Duality of the alternating integral for quasi-linear differential games

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Abstract. The concept of alternated integrals proved to be useful in differential games and in control theory under the conditions of uncertainty. In this article the connection between upper and lower alternating integrals for quasi-linear differential games is established and its applications to the problem of pursuit is studied.

Keywords: differential games, alternating integral, multi-valued mapping, pursuer, evader, strategy.

1 Introduction

To solve the problem of pursuit in linear differential games, L.S. Pontryagin suggested two direct methods [1, 2]. Pontryagin's second direct method, based on concept of the alternating integral, which has no analogs in integration of real function. In definition of alternating integral participate operations of integration of set-valued mappings and geometric difference (Minkovski difference) of sets. These operations make difficulties for computation of alternating integral. A simplified schemes for constructing of alternating integral were proposed in [3, 4]. An evaluation error of the numerical constructing of the alternating integral has been described in [5]. This elegant construction admits a generalization, which makes possible to define a differential and an integral of a set-valued mapping in such way that these two operations become mutually reverse. Such a generalization based on quasi-affine mapping has been proposed in [6, 7]. Numerical algorithms for evaluation of a generalized alternating integral were proposed in [8, 9]. An important facts has been established in [10, 11]: *A generalized alternating integral describes the epigraph of a function, which is the viscosity solution to a Hamilton–Jacobi equation.*

Another set-valued integration procedure unifying Rieman-type integral, Auman's integral and Pontryagin's alternating integral were proposed in [12,13]. In these works the following important observation has been made: an alternating integral may serve crosscuts of "bridge of Krasovski" and a level set of value functions for related Hamilton–Jacobi–Bellman–Isaacs equations. Some aspects of computation alternating integral basic on ellipsoidal calculation is studied. This procedure allowed to apply an alternating integral to the control synthesis problem.

Pontryagin's second direct method has played the great role in development of the differential games and control theory. Therefore, many works are devoted to investigation of this method (see [14–22]). In particular, a lower analogue of the Pontryagin alternating integral for linear differential games was introduced in [14] (to complete the symmetry the alternating integral defined in [2] was called the upper Pontryagin alternating integral). The lower alternating integral has been found useful in solving problems of pursuit under certain information discrimination of the pursuer compared to the evader. In [14], a connection was also established between these concepts, and it was applied to the problem of informality [4]. In this article a connection between the upper and lower alternating integrals for quasi-linear differential games and its applications to the problem of pursuit are established.

We shall use the following notations: $I = [0, \tau]$ is the fixed segment of time; Δ is a subsegment of I; $|\Delta|$ is the length of Δ; $\mathrm{cl}(\mathbb{R}^d)$ ($\mathrm{Ccl}(\mathbb{R}^d)$, respectively) is the collection of all nonempty closed (convex closed, respectively) subsets of $\mathbb{R}^d$; $\mathrm{cm}(\mathbb{R}^d)$ ($\mathrm{Ccm}(\mathbb{R}^d)$, respectively) is the collection of all nonempty compact (convex compact, respectively) subsets of $\mathbb{R}^d$; $H = \{z \in \mathbb{R}^d \mid |z| \leq 1\}$ is the unit closed ball in $\mathbb{R}^d$; $h(A, B) = \min\{r \geq 0 \mid A \subset B + rH,\, B \subset A + rH\}$ – Hausdorff metric; $\omega_n = \{0, \delta, 2\delta, \dots, n\delta = \tau\}$ is an uniform partition of the segment $[0, \tau]$; $\int_i$ is an integral by the segment $[(i-1)\delta, i\delta]$, and $X(i)$ is the collection of all measurable functions $x(\cdot) : [(i-1)\delta, i\delta] \to X$, $X \subset \mathbb{R}^d$.

A differential game described by the equation

$$\dot{z} = -f(t, u, v) \tag{1}$$

is considered, where $z \in \mathbb{R}^d$, $t \in I$, u and v are control parameters, u is the pursuer parameter, v is the evader parameter, $u \in P \in \mathrm{Ccm}(\mathbb{R}^d)$, $v \in Q \in \mathrm{Ccm}(\mathbb{R}^d)$, and $f : I \times P \times Q \to \mathbb{R}^d$ is a continuous function. The terminal set is M, $M \subset \mathbb{R}^d$. An arbitrary measurable function $u(\cdot) \in P(I)$ ($v(\cdot) \in Q(I)$) is called admissible control parameter of the pursuer (evader).

A given initial point z_0 and a pair of controlling parameters $u(\cdot) \in P(I)$, $v(\cdot) \in Q(I)$ give rise to a unique trajectory $z(t) = z(t, z_0, u(\cdot), v(\cdot)), t \geq 0$, of the system (1) (precisely definition of a trajectory is given in Section 4). Pursuit starts from a point $z_0 \in \mathbb{R}^d \setminus M$ and it is considered to be ended, when the phase point hits the set M. In other words, the pursuer aims to realize the inclusion $z(\tau) \in M$. Then, we say that pursuit from a point z_0 is completed at the time τ in the game (1).

Naturally, there is a question: *From which initial points z_0 pursuit can be completed at the time τ in the game* (1)*?*

To solve this problem L.S. Pontryagin has introduced the second method of pursuit in a linear differential game. The second method of pursuit is formulated in terms of an alternating integral.

To each partition ω_n of the interval I we assign upper and lower alternating sums $S^\tau(M,\omega_n)$ and $S_\tau(M,\omega_n)$ as described below:

$$S^n = M, \quad S^{i-1} = \bigcap_{v(\cdot)\in Q(i)} \bigcup_{u(\cdot)\in P(i)} \left[S^i + \int_i f\big(t,u(t),v(t)\big)\,\mathrm{d}t\right], \quad S^\tau(M,\omega_n) = S^0,$$

$$S_n = M, \quad S_{i-1} = \bigcup_{u(\cdot)\in P(i)} \bigcap_{v(\cdot)\in Q(i)} \left[S_i + \int_i f\big(t,u(t),v(t)\big)\,\mathrm{d}t\right], \quad S_\tau(M,\omega_n) = S_0.$$

The set $W^\tau(M) = \bigcap_{\omega_n} S^\tau(M,\omega_n)$ is said to be the *upper Pontryagin alternating integral* [1, 2, 16]. The set $W_\tau(M) = \bigcup_{\omega_n} S_\tau(M,\omega_n)$ is called the *lower Pontryagin alternating integral* [14, 15].

Further, if it will be necessary, we shall indicate in notations, the dependence of sums and integrals not only of ω or τ, but also of other initial data.

A concepts of the upper and lower alternating integrals have the following role in a quasi-linear differential games: *For points z_0 with $z_0 \in W^\tau(M)$ ($z_0 \in W_\tau(M)$) the pursuit can be completed at the time τ with* (*without*) *discriminating against the evader controls* [1, 2, 14–16].

Now we formulate the basic results. The following theorems establish connection between the upper and lower alternating integrals.

Theorem 1. *Let $M \in \mathrm{Ccl}(\mathbb{R}^d)$ and the set $f(t,P,v)$ is convex for any $t \in I$ and $v \in Q$. Then the equality*

$$W^\tau(M) = \bigcap_{\varepsilon>0} W_\tau(M+\varepsilon H) \tag{2}$$

holds.

Theorem 2. *Let the set $M \subset \mathbb{R}^d$ has the convex closed complement, and $f(t,u,Q)$ is convex for any $t \in I$ and $u \in P$. Then the equality*

$$W_\tau(M) = \bigcup_{\varepsilon>0} W^\tau(M \mathbin{\underline{*}} \varepsilon H) \tag{3}$$

holds. (*Here $A \mathbin{\underline{*}} B = \{z \in \mathbb{R}^d \mid z + B \subset A\}$.*)

Remark 1. Note that if the set M is open (closed), then the sets S_n, W_τ (S^n, W^τ) are open (closed) [14, 19].

Remark 2. It should be noted (see proof of the Theorem 1 and Theorem 2), for any set $M \subset \mathbb{R}^d$, the following relations are valid:

$$W^\tau(M) \subset \bigcap_{\varepsilon>0} W_\tau(M+\varepsilon H) \subset W^\tau(\mathrm{cl}\, M),$$
$$W_\tau\big(\big(\mathrm{cl}\big(\mathbb{R}^d \setminus M\big)\big)^c\big) \subset \bigcap_{\varepsilon>0} W_\tau(M \mathbin{\underline{*}} \varepsilon H) \subset \bigcap_{\varepsilon>0} W_\tau(M),$$

where A^c is a complement of a set A.

2 Preliminary lemmas

In this section we give some preliminary facts which are necessary for proving basic theorems.

Lemma 1. (*See* [14].) *Let a sequence $X_k \in \mathrm{cl}(\mathbb{R}^d)$ decreases monotonically by inclusion and $Y \in \mathrm{cm}(\mathbb{R}^d)$. Then the equality*

$$\left(\bigcap_{k=1}^{\infty} X_k\right) + Y = \bigcap_{k=1}^{\infty}(X_k + Y) \tag{4}$$

is valid.

It should be noted, for any family X_α and a set $Y \subset \mathbb{R}^d$, the following relations

$$\left(\bigcap_{\alpha} X_\alpha\right) + Y \subset \bigcap_{\alpha}(X_\alpha + Y), \quad \bigcup_{\alpha}(X_\alpha + Y) = \left(\bigcup_{\alpha} X_\alpha\right) + Y \tag{5}$$

hold.

Lemma 2. (*See* [14].) *If $M \in \mathrm{Ccl}(\mathbb{R}^d)$, then $W_\tau(M) \subset W^\tau(M)$.*

Further, we assume that the set $f(t, P, v)$ is convex for any $t \in I$ and $v \in Q$.

One can easily verify the validity of the following

Lemma 3. *If $f : I \times P \times Q \to \mathbb{R}^d$ is continuous, then the multi-valued mapping $F(t, v) = f(t, P, v)$ is also continuous on $I \times Q$ in metric Hausdorff.*

Let $\Gamma(\delta) = \max\{h(F(t_1, v_1), F(t_2, v_2)),\ |t_1 - t_2| < \delta,\ |v_1 - v_2| < \delta,\ t_1, t_2 \in I,$ $v_1, v_2 \in Q\}$ is the module of continuity for mappings $F(t, v) = f(t, P, v)$.

If $\xi \in \Delta \subset I$, $v \in Q$, then

$$\int_{\Delta} f(t, P, v)\,\mathrm{d}t \subset \delta f(\xi, P, v) + \delta\Gamma(\delta), \quad \delta f(\xi, P, v) \subset \int_{\Delta} f(t, P, v)\,\mathrm{d}t + \delta\Gamma(\delta). \tag{6}$$

Lemma 4. *Let L is arbitrary subset of $\mathbb{R}^d$, and $\Delta \subset I$. Then*

$$\begin{aligned} &\bigcap_{v(\cdot)\in Q(\Delta)} \bigcup_{u(\cdot)\in P(\Delta)} \left[L + \int_{\Delta} f\big(t, u(t), v(t)\big)\,\mathrm{d}t\right] \\ &\quad \subset \bigcap_{\xi\in\Delta} \bigcap_{v\in Q} \bigcup_{u\in P} \left[L + \Gamma(\delta)\delta H + \delta f(\xi, u, v)\right]. \end{aligned} \tag{7}$$

Proof. It is obvious,

$$\bigcap_{v(\cdot)\in Q(\Delta)} \bigcup_{u(\cdot)\in P(\Delta)} \left[L + \int_{\Delta} f\big(t, u(t), v(t)\big)\,\mathrm{d}t\right] \subset \bigcap_{v\in Q} \left[L + \int_{\Delta} f(t, P, v)\,\mathrm{d}t\right]. \tag{8}$$

Applying the inequality (6) to the right side of (8), we obtain

$$\bigcap_{v\in Q}\left[L+\int_{\Delta} f(t,P,v)\,\mathrm{d}t\right]\subset\bigcap_{v\in Q}\bigl[L+\Gamma(\delta)\delta H+\delta f(\xi,P,v)\bigr].$$

By virtue of (8), we get from here (7). □

Let $\omega_n\in\Omega$, $\xi_i\in\Delta_i$, and

$$B^n=M,\quad B^{i-1}=\bigcap_{v\in Q}\bigcup_{u\in P}\bigl[B^i+\delta f(\xi_i,u,v)\bigr],\quad B^\tau(M,\omega_n)=B^0.$$

Lemma 5. *For any $\varepsilon>0$, there exists a positive integer N such that for all $n\geq N$, the inclusion*

$$S^\tau(M,\omega_n)\subset B^\tau\left(M+\frac{\varepsilon}{3}H,\omega_n\right)$$

takes place.

Proof. Let the partition ω_n satisfies condition $\Gamma(\delta)<\varepsilon/(3\tau)$. By definition,

$$S^{n-1}=\bigcap_{v(\cdot)\in Q(n)}\ \bigcup_{u(\cdot)\in P(n)}\left[S^n+\int_n f\bigl(t,u(t),v(t)\bigr)\,\mathrm{d}t\right].$$

Applying Lemma 4 to the right side of the last equality, we obtain

$$S^{n-1}\subset\bigcap_{v\in Q}\bigcup_{u\in P}\bigl[M+\delta\Gamma(\delta)H+\delta f(\xi_n,u,v)\bigr]=B^n\bigl(M+\delta\Gamma(\delta)H\bigr),$$

where ξ_n is arbitrary point from the segment Δ_n.

Suppose

$$S^{n-k}(M)\subset B^{n-k}\bigl(M+k\delta\Gamma(\delta)H\bigr).$$

We shall show that

$$S^{n-(k+1)}(M)\subset B^{n-(k+1)}\bigl(M+(k+1)\delta\Gamma(\delta)H\bigr).$$

By definition,

$$S^{k-1}=\bigcap_{v(\cdot)\in Q(k-1)}\ \bigcup_{u(\cdot)\in P(k-1)}\left[S^k+\int_{n-k} f\bigl(t,u(t),v(t)\bigr)\,\mathrm{d}t\right].$$

Using Lemma 4, we have

$$S^{n-(k+1)}\subset\bigcap_{v\in Q}\bigcup_{u\in P}\bigl[S^{n-k}+\delta\Gamma(\delta)H+\delta f(\xi_{n-k},u,v)\bigr],$$

where ξ_{n-k} is arbitrary point from the segment Δ_{n-k}.

By virtue of assumption,

$$S^{n-(k+1)} \subset \bigcap_{v\in Q} \bigcup_{u\in P} \bigl[B^{n-k}\bigl(M + k\delta\Gamma(\delta)H\bigr) + \delta\Gamma(\delta)H + \delta f(\xi_{n-k}, u, v)\bigr].$$

Now, using (5), we obtain

$$\begin{aligned} S^{n-(k+1)} &\subset \bigcap_{v\in Q} \bigcup_{u\in P} \bigl[B^{n-k}\bigl(M + (k+1)\delta\Gamma(\delta)H\bigr) + \delta f(\xi_{n-k}, u, v)\bigr] \\ &= B^{n-(k+1)}\bigl(M + (k+1)\delta\Gamma(\delta)H\bigr). \end{aligned}$$

This implies

$$S^0(M) \subset B^0\bigl(M + \tau\Gamma(\delta)H\bigr).$$

Since $\Gamma(\delta) < \varepsilon/(3\tau)$ by condition, we have

$$S^\tau(M, \omega_n) \subset B^\tau\left(M + \frac{\varepsilon}{3}H, \omega_n\right). \qquad \square$$

Let now

$$B_n = M, \quad B_{i-1} = \bigcup_{u\in P} \bigcap_{v\in Q} \bigl[B^i + \delta f(\xi_i, u, v)\bigr], \quad B_\tau(M, \omega_n) = B_0.$$

Lemma 6. *For any $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$ the relation*

$$B^\tau(M, \omega_n) \subset B_\tau\left(M + \frac{\varepsilon}{3}H, \omega_n\right)$$

holds. Here $\gamma(\cdot)$ is the continuity module of a function $f(t, u, v)$.

Proof. Let the partition ω_n satisfies the following conditions:

1. $|\delta f(\xi, u, v)| < \varepsilon/9$ for any $\xi \in I$, $u \in P$, $v \in Q$;
2. $\tau\gamma(2\delta) < \varepsilon/9$.

Then by virtue of (5) we have

$$\begin{aligned} &B^\tau(M, \omega_n) \\ &\quad= \bigcap_{v\in Q} \bigcup_{u\in P} \Biggl[\bigcap_{v\in Q} \bigcup_{u\in P} \cdots \Biggl[\bigcap_{v\in Q} \bigcup_{u\in P} \Biggl[\bigcap_{v\in Q} \bigcup_{u\in P} \bigl(M + \delta f\bigl(\xi_n, u, v\bigr)\bigr) + \delta f(\xi_{n-1}, u, v)\Biggr] \\ &\qquad + \cdots + \delta f(\xi_2, u, v)\Biggr] + \delta f(\xi_1, u, v)\Biggr] \\ &\quad\subset \bigcup_{u\in P} \bigcap_{v\in Q} \Biggl[\bigcup_{u\in P} \bigcap_{v\in Q} \cdots \Biggl[\bigcup_{u\in P} \bigcap_{v\in Q} \Biggl[\bigcup_{u\in P} \bigcap_{v\in Q} \Biggl[\bigcup_{u\in P} \bigl(M + \delta f(\xi_n, u, v)\bigr) \\ &\qquad + \delta f(\xi_{n-1}, u, v)\Biggr] + \delta f(\xi_{n-3}, u, v)\Biggr] + \cdots + \delta f(\xi_1, u, v)\Biggr] + \delta f(\xi_1, u, v) + \frac{\varepsilon}{9}H\Biggr]. \end{aligned}$$

(Here we considered the fact $0 \in \delta f(\xi_1, u, v) + (\varepsilon/9)H$.)

Now we replace any vector $\delta f(\xi_i, u, v)$ in the right side of the equality on $\delta f(\xi_{i+1}, u, v) + \delta\gamma(2\delta)H$, $i = 1, 2, \dots, n-1$, the set $\bigcup_{u \in P}(M + \delta f(\xi_n, u, v))$ on the set $M + (\varepsilon/9)H$, and using the inclusion (5) one more time, we obtain

$$B^\tau(M, \omega_n) \subset B_\tau\biggl(M + \biggl(\frac{2\varepsilon}{9} + \tau\gamma(2\delta)\biggr)H, \omega_n\biggr).$$

By choose of partitions we have $\tau\gamma(2\delta) < \varepsilon/9$. Therefore,

$$B^\tau(M, \omega_n) \subset B_\tau\biggl(M + \frac{\varepsilon}{3}H, \omega_n\biggr). \qquad \square$$

Further, we denote the collection of piecewise constant functions $\hat{v}(\cdot) : \Delta \to Q$ as $\hat{Q}(\Delta)$.

Lemma 7. *If $L \in \mathrm{cl}(\mathbb{R}^d)$, then*

$$\bigcup_{u \in P} \bigcap_{v(\cdot) \in Q(\Delta)} \biggl[L + \int\limits_\Delta f\bigl(t, u, v(t)\bigr)\,\mathrm{d}t\biggr] = \bigcup_{u \in P} \bigcap_{\hat{v}(\cdot) \in \hat{Q}(\Delta)} \biggl[L + \int\limits_\Delta f\bigl(t, u, \hat{v}(t)\bigr)\,\mathrm{d}t\biggr]. \tag{9}$$

Proof. Obviously, the left side of (9) is contained in its right side. Prove the opposite.

Let $v(\cdot)$ be arbitrary function from $Q(\Delta)$. Then there exists a sequence of piecewise constant functions $v_k(\cdot) \in Q(\Delta)$ convergent almost everywhere to $v(\cdot)$ on Δ. Then

$$x \in L + \int\limits_\Delta f\bigl(t, u, \hat{v}_k(t)\bigr)\,\mathrm{d}t. \tag{10}$$

By given $\varepsilon > 0$, we find $\eta > 0$ such that

$$\bigl|f(t, u, v_1) - f(t, u, v_2)\bigr| < \varepsilon, \tag{11}$$

$t \in \Delta$, $u \in P$, $v_1, v_2 \in Q$, $|v_1 - v_2| < \eta$. By the Egorov theorem, there exists an open subset $\Delta_1 \subset \Delta$ with the measure $\mu(\Delta_1) < \varepsilon$ such that $v_k(\cdot) \to v(\cdot)$ uniformly on the closed set $\Delta \setminus \Delta_1$. Then $|v_k(t) - v(t)| < \eta$ for all $t \in \Delta \setminus \Delta_1$ and $k > N$ if N is sufficiently large. Hence, by virtue of (11) we have

$$f\bigl(t, u, \hat{v}_k(t)\bigr) \subset f\bigl(t, u, v(t)\bigr) + \varepsilon H \tag{12}$$

for all $t \in \Delta \setminus \Delta_1$ and $k > N$.

Let $\lambda = \max\{|f(t, u, v)|,\ t \in \Delta,\ u \in P,\ v \in Q\}$, and $|\Delta|$ is the length of the segment Δ. Then by the inclusion (12) we have

$$\int\limits_\Delta f\bigl(t, u, \hat{v}_k(t)\bigr)\,\mathrm{d}t \subset \int\limits_{\Delta \setminus \Delta_1} f\bigl(t, u, v(t)\bigr)\,\mathrm{d}t + \int\limits_{\Delta_1} f\bigl(t, u, \hat{v}_k(t)\bigr)\,\mathrm{d}t + \varepsilon|\Delta|H.$$

Moreover,

$$\int\limits_{\Delta} f\big(t,u,\hat{v}_k(t)\big)\,\mathrm{d}t \subset \int\limits_{\Delta\setminus\Delta_1} f\big(t,u,v(t)\big)\,\mathrm{d}t + \big(\lambda+|\Delta|\big)\varepsilon H. \tag{13}$$

Note, $\int_{\Delta_1} f(t,u,v(t))\,\mathrm{d}t \subset \lambda\varepsilon H$. Therefore

$$0 \in \int\limits_{\Delta_1} f\big(t,u,v(t)\big)\,\mathrm{d}t + \lambda\varepsilon H. \tag{14}$$

Adding (13) and (14), we obtain

$$\int\limits_{\Delta} f\big(t,u,\hat{v}_k(t)\big)\,\mathrm{d}t \subset \int\limits_{\Delta} f\big(t,u,v(t)\big)\,\mathrm{d}t + \big(2\lambda+|\Delta|\big)\varepsilon H.$$

Thus, by virtue of (10), we have

$$x \in L + \int\limits_{\Delta} f\big(t,u,\hat{v}_k(t)\big)\,\mathrm{d}t \subset L + \int\limits_{\Delta} f\big(t,u,v(t)\big)\,\mathrm{d}t + \big(2\lambda+|\Delta|\big)\varepsilon H. \tag{15}$$

Since the set L is closed and ε is arbitrary, inclusion (15) implies

$$x \in L + \int\limits_{\Delta} f\big(t,u,v(t)\big)\,\mathrm{d}t. \qquad \square$$

Lemma 8. *If $L \in \mathrm{Ccl}(\mathbb{R}^d)$, then*

$$\bigcup_{u\in P}\bigcap_{v\in Q}\Big[L + \int\limits_{\Delta} f(t,u,v)\,\mathrm{d}t\Big] \subset \bigcup_{u\in P}\bigcap_{v(\cdot)\in Q(\Delta)}\Big[L + \int\limits_{\Delta} f\big(t,u,v(t)\big)\,\mathrm{d}t + 2\delta\gamma(\delta)H\Big].$$

Proof. By Lemma 7, it is sufficient to prove the following inclusion

$$\bigcup_{u\in P}\bigcap_{v\in Q}\Big[L + \int\limits_{\Delta} f(t,u,v)\,\mathrm{d}t\Big] \subset \bigcup_{u\in P}\bigcap_{\hat{v}(\cdot)\in \hat{Q}(\Delta)}\Big[L + \int\limits_{\Delta} f\big(t,u,\hat{v}(t)\big)\,\mathrm{d}t + 2\delta\gamma(\delta)\Big]. \tag{16}$$

Let x be arbitrary element from the left side of (16). Then

$$x \in \bigcap_{v\in Q}\Big[L + \int\limits_{\Delta} f(t,u,v)\,\mathrm{d}t\Big] \tag{17}$$

for some $u \in P$. Choose arbitrary piecewise constant function $\hat{v}(t)$ from the collection $\hat{Q}(\Delta)$. We shall consider $\hat{v}(t) = v_j$ at $[t_{j-1},t_j]$, where $t_0 < t_1 < \cdots < t_m$, $\Delta = [t_0,t_m]$.

Set $\delta = |\Delta|$, $\delta_j = |t_j - t_{j-1}|$. Obviously, $\sum_{j=1}^{n} \delta_j = \delta$. From each $[t_{j-1}, t_j]$ we choose point ξ_j and apply twice the following inequalities

$$\int\limits_{\Delta} f(t,u,v)\,\mathrm{d}t \subset \delta f(\xi,u,v) + \delta\gamma(\delta)H,$$

$$\delta f(\xi,u,v) \subset \int\limits_{\Delta} f(t,u,v)\,\mathrm{d}t + \delta\gamma(\delta)H, \quad \xi \in \Delta,$$

at first to the segment Δ, then to the segment $[t_{j-1}, t_j]$. Then we get

$$\int\limits_{\Delta} f(t,u,v_j)\,\mathrm{d}t \subset \delta f(\xi_j,u,v_j) + \delta\gamma(\delta)H \subset \frac{\delta}{\delta_j}\int\limits_{\Delta_j} f(t,u,v_j)\,\mathrm{d}t + 2\delta\gamma(\delta)H,$$

from which

$$x \in L + \frac{\delta}{\delta_j}\int\limits_{\Delta_j} f(t,u,v_j)\,\mathrm{d}t + 2\delta\gamma(\delta)H$$

follows. Now we multiply these inclusions by δ_j/δ and add term by term. Since, the set L is convex and by virtue of (17) we have

$$x \in L + \int\limits_{\Delta} f\bigl(t,u,\hat{v}(t)\bigr)\,\mathrm{d}t + 2\delta\gamma(\delta)H. \qquad \square$$

Let $\omega_n \in \Omega$ and

$$C_n = M, \quad C_{i-1} = \operatorname{co}\bigcup_{u\in P}\ \bigcap_{v(\cdot)\in Q(\Delta)}\left[\operatorname{cl} C_i + \int\limits_{i} f\bigl(t,u,v(t)\bigr)\,\mathrm{d}t\right], \quad C_\tau(M,\omega_n) = C_0.$$

Applying Lemma 8 sequentially to the partial sums B^i, $i = n, n-1, \dots, 1$, we get the following

Lemma 9. *If $M \in \mathrm{Ccl}(\mathbb{R}^d)$, then for any $\varepsilon > 0$, there exists a positive integer N such that the inclusion*

$$B_\tau(M,\omega_n) \subset C_\tau\left(M + \frac{\varepsilon}{6}H\right)$$

is valid at all $n \geq N$.

Lemma 10. *If L is a convex subset of $\mathbb{R}^d$, then the set*

$$\bigcup_{u(\cdot)\in P(\Delta)}\ \bigcap_{v(\cdot)\in Q(\Delta)}\left[L + \int\limits_{\Delta} f\bigl(t,u(t),v(t)\bigr)\,\mathrm{d}t\right]$$

is convex.

Proof of Lemma 10 is obvious. □

By virtue of Lemma 10, Lemma 9 implies the following

Lemma 11. *Let $M \in \mathrm{Ccl}(\mathbb{R}^d)$. Then for any $\varepsilon > 0$, there exists a positive integer N such that the relation*

$$B_\tau(M, \omega_n) \subset S_\tau\left(M + \frac{\varepsilon}{3}H, \omega_n\right)$$

holds at all $n \geq N$.

Now Lemmas 5, 6, and 11 imply

Lemma 12. *If $M \in \mathrm{Ccl}(\mathbb{R}^d)$, then for any $\varepsilon > 0$, there exists a positive integer N such that the inclusion*

$$S^\tau(M, \omega_n) \subset S_\tau(M + \varepsilon H, \omega_n)$$

takes place for all $n \geq N$.

3 Proof of basic theorems

Proof of Theorem 1. It follows from Lemma 12,

$$W^\tau(M) \subset \bigcap_{\varepsilon>0} W_\tau(M + \varepsilon H).$$

On the other hand, we have on the base of Lemmas 2 and 1

$$\begin{aligned}\bigcap_{\varepsilon>0} W_\tau(M + \varepsilon H) &\subset \bigcap_{\varepsilon>0} W^\tau(M + \varepsilon H) = \bigcap_{\varepsilon>0}\bigcap_{n} S^\tau(M + \varepsilon H, \omega_n)\\ &= \bigcap_{n}\bigcap_{\varepsilon>0} S^\tau(M + \varepsilon H, \omega_n) = \bigcap_{n} S^\tau(M, \omega_n) = W^\tau(M).\end{aligned}$$

Theorem 1 is proved. □

Let L^c be a notation of the complement for the set $L \subset \mathbb{R}^d$.

From the duality of operations of intersection and join it follows that

$$\begin{aligned}&\bigcap_{v(\cdot)\in Q(\Delta)} \bigcup_{u(\cdot)\in P(\Delta)} \left[L + \int_\Delta f\big(t, u(t), v(t)\big)\,\mathrm{d}t\right]\\ &\quad= \bigcup_{v(\cdot)\in Q(\Delta)} \bigcap_{u(\cdot)\in P(\Delta)} \left[L^c + \int_\Delta f\big(t, u(t), v(t)\big)\,\mathrm{d}t\right].\end{aligned} \tag{18}$$

Applying formulas (18) to the upper alternating sum, we obtain

$$\big(S^\tau(M, P, Q, \omega_n)\big)^c = S_\tau\big(M^c, Q, P, \omega_n\big). \tag{19}$$

Analogously on can verify

$$\big(S_\tau(M, P, Q, \omega_n)\big)^c = S^\tau\big(M^c, Q, P, \omega_n\big). \tag{20}$$

Now, relations (19) and (20) imply

Lemma 13. *The following equalities*

$$\big[W^\tau(M,P,Q)\big]^c = W_\tau\big(M^c,Q,P\big), \tag{21}$$

$$\big[W_\tau(M,P,Q)\big]^c = W^\tau\big(M^c,Q,P\big) \tag{22}$$

take place.

Proof of Theorem 2. Theorem 1 implies

$$W^\tau\big(M^c,Q,P\big) = \bigcap_{\varepsilon>0} W_\tau\big(M^c+\varepsilon H,Q,P\big).$$

Applying the operation of complement to the both parts of this equality, using relations (21) and (22), we obtain

$$\big(W^\tau\big(M^c,Q,P\big)\big)^c = \Big(\bigcap_{\varepsilon>0} W_\tau\big(M^c+\varepsilon H,Q,P\big)\Big)^c.$$

Hence,

$$W_\tau(M) = \bigcup_{\varepsilon>0} W^\tau(M \underset{-}{*} \varepsilon H).$$

Theorem 2 is proved. □

4 Application of duality of the alternating integral to differential games of pursuit

Applications of the upper and lower alternating integral to quasi-linear differential games are similarly to the linear case [3]. Therefore, in this section we turn our attention to definitions of basic concepts and restrict ourself to state basic results in connection with the system (1).

Let $\theta > 0$, $X \subset \mathbb{R}^d$. We denote by $X(\theta)$ the family of all measurable functions $x(\cdot) : [0,\theta] \to X$.

Definition 1. The mapping $V_\delta^* : \mathbb{R}^d \to Q(\delta)$ is said to be δ-strategy of the evader in the upper game. The mapping $U_\delta^* : \mathbb{R}^d \times Q(\delta) \to P(\delta)$ is said to be δ-strategy of the pursuer in the upper game.

Definition 2. The mapping $U_*^\delta : \mathbb{R}^d \to P(\delta)$ is said to be δ-strategy of the pursuer in the lower game. The mapping $V_*^\delta : \mathbb{R}^d \times P(\delta) \to Q(\delta)$ is said to be δ-strategy of the evader in the lower game.

A given starting point z_0 and a pair of strategies U_δ^*, V_δ^* correspond to the unique absolutely continuous trajectory $z(t) = z(t, z_0, U_\delta^*, V_\delta^*)$, $t \geq 0$, defined in a following way.

The trajectory $z(t)$ on the segment $[0, \delta]$ is defined as the solution of the Cauchy problem

$$\dot{z}(t) = f\big(t, u_0(t), v_0(t)\big), \quad z(0) = z_0,$$

where $v_0(\cdot) = V_\delta^*(z_0)$ and $u_0(\cdot) = U_\delta^*(z_0, v_0(\cdot))$.

The trajectory $z(t)$ is continued from the segment $[0, k\delta]$ on the segment $[0, (k+1)\varepsilon]$ as the solution of the following Cauchy problem

$$\dot{z}(t) = f\big(t, u_k(t), v_k(t)\big), \quad z(k\delta) = z_k,$$

here $v_k(t) = v(t - k\delta)$, $v(\cdot) = V_\delta^*(z(k\delta))$ and $u_k(t) = U^*(z(k\delta), v(\cdot))(t - k\delta)$, $t \in [k, (k+1)\delta]$.

The trajectory $z(t) = z(t, z_0, U_*^\delta, V_*^\delta)$ is defined similarly. It corresponds to the given starting point z_0 and the pair of strategies U_*^δ, V_*^δ.

Definition 3. Pursuit from a point z_0 can be completed at the time τ in the upper game if for any $\delta = \tau/n$, there exists δ-strategy of the pursuer U_δ^* such that $z(\tau, z_0, U_\delta^*, V_\delta^*) \in M$ for any δ-strategy of the evader V_δ^*.

Concept of possibility to complete pursuit at the time τ in the lower game can be introduced similarly.

On the base of these definitions, Theorem 1 can be interpreted as follows. If $M \in \mathrm{Ccl}(\mathbb{R}^d)$, then pursuit from a point z_0 can be completed at the time τ in the upper game if and only if the pursuer in the lower game can transfer the phase point from the starting state z_0 into any neighborhood of the terminal set at the time τ (see [2, 14, 18, 23–25]).

Theorem 2 admits similarly interpretation.

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