that when $\gamma \in (0, 1/2]$ the central limit theorem is false for fractional Brownian motion and holds in other cases. Since the Brownian motion is a special case of both fractional and sub-fractional Brownian motion and fractional Brownian motion is a special case of bifractional Brownian motion one cannot expect to prove the central limit theorem for these processes when $\gamma \in (0, 1/2]$. In fact, it can be seen from Corrolary 2 that $H_n^2(\gamma)/\mathrm{p}(\pi_n) = o(1)$ is false for any sequence of partitions when $\gamma \in (0, 1/2]$ for the sub-fractional Brownian motion. For the bifractional Brownian motion Corrolary 1 shows that $H_n^2(\gamma)/\mathrm{p}(\pi_n) = o(1)$ is true only when $\gamma \in (1, 2)$.

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