

## The Laplace transform of Dirichlet $L$ -functions

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**Abstract.** In the paper, a formula for the Laplace transform of the square of Dirichlet  $L$ -functions on the critical line is obtained.

**Keywords:** Dirichlet  $L$ -function, functional equation, Laplace transform, Riemann zeta-function.

### 1 Introduction

Let  $s = \sigma + it$  be a complex variable. The Laplace transform  $\mathfrak{L}(s)$  of the function  $f(x)$  is defined by

$$\mathfrak{L}(s) = \int_0^{\infty} f(x)e^{-sx} dx$$

provided the integral exists for  $\sigma > \sigma_0$  with some  $\sigma_0$ . The function  $\mathfrak{L}(s)$  can be applied for the investigation of the asymptotics for the mean value

$$M_T(f) \stackrel{\text{def}}{=} \int_0^T f(x) dx.$$

Suppose that  $f(x) \geq 0$  for all  $x \in [0, \infty)$  and, for a given  $m > 0$ ,

$$\mathfrak{L}(\delta) \sim \frac{1}{\delta} \log^m \frac{1}{\delta}$$

as  $\delta \rightarrow 0$ . Then [1, Chap. VII],

$$M_T(f) \sim T \log^m T$$

as  $T \rightarrow \infty$ . The latter example shows that Laplace transforms can be used in the moment problem of zeta and  $L$ -functions. Let  $\chi$  be a Dirichlet character mod  $q$ , and let  $L(s, \chi)$  denote the corresponding  $L$ -function which is defined, for  $\sigma > 1$ , by

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

Denote by  $\chi_0$  the principal character mod  $q$ . Then it is well known that the function  $L(s, \chi_0)$  is analytically continued to the whole complex plane, except for a simple pole at  $s = 1$  with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right),$$

where  $p$  denotes a prime number. If  $\chi \neq \chi_0$ , then  $L(s, \chi)$  is analytically continued to an entire function.

In the theory of Dirichlet  $L$ -functions, usually the moments

$$\sum_{\chi=\chi \pmod{q}} \int_0^T |L(\sigma + it, \chi)|^{2k} dt, \quad k \geq 0, \quad \sigma \geq \frac{1}{2},$$

are considered, see, for example, [2]. This corresponds to the Laplace transform

$$\sum_{\chi=\chi \pmod{q}} \int_0^{\infty} |L(\sigma + it, \chi)|^{2k} e^{-sx} dx.$$

The aim of this note is an explicit formula for the individual Laplace transform  $\mathfrak{L}(s, \chi)$  of  $|L(1/2 + ix, \chi)|^2$ , i.e., for the function

$$\mathfrak{L}(s, \chi) = \int_0^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 e^{-sx} dx.$$

In forthcoming papers, this formula will be applied for the investigation of the Mellin transform of  $|L(1/2 + ix, \chi)|^2$ , for the case of  $|\zeta(\rho + ix)|^2$  with  $1/2 < \rho < 1$ , see [3]. In a series of papers, A. Ivič, M. Jutila and Y. Motohashi developed the method of Mellin transforms for the moment problem in the theory of zeta-functions, see, for example, [4]. These remarks show the motivation of the paper. For the statement of the formula, we need some notation.

Let  $G(\chi)$  denote the Gauss sum, i.e.,

$$G(\chi) = \sum_{l=1}^q \chi(l) e^{2\pi i l/q}.$$

Moreover, let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if } a = 0, \\ \epsilon_1(\chi) & \text{if } a = 1, \end{cases}$$

where

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \quad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}}.$$

As usual, denote by  $d(m)$  the divisor function

$$d(m) = \sum_{d|m} 1.$$

**Theorem 1.** *Suppose that  $\chi$  is a primitive character mod  $q$ ,  $q > 1$ . Then*

$$\mathfrak{L}(s, \chi) = \frac{2\pi i^a e^{-is/2}}{\sqrt{q}E(\chi)} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi im}{q}e^{-is}\right\} + \lambda(s, \chi),$$

where the function  $\lambda(s, \chi)$  is analytic in the strip  $\{s \in \mathbb{C}: |\sigma| < \pi\}$ , and, for  $|\sigma| \leq \theta$ ,  $0 < \theta < \pi$ , the estimate

$$\lambda(s, \chi) = O((1 + |s|)^{-1})$$

is valid.

Let  $\mu(m)$  be the Möbius function, and let  $\gamma_0$  denote the Euler constant.

**Theorem 2.** *Suppose that  $\chi_0$  is the principal character mod  $q$ ,  $q > 1$ . Then*

$$\begin{aligned} &\mathfrak{L}(s, \chi_0) \\ &= ie^{is/2} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i + \sum_{p|q} \frac{\log p}{p-1} + \sum_{m|q} \mu(m) \log m\right) \\ &\quad + 2\pi e^{-is/2} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \exp\left\{-\frac{2\pi i kn}{m}e^{-is}\right\} + \lambda(s, \chi_0), \end{aligned}$$

where the function  $\lambda(s, \chi_0)$  has the same properties as  $\lambda(s, \chi)$  in Theorem 1.

The case  $q = 1$  corresponds the Riemann zeta-function. Let  $\mathfrak{L}_\zeta(s)$  be the Laplace transform of  $|\zeta(1/2 + ix)|^2$ .

**Corollary 1.** *We have*

$$\begin{aligned} \mathfrak{L}_\zeta(s) &= ie^{is/2} \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i\right) \\ &\quad + 2\pi e^{-is/2} \sum_{m=1}^{\infty} d(m) \exp\{-2\pi ime^{-is}\} + \lambda(s), \end{aligned}$$

where function  $\lambda(s)$  has the same properties as  $\lambda(s, \chi)$ .

## 2 Lemmas

First we remind the functional equation for  $L(s, \chi)$ . We preserve the notation used in introduction. Let

$$l(s, \chi) = \left(\frac{\pi}{q}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi).$$

**Lemma 1.** *Suppose that  $\chi$  is a primitive character mod  $q$ ,  $q > 1$ . Then*

$$l(s, \chi) = E(\chi)l(1-s, \bar{\chi}).$$

Proof of the lemma is given, for example, in [5].

**Lemma 2.** *For  $\sigma > 1$ ,*

$$L^2(s, \chi) = \sum_{m=1}^{\infty} \frac{d(m)\chi(m)}{m^s}.$$

*Proof.* For  $\sigma > 1$ , we have that

$$L^2(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

where

$$a(m) = \sum_{d|m} \chi(d)\chi\left(\frac{m}{d}\right) = \chi(m)d(m). \quad \square$$

Denote by  $\varphi(q)$  the Euler totient function.

**Lemma 3.** *Let  $\sigma_0$  be arbitrary real number. Then, for  $\sigma \geq \sigma_0$ ,*

$$L(s, \chi) = \frac{E_0\varphi(q)}{q(s-1)} + O_{\sigma_0}(q^c(|t|+2)^c),$$

where  $c = c(\sigma_0) > 0$ , and

$$E_0 = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

The lemma is Theorem 7.3.2 from [6].

**Lemma 4.** *The estimate*

$$L'\left(\frac{1}{2} + it, \chi\right) = O(q^{c_1}(|t|+2)^{c_1})$$

*holds with some  $c_1 > 0$ .*

*Proof.* The lemma is a corollary of Lemma 3 and the integral Cauchy formula. □

**Lemma 5.** *Suppose that  $a > 0$ . Then*

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s} ds = e^{-b}.$$

The lemma is the well-known Mellin formula, see, for example, [7].

### 3 Proof of Theorems

Define the function  $\lambda(s, \chi)$  by

$$\begin{aligned} \lambda(s, \chi) &= \int_0^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 e^{-sx} dx \\ &\quad - \frac{e^{-is/2}}{2i^{1-a}} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z, \chi)L(1-z, \bar{\chi})e^{-iz(\pi/2-s)}}{\cos(\frac{\pi a}{2} - \frac{\pi z}{2})} dz. \end{aligned} \quad (1)$$

First suppose that  $a = 0$ . Then the formulae

$$\cos s = \frac{e^{is} + e^{-is}}{2} \quad \text{and} \quad \overline{L(s, \chi)} = L(\bar{s}, \bar{\chi})$$

give

$$\begin{aligned} &\frac{e^{-is/2}}{2i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z, \chi)L(1-z, \bar{\chi})e^{-iz(\pi/2-s)}}{\cos \frac{\pi z}{2}} dz \\ &= e^{-is/2} \int_{-\infty}^{\infty} \frac{L(\frac{1}{2} + ix, \chi)L(\frac{1}{2} - ix, \bar{\chi}) \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2} + \frac{is}{2} - xs\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &= \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \chi)|^2 \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2} - xs\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &\quad + \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \bar{\chi})|^2 \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2} + xs\}}{\exp\{\frac{\pi i}{4} + \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}} dx. \end{aligned} \quad (2)$$

Since

$$1 - \frac{\exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} = \frac{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}},$$

we find from (1) and (2) that

$$\begin{aligned} \lambda(s, \chi) &= \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \chi)|^2 e^{-xs} \exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &\quad - \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \bar{\chi})|^2 e^{xs} \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} + \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}} dx. \end{aligned} \quad (3)$$

Now let  $a = 1$ . In this case, in view of the formula

$$\sin s = \frac{e^{is} - e^{-is}}{2i},$$

we find that

$$\begin{aligned} &\frac{e^{-is/2}}{2} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z, \chi)L(1-z, \bar{\chi})e^{-iz(\pi/2-s)}}{\sin \frac{\pi z}{2}} dz \\ &= -e^{-is/2} \int_{-\infty}^{\infty} \frac{L(\frac{1}{2} + ix, \chi)L(\frac{1}{2} - ix, \bar{\chi}) \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2} + \frac{is}{2} - xs\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &= - \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \chi)|^2 \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2} - xs\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &\quad - \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \bar{\chi})|^2 \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2} + xs\}}{\exp\{\frac{\pi i}{4} + \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}} dx. \end{aligned} \quad (4)$$

Clearly, we have

$$1 + \frac{\exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} = \frac{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}},$$

thus, from (1) and (4), it follows that

$$\begin{aligned} \lambda(s, \chi) &= \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \chi)|^2 e^{-xs} \exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} + \frac{\pi x}{2}\}} dx \\ &\quad + \int_0^{\infty} \frac{|L(\frac{1}{2} + ix, \bar{\chi})|^2 e^{xs} \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4} + \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{4} - \frac{\pi x}{2}\}} dx. \end{aligned} \quad (5)$$

By estimates for  $L(1/2 + ix, \chi)$  of Lemma 3, we have that the integrals in (3) and (5) converge uniformly in compact subsets of the strip  $\{s \in \mathbb{C}: |\sigma| < \pi\}$ , thus, the function  $\lambda(s, \chi)$  is analytic in that strip.

It remains to estimate the function  $\lambda(s, \chi)$ . Suppose that  $|\sigma| \leq \theta$ , where  $0 < \theta < \pi$ . First let  $|s|$  is small. Then the integrals in (3) and (5) are bounded by a constant. If  $|s|$  is large, then integrating by parts with respect to  $e^{\pm sx}$  and using the estimate

$$\begin{aligned} & \left( \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 \right)' \\ &= \left( L\left(\frac{1}{2} + ix, \chi\right) L\left(\frac{1}{2} - ix, \bar{\chi}\right) \right)' \\ &= iL'\left(\frac{1}{2} + ix, \chi\right) L\left(\frac{1}{2} - ix, \bar{\chi}\right) - iL\left(\frac{1}{2} + ix, \chi\right) L'\left(\frac{1}{2} - ix, \bar{\chi}\right) \\ &= O(q(|x| + 2))^{c_2} \end{aligned}$$

with some  $c_2 > 0$ , which follows from Lemmas 3 and 4, we obtain that

$$\lambda(s, \chi) = O(|s|^{-1}).$$

So, in all cases, we have that, for  $|\sigma| \leq \theta$ ,  $0 < \theta < \pi$ ,

$$\lambda(s, \chi) = O((1 + |s|)^{-1}).$$

Equality (1) shows that

$$\mathfrak{L}(s, \chi) = \frac{e^{-is/2}}{2i^{1-a}} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z, \chi)L(1-z, \bar{\chi})e^{-iz(\pi/2-s)}}{\cos(\frac{\pi a}{2} - \frac{\pi z}{2})} dz + \lambda(s, \chi), \tag{6}$$

where the function  $\lambda(s, \chi)$  is analytic in the strip  $\{s \in \mathbb{C} : |\sigma| < \pi\}$ , and, for  $|\sigma| \leq \theta$ ,  $0 < \theta < \pi$ , the estimate

$$\lambda(s, \chi) = O((1 + |s|)^{-1})$$

is valid.

It remains to calculate the integral in (6). Using Lemma 1, we find

$$L(1-z, \bar{\chi}) = E^{-1}(\chi) \left(\frac{\pi}{q}\right)^{-z+1/2} \frac{\Gamma(\frac{z}{2} + \frac{a}{2})}{\Gamma(\frac{1}{2} + \frac{a}{2} - \frac{z}{2})} L(z, \chi). \tag{7}$$

Taking into account the formulas, see [8],

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z},$$

and

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2\sqrt{\pi}2^{-2z}\Gamma(2z),$$

we obtain that

$$\begin{aligned} \frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{1}{2} - \frac{z}{2})} &= \frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2} + \frac{z}{2})}{\Gamma(\frac{1}{2} - \frac{z}{2})\Gamma(\frac{1}{2} + \frac{z}{2})} = \frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2} + \frac{z}{2}) \cos \frac{\pi z}{2}}{\pi} \\ &= \frac{2\sqrt{\pi}2^{-z}\Gamma(z) \cos \frac{\pi z}{2}}{\pi} = 2^{1-z}\pi^{-1/2}\Gamma(z) \cos \frac{\pi z}{2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\Gamma(\frac{1}{2} + \frac{z}{2})}{\Gamma(1 - \frac{z}{2})} &= \frac{\Gamma(\frac{1}{2} + \frac{z}{2})\Gamma(\frac{1}{2} - \frac{z}{2})}{\Gamma(1 - \frac{z}{2})\Gamma(\frac{1}{2} - \frac{z}{2})} = \frac{\pi}{2\sqrt{\pi}2^{-1+z}\Gamma(1-z) \cos \frac{\pi z}{2}} \\ &= \frac{\pi^{-1/2}\Gamma(z) \sin \pi z}{2^z \cos \frac{\pi z}{2}} = 2^{1-z}\pi^{-1/2}\Gamma(z) \sin \frac{\pi z}{2}. \end{aligned}$$

Since  $a = 0$  or  $a = 1$ , hence we have that

$$\frac{\Gamma(\frac{z}{2} + \frac{a}{2})}{\Gamma(\frac{1}{2} + \frac{a}{2} - \frac{z}{2})} = 2^{1-z}\pi^{-1/2}\Gamma(z) \cos\left(\frac{\pi a}{2} - \frac{\pi z}{2}\right),$$

and, in view of (7), this gives

$$L(1-z, \bar{\chi}) = E^{-1}(\chi)2^{1-z}\pi^{-z}q^{z-1/2}\Gamma(z) \cos\left(\frac{\pi a}{2} - \frac{\pi z}{2}\right)L(z, \chi).$$

Hence, we find that

$$\begin{aligned} &\frac{e^{-is/2}}{2i^{1-a}} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z, \chi)L(1-z, \bar{\chi})e^{-iz(\pi/2-s)}}{\cos(\frac{\pi a}{2} - \frac{\pi z}{2})} dz \\ &= \frac{e^{-is/2}}{i^{1-a}\sqrt{q}E(\chi)} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(z)L^2(z, \chi) \left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz. \end{aligned} \quad (8)$$

Next we prove Theorem 1.

*Proof.* Let  $\chi$  be a non-principal character. Then the integrand in (8) is a regular function in the strip  $\{z \in \mathbb{C}: 1/2 < \Re z < 2\}$ . Therefore,

$$\begin{aligned} &\frac{e^{-is/2}}{i^{1-a}\sqrt{q}E(\chi)} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(z)L^2(z, \chi) \left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz \\ &= \frac{e^{-is/2}}{i^{1-a}\sqrt{q}E(\chi)} \int_{2-i\infty}^{2+i\infty} \Gamma(z)L^2(z, \chi) \left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz. \end{aligned} \quad (9)$$



Moreover, in view of Lemmas 2 and 5,

$$\begin{aligned} \int_{2-i\infty}^{2+i\infty} \Gamma(z)L^2(z, \chi) \left(\frac{2\pi i}{q} e^{-is}\right)^{-z} dz &= \sum_{m=1}^{\infty} d(m)\chi(m) \int_{2-i\infty}^{2+i\infty} \Gamma(z) \left(\frac{2\pi im}{q} e^{-is}\right)^{-z} dz \\ &= 2\pi i \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi im}{q} e^{-is}\right\}. \end{aligned}$$

This, (8), (9) and (6) prove Theorem 1. □

Next we prove Theorem 2.

*Proof.* Let  $\chi_0$  be the principal character mod  $q$ . Then we have that

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

where  $\zeta(s)$  is the Riemann zeta-function. Using the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

we find that

$$\begin{aligned} L(1-z, \chi_0) &= \zeta(1-z) \prod_{p|q} \left(1 - \frac{1}{p^{1-z}}\right) \\ &= \zeta(z) 2^{1-z} \pi^{-z} \Gamma(z) \cos \frac{\pi z}{2} \prod_{p|q} \left(1 - \frac{1}{p^{1-z}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} L(z, \chi_0)L(1-z, \chi_0)e^{-iz(\pi/2-s)} &= \zeta^2(z) 2^{1-z} \pi^{-z} \Gamma(z) \cos \frac{\pi z}{2} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \prod_{p|q} \left(1 - \frac{1}{p^{1-z}}\right), \end{aligned}$$

and, similarly to the case of non-principal character, we obtain

$$\begin{aligned} \mathfrak{L}(s, \chi_0) &= \frac{e^{-is/2}}{i} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \prod_{p|q} \left(1 - \frac{1}{p^{1-z}}\right) dz \\ &\quad + \lambda(s, \chi_0). \end{aligned} \tag{10}$$

The product

$$\prod_q(z) = \prod_{p|q} \left(1 - \frac{1}{p^{1-z}}\right)$$

has zeroes lying on the line  $\sigma = 1$ . Therefore, we write  $\prod_q(z)$  as a sum. Using the Möbius function  $\mu(m)$ , we obtain that

$$\prod_q(z) = \sum_{p|q} \mu(m) m^{z-1}.$$

The function

$$\Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1}$$

has the pole of order 2 at  $z = 1$ . Moreover, since

$$\zeta(z) = \frac{1}{z-1} + \gamma_0 + \gamma_1(z-1) + \dots,$$

and  $\Gamma(1) = 1$ ,  $\Gamma'(1) = -\gamma_0$ , we find that

$$\begin{aligned} & \operatorname{Res}_{z=1} \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \\ &= \lim_{z \rightarrow 1} ((z-1)^2 \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z})' = -\frac{e^{is}}{2\pi i} \left( \gamma_0 - \log 2\pi - \left( \frac{\pi}{2} - s \right) i \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \operatorname{Res}_{z=1} \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\ &= \lim_{z \rightarrow 1} \left( (z-1)^2 \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \right)' \\ &= \lim_{z \rightarrow 1} \left( (z-1)^2 \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \right)' \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\ &\quad + \lim_{z \rightarrow 1} (z-1)^2 \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \sum_{p|q} \frac{\frac{1}{p^z} \log p}{1 - \frac{1}{p^z}} m^{z-1} \\ &\quad + \lim_{z \rightarrow 1} (z-1)^2 \Gamma(z)\zeta^2(z)(2\pi i e^{-is})^{-z} \log m \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\ &= \frac{e^{is}}{2\pi i} \left( \gamma_0 - \log 2\pi - \left( \frac{\pi}{2} - s \right) i \right) \prod_{p|q} \left(1 - \frac{1}{p}\right) \\ &\quad + (2\pi i e^{-is})^{-1} \prod_{p|q} \left(1 - \frac{1}{p}\right) \sum_{p|q} \frac{\log p}{p-1} + (2\pi i e^{-is})^{-1} \log m \prod_{p|q} \left(1 - \frac{1}{p}\right) \\ &= \frac{e^{is}}{2\pi i} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left( \gamma_0 - \log 2\pi - \left( \frac{\pi}{2} - s \right) i + \sum_{p|q} \frac{\log p}{p-1} + \log m \right). \end{aligned}$$

Thus, in view of (10),

$$\begin{aligned} \mathfrak{L}(s, \chi_0) &= ie^{is/2} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i + \sum_{p|q} \frac{\log p}{p-1} + \sum_{m|q} \mu(m) \log m\right) \\ &\quad - ie^{-is/2} \int_{2-i\infty}^{2+i\infty} \Gamma(z) \zeta^2(z) (2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \left(1 - \frac{1}{p^{1-z}}\right) dz \\ &\quad + \lambda(s, \chi_0). \end{aligned} \tag{11}$$

We have that

$$\prod_{p|q} \left(1 - \frac{1}{p^z}\right) \left(1 - \frac{1}{p^{1-z}}\right) = \sum_{n|q} \sum_{m|q} \frac{\mu(m)}{m} \mu(n) \left(\frac{m}{n}\right)^z.$$

Moreover, for  $\sigma > 1$ ,

$$\zeta^2(s) = \sum_{k=1}^{\infty} \frac{d(k)}{k^s}.$$

Therefore, the application of Lemma 5, leads to

$$\begin{aligned} &\int_{2-i\infty}^{2+i\infty} \Gamma(z) \zeta^2(z) (2\pi i e^{-is})^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \left(1 - \frac{1}{p^{1-z}}\right) dz \\ &= 2\pi i \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \exp\left\{-\frac{2\pi i k n}{m} e^{-is}\right\}. \end{aligned}$$

This together with (11) proves the theorem.  $\square$

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