On a third order initial boundary value problem in a plane domain

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Received: 13 January 2012 / **Revised:** 26 May 2012 / **Published online:** 5 July 2012

Abstract. Third order initial boundary value problem is studied in a bounded plane domain σ with C^4 smooth boundary $\partial\sigma$. The existence and uniqueness of the solution is proved using Galerkin approximations and a priory estimates. The problem under consideration appear as an auxiliary problem by studying a second grade fluid motion in an infinite three-dimensional pipe with non-circular cross-section.

Keywords: initial boundary value problem, third order evolutionary equation, second grade fluid flow, infinite pipe flow.

1 Introduction

In the paper we study the following initial boundary value problem in a bounded two–dimensional domain σ:

$$\begin{aligned} &\partial_t(v-\alpha\Delta v)-\nu\Delta v+(\mathbf{U}'\cdot\nabla)(v-\alpha\Delta v)=f,\\ &v|_{\partial\sigma}=0, \quad v(x',0)=v_0(x'), \end{aligned} \tag{1}$$

where ∂_t denotes $\partial/\partial t$. Problem (1) appears as auxiliary by studying the incompressible non-Newtonian second grade fluid flow in a three-dimensional pipe $\Pi=\{x=(x',x_3)\in\mathbb{R}^3\colon\, x'\in\sigma,\, x_3\in\mathbb{R}\}$ with cross-section σ. The corresponding equations have the form

$$\begin{aligned} &\partial_t(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla p=\mathbf{f},\\ &\operatorname{div}\mathbf{u}=0,\\ &\mathbf{u}|_{\partial\Pi}=0, \quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\\ &\int_\sigma u_3(x',x_3,t)\,\mathrm{d}x'=F(t). \end{aligned} \tag{2}$$

Here $\mathbf{u}$ is the velocity of the fluid, p is the pressure, $\mathbf{f}$ is the external force, $\mathbf{u}_0$ is the initial velocity, α is the normal stress module, ν is the kinematic viscosity (α and ν are positive constants) and the last condition in (2) prescribes the flux $F(t)$ over the cross-section σ.

In two-dimensional channels and three-dimensional pipes with rotational symmetry problem (2) has the unidirectional Poiseuille type solutions [1]. However, in a three-dimensional pipe with non-circular section σ secondary flows appear and the velocity field has all three components. Let us suppose that the data $\mathbf{f}$ and $\mathbf{u}_0$ do not depend on the coordinate x_3 and have the form

$$\begin{aligned}\mathbf{u}_0(x) &= \big(u_{01}(x'), u_{02}(x'), u_{03}(x')\big),\\ \mathbf{f}(x,t) &= \big(f_1(x',t), f_2(x',t), f_3(x',t)\big).\end{aligned}$$

Moreover, suppose that the necessary compatibility condition

$$\int\limits_{\sigma} v_0(x')\,\mathrm{d}x' = F(0) \tag{3}$$

holds. Then we can look for the solution $(\mathbf{u}(x,t), p(x,t))$ of system (2) in the form

$$\begin{aligned}\mathbf{u}(x,t) &= \big(U_1(x',t), U_2(x',t), U_3(x',t)\big),\\ p(x,t) &= \widetilde{p}(x',t) - q(t)x_3 + p_0(t),\end{aligned} \tag{4}$$

where $p_0(t)$ is an arbitrary function. Substituting (4) into (2) we get the following problem on $\sigma^T = \sigma \times (0,T)$:

$$\begin{aligned}&\partial_t(\mathbf{U}' - \alpha\Delta\mathbf{U}') - \nu\Delta\mathbf{U}' + \operatorname{curl}(\mathbf{U}' - \alpha\Delta\mathbf{U}') \times \mathbf{U}'\\ &\quad - U_3\big(\nabla(U_3 - \alpha\Delta U_3)\big) + \nabla\widetilde{p} = \mathbf{f}',\\ &\partial_t(U_3 - \alpha\Delta U_3) - \nu\Delta U_3 + (\mathbf{U}'\cdot\nabla)(U_3 - \alpha\Delta U_3) = f + q(t),\\ &\operatorname{div}_{x'}\mathbf{U}' = 0,\\ &\mathbf{U}'|_{\partial\sigma} = 0, \quad U_3|_{\partial\sigma} = 0, \quad \mathbf{U}'(x',0) = \mathbf{u}_0'(x'), \quad U_3(x',0) = v_0(x'),\\ &\int\limits_{\sigma} U_3(x',t)\,\mathrm{d}x' = F(t),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\mathbf{f}'(x',t) &= \big(f_1(x',t), f_2(x',t)\big), \quad \mathbf{u}_0'(x) = \big(u_{01}(x'), u_{02}(x')\big),\\ f(x',t) &= f_3(x',t), \quad v_0(x') = u_{03}(x'), \quad \mathbf{U}'(x',t) = \big(U_1(x',t), U_2(x',t)\big).\end{aligned}$$

Notice that in (5) functions $\mathbf{u}_0(x')$, $\mathbf{f}(x',t)$ and $F(t)$ are given, while $\mathbf{U}(x',t)$, $\widetilde{p}(x',t)$ and $q(t)$ are unknown and have to be found. For small data problem (5) can be solved by iterations dividing it into two problems:

$$\begin{aligned}&\partial_t(\mathbf{U}' - \alpha\Delta\mathbf{U}') - \nu\Delta\mathbf{U}' + \operatorname{curl}(\mathbf{U}' - \alpha\Delta\mathbf{U}') \times \mathbf{U}' + \nabla\widetilde{p}\\ &\quad = U_3\big(\nabla(U_3 - \alpha\Delta U_3)\big) + \mathbf{f}',\\ &\operatorname{div}_{x'}\mathbf{U}' = 0,\\ &\mathbf{U}'|_{\partial\sigma} = 0, \quad \mathbf{U}'(x',0) = \mathbf{u}_0'(x'),\end{aligned} \tag{6}$$

with given U_3 and

$$\begin{aligned} &\partial_t(U_3 - \alpha\Delta U_3) - \nu\Delta U_3 + (\mathbf{U}' \cdot \nabla)(U_3 - \alpha\Delta U_3) = f + q(t), \\ &U_3|_{\partial\sigma} = 0, \quad U_3(x', 0) = v_0(x'), \\ &\int\limits_\sigma U_3(x', t)\,\mathrm{d}x' = F(t), \end{aligned} \tag{7}$$

with given $\mathbf{U}'$. Problem (6) with the given right-hand side is the standard initial boundary value problem describing the motion of the second grade fluid in a bounded plane domain. Such two- and three-dimensional problems has been studied by several authors (e.g. [2–17], etc.). Problem (7) is the inverse problem (the function q in the right-hand side is unknown). In the case $\alpha = 0$ system (2) coincide with the Navier–Stokes system. The corresponding inverse problem was studied in [12]. If $\mathbf{U}'(x', t) = 0$, we have equation describing unidirectional flow of the second grade fluid. The existence of a unique solution in this case is proved in [1].

By studying the inverse problem (7) it is convenient to reduce it to the case of zero f and the homogeneous initial condition. This could be done by subtracting the solution v of problem (1) which we study here. We prove the existence of the solution to (1) using the Galerkin method and choosing the special sufficiently smooth basis. Constructing this basis and getting a priori estimates of Galerkin approximations we follow ideas proposed by D. Cioranescu and E. Quazar (see [2–4]).

The inverse problem (7) and complete analysis of problem (5) will be done in the forthcoming paper.

2 Main notations and a special basis

Let $\sigma \subset \mathbb{R}^2$ be a bounded domain. In the paper we use the standard notations for Sobolev spaces [18]. Denote by $W_2^{-1}(\sigma)$ the adjoint space to $\mathring{W}_2^1(\sigma)$. The notation $\mathring{V}(\sigma)$ (or $\mathring{V}(\sigma^T)$, $\sigma^T = \sigma \times (0, T)$) means subspace of the space $V(\sigma)$ (or of $V(\sigma^T)$) consisting of functions equal to zero (in the sense of traces) on $\partial\sigma$. Vector-valued functions are denoted by bold letters; spaces of scalar and vector-valued functions are not distinguished in notations. The vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \dots, n$, and $\|\mathbf{u}\|_V = \sum_{i=1}^n \|u_i\|_V$. Below we will need the following spaces:

$$\mathring{X}(\sigma) = \bigl\{v \in \mathring{W}_2^1(\sigma)\colon\ \nabla(v - \alpha\Delta v) \in L_2(\sigma)\bigr\},$$

with the norm $\|u\|^2_{\mathring{X}(\sigma)} = \|u\|^2_{W_2^1(\sigma)} + \|\nabla(u - \alpha\Delta u)\|^2_{L_2(\sigma)}$;

$$\mathring{\mathcal{W}}\bigl(\sigma^T\bigr) = \bigl\{v\colon D_x^\alpha v \in L_2\bigl(\sigma^T\bigr),\ |\alpha| \le 3;\ \partial_t v \in L_2\bigl(\sigma^T\bigr);\ \nabla\partial_t v \in L_2\bigl(\sigma^T\bigr);\ v|_{\partial\sigma} = 0\bigr\},$$

with the norm $\|v\|^2_{\mathring{\mathcal{W}}(\sigma^T)} = \sum_{|\alpha|\le 3} \|D_x^\alpha v\|^2_{L_2(\sigma^T)} + \|\partial_t v\|^2_{L_2(\sigma^T)} + \|\partial_t \nabla v\|^2_{L_2(\sigma^T)}$; and

$$\mathring{\mathcal{V}}\bigl(\sigma^T\bigr) = \bigl\{\mathbf{v} \in \mathring{\mathcal{W}}\bigl(\sigma^T\bigr)\colon\ \operatorname{div}\mathbf{v} = 0\bigr\}.$$

Note that in the case $\partial\sigma \in C^3$ the space $\mathring{X}(\sigma)$ is equivalent to $W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$. Indeed, if $u \in W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$, then, obviously, $u \in \mathring{X}(\sigma)$ and $\|u\|_{\mathring{X}(\sigma)} \le c\|u\|_{W_2^3(\sigma)}$. On the other hand, according to the Nečas inequality (see [15])

$$\|\Delta u\|_{L_2(\sigma)} \le c\big(\|\Delta u\|_{W_2^{-1}(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\big) \le c\big(\|\nabla u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\big).$$

Therefore, considering u as a solution to the Poisson equation

$$\begin{aligned} -\Delta u &= -\Delta u, \\ u|_{\partial\sigma} &= 0, \end{aligned}$$

we get the estimate

$$\begin{aligned} \|u\|_{W_2^3(\sigma)} &\le c\big(\|\Delta u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\big) \\ &\le c\big(\|\nabla u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\big) \le c\|u\|_{\mathring{X}(\sigma)}. \end{aligned} \tag{8}$$

Let us construct a special basis in the space $\mathring{X}(\sigma)$. Let $\{\lambda_k\}$ and $\{w_k(x')\} \subset \mathring{X}(\sigma)$ be eigenvalues and eigenfunctions[1] of the following problem:

$$\begin{aligned} &\int\limits_\sigma \nabla\big(w_k(x') - \alpha\Delta w_k(x')\big) \cdot \nabla\big(\rho(x') - \alpha\Delta\rho(x')\big)\,\mathrm{d}x' \\ &\quad = (\lambda_k - 1)\int\limits_\sigma \big(w_k\rho(x') + \alpha\nabla w_k(x')\cdot\nabla\rho(x')\big)\,\mathrm{d}x' \quad \forall\rho \in \mathring{X}(\sigma). \end{aligned} \tag{9}$$

Theorem 1. *Let $\sigma \subset \mathbb{R}^2$ be a bounded simply connected domain with the boundary $\partial\sigma \in C^4$. Then*

- (9) *defines a countable set of eigenvalues $\lambda_k > 1$, $k = 1, 2, \dots$; the corresponding eigenfunctions w_k constitute bases in $\mathring{X}(\sigma)$, $\mathring{W}_2^1(\sigma)$ and $L_2(\sigma)$.*
- *The eigenfunctions w_k can be orthonormalized*

$$\int\limits_\sigma \big(w_k(x')w_l(x') + \alpha\nabla w_k\cdot\nabla w_l\big)\,\mathrm{d}x' = \begin{cases} 0, & k \ne l, \\ 1, & k = l. \end{cases} \tag{10}$$

  Then

$$\int\limits_\sigma \nabla\big(w_k(x') - \alpha\Delta w_k\big)\cdot\nabla(w_l - \alpha\Delta w_l)\,\mathrm{d}x' = \begin{cases} 0, & k \ne l, \\ \lambda_k - 1, & k = l. \end{cases} \tag{11}$$

- *The eigenfunctions of* (9) *belongs to $W_2^4(\sigma)$.*

Proof. The proof of the first property is standard for elliptic equations (e.g. [7]). The second property follows from the identity (9). To prove the third property, we denote $\rho - \alpha\Delta\rho = \varphi$ and rewrite integral identity (9) in the form

$$\int\limits_\sigma \nabla\big(w_k(x') - \alpha\Delta w_k(x')\big)\cdot\nabla\varphi\,\mathrm{d}x' = (\lambda_k - 1)\int\limits_\sigma w_k\varphi\,\mathrm{d}x'.$$

[1] i.e. $w_k(x')$ are nontrivial solutions of (9)

Obviously, if $\rho \in \overset{\circ}{X}(\sigma)$, then $\varphi \in W_2^1(\sigma)$, and, oppositely, for any $\varphi \in W_2^1(\sigma)$ there exists a unique $\rho \in \overset{\circ}{X}(\sigma)$ such that $\rho - \alpha\Delta\rho = \varphi$. Therefore, $w_k - \alpha\Delta w_k \in W_2^1(\sigma)$ can be interpreted as a weak solution to the following Neumann problem:

$$-\Delta(w_k - \alpha\Delta w_k) = (\lambda_k - 1)w_k,$$
$$\frac{\partial(w_k - \alpha\Delta w_k)}{\partial n}\bigg|_{\partial\sigma} = 0.$$

Since $w_k \in L_2(\sigma)$, we conclude that $w_k - \alpha\Delta w_k \in W_2^2(\sigma)$. Consider now $w_k - \alpha\Delta w_k$ as a solution of the Dirichlet problem:

$$w_k - \alpha\Delta w_k = w_k - \alpha\Delta w_k,$$
$$w_k|_{\partial\sigma} = 0.$$

Since $\sigma \in C^4$, we conclude that $w_k \in W_2^4(\sigma) \cap \overset{\circ}{W}{}_2^1(\sigma)$. □

In the paper we suppose that the function $\mathbf{U}' \in \overset{\circ}{\mathcal{V}}(\sigma^T)$ is given and satisfies the following condition:

$$\sup_{t\in[0,T]} \|\mathbf{U}'\|_{\overset{\circ}{X}(\sigma)} + \|\mathbf{U}'\|_{\overset{\circ}{\mathcal{W}}(\sigma^T)} \le \delta_0, \tag{12}$$

where δ_0 is a sufficiently small constant.

3 Construction of an approximate solution

The function $v \in \overset{\circ}{\mathcal{W}}(\sigma^T)$ is called a weak solution of problem (1) if it satisfies for all $t \in [0,T]$ the integral identity

$$\int_0^t \int_\sigma (\partial_\tau v\eta + \alpha\partial_\tau\nabla v \cdot \nabla\eta)\,\mathrm{d}x'\mathrm{d}\tau + \nu\int_0^t\int_\sigma \nabla v\cdot\nabla\eta\,\mathrm{d}x'\,\mathrm{d}\tau$$
$$= \int_0^t\int_\sigma f\eta\,\mathrm{d}x'\,\mathrm{d}\tau + \int_0^t\int_\sigma (\mathbf{U}'\cdot\nabla)\eta(v - \alpha\Delta v)\,\mathrm{d}x'\mathrm{d}\tau \quad \forall\eta\in \overset{\circ}{W}{}_2^{1,0}(\sigma^T) \tag{13}$$

and the initial condition $v(x',0) = v_0(x')$.

Let $f \in L_2(\sigma^T)$, $v_0 \in \overset{\circ}{W}{}_2^1(\sigma)$. Then we can express them by the Fourier series

$$f(x',t) = \sum_{k=1}^\infty f_k(t)w_k(x'), \quad v_0(x') = \sum_{k=1}^\infty a_k w_k(x'),$$

where $f_k(t) = \int_\sigma f_3(x',t)w_k(x')\,\mathrm{d}x'$, $a_k = \int_\sigma v_0(x')w_k(x')\,\mathrm{d}x'$.

We look for the approximate solutions $v^{(N)}(x',t)$ in the form

$$v^{(N)}(x',t) = \sum_{k=1}^N y_k^{(N)}(t)w_k(x'),$$

where coefficients $y_k^{(N)}(t)$ are found from the integral equalities

$$\int_\sigma \big(\partial_t v^{(N)} w_k + \alpha\nabla\partial_t v^{(N)}\cdot\nabla w_k\big)\,\mathrm{d}x' + \nu\int_\sigma \nabla v^{(N)}\cdot\nabla w_k\,\mathrm{d}x' \\ = \int_\sigma f^{(N)} w_k\,\mathrm{d}x' + \int_\sigma \mathbf{U}'\cdot\nabla w_k\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\,\mathrm{d}x', \quad k = 1,\dots,N, \tag{14}$$

and the initial condition $v^{(N)}(x',0) = v_0^{(N)}(x')$, where $f^{(N)}(x',t) = \sum_{k=1}^N f_k(t)w_k(x')$, $v_0^{(N)}(x') = \sum_{k=1}^N a_k w_k(x')$.

Since the eigenfunctions w_k are smooth ($w_k \in W_2^4(\sigma)$), the approximations $v^{(N)}(x',t)$ are solutions to the following initial boundary value problems:

$$\partial_t\big(v^{(N)} - \alpha\Delta v^{(N)}\big) - \nu\Delta v^{(N)} + (\mathbf{U}'\cdot\nabla)\big(v^{(N)} - \alpha\Delta v^{(N)}\big) = f^{(N)}, \\ v^{(N)}|_{\partial\sigma} = 0, \quad v^{(N)}(x',0) = v_0^{(N)}(x'). \tag{15}$$

Using (9), (10) we derive from (14) the Cauchy problem for the system of ordinary linear differential equations:

$$y_k^{(N)\prime}(t) + \sum_{j=1}^N\left(\frac{\nu}{\alpha} + m_{kj}(t)\right)y_k^{(N)}(t) = f_k(t), \quad k = 1,\dots,N, \\ y_k^{(N)}(0) = a_k, \quad k = 1,\dots,N,$$

where $m_{kj}(t) = -\int_\sigma((\nu/\alpha)w_k w_j - (\mathbf{U}'\cdot\nabla)w_j(w_k - \alpha\Delta w_k))\,\mathrm{d}x'$. The last system can be rewritten in the vector form:

$$\mathbf{Y}^{(N)\prime}(t) + \big(\mathfrak{J}^{(N)} + \mathfrak{A}^{(N)}(t)\big)\mathbf{Y}^{(N)}(t) = \mathbf{f}(t), \\ \mathbf{Y}^{(N)}(0) = \mathbf{a}. \tag{16}$$

Here

$$\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \vdots \\ y_N^{(N)}(t)\end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N\end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N\end{pmatrix},$$

$\mathfrak{J}^{(N)} = \operatorname{diag}(\nu/\alpha,\dots,\nu/\alpha)$ – diagonal matrix, $\mathfrak{A}^{(N)}(t)$ is $(N\times N)$ matrix with elements $m_{kj}(t)$.

Lemma 1. *Let $f \in L_2(\sigma^T)$, $v_0 \in \mathring{W}_2^1(\sigma)$. Suppose that $\mathbf{U}' \in \mathring{\mathcal{V}}(\sigma^T)$ is given and satisfies* (12)*. Then there exist a unique solution $Y^{(N)} \in W_2^1(0,T)$ of system* (16)*.*

Proof. Let us prove that the elements $m_{kj}(t)$ of the matrix $\mathfrak{A}^{(N)}(t)$ are bounded. We have

$$\big|m_{kj}(t)\big| \\ = \left|\int_\sigma -\frac{\nu}{\alpha}w_k(x')w_j(x') - (\mathbf{U}'(x',t)\cdot\nabla)w_j(x')\big(w_k(x') - \alpha\Delta w_k(x')\big)\,\mathrm{d}x'\right|$$

$$\leq \frac{\nu}{\alpha} + \left| \int\limits_{\sigma} \big(\mathbf{U}'(x',t)\cdot\nabla\big)\big(w_k(x') - \alpha\Delta w_k(x')\big) w_j(x')\,\mathrm{d}x' \right|$$
$$\leq \frac{\nu}{\alpha} + \big\|\mathbf{U}'(\cdot,t)\big\|_{L_4(\sigma)} \big\|\nabla(w_k - \alpha\Delta w_k)\big\|_{L_2(\sigma)} \|w_j\|_{L_4(\sigma)}$$
$$\leq \frac{\nu}{\alpha} + c\big\|\mathbf{U}'(\cdot,t)\big\|^{1/2}_{L_2(\sigma)} \big\|\nabla\mathbf{U}'(\cdot,t)\big\|^{1/2}_{L_2(\sigma)} \big\|\nabla(w_k - \alpha\Delta w_k)\big\|_{L_2(\sigma)} \|\nabla w_j\|_{L_2(\sigma)}$$
$$\leq \frac{\nu}{\alpha} + c\left(\frac{\lambda_k - 1}{\alpha}\right)^{1/2} \sup_{t\in[0,T]} \big\|\mathbf{U}'(\cdot,t)\big\|^{1/2}_{L_2(\sigma)} \sup_{t\in[0,T]} \big\|\nabla\mathbf{U}'(\cdot,t)\big\|^{1/2}_{L_2(\sigma)} \leq \frac{\nu}{\alpha} + c\delta_0.$$

Here we have used equalities (10), (11), the well known inequality

$$\|u\|^4_{L_4(\sigma)} \leq c\|u\|^2_{L_2(\sigma)} \|\nabla u\|^2_{L_2(\sigma)} \leq c\|\nabla u\|^4_{L_2(\sigma)}$$

which holds for any function $u \in \mathring{W}^1_2(\sigma)$ and the condition (12). Thus, all elements of the matrix $\mathfrak{A}^{(N)}(t)$ are bounded functions and, therefore, the existence of the unique solution to problem (16) follows from standard results of the theory of ordinary differential equations (e.g., [16]). □

4 A priori estimates

Lemma 2. *Let $\mathbf{U}' \in \mathring{\mathcal{V}}(\sigma^T)$ satisfies condition* (12) *with sufficiently small δ_0 (δ_0 is subject to inequalities* (27), (29), (32) *below). Suppose that $\partial\sigma \in C^4$, $f \in W^1_2(\sigma)$ and $v_0 \in W^3_2(\sigma) \cap \mathring{W}^1_2(\sigma)$. Then for the approximate solution $v^{(N)}(x',t)$ the following estimate:*

$$\sup_{t\in[0,T]} \big\|v^{(N)}\big\|_{\mathring{X}(\sigma)} + \big\|v^{(N)}\big\|_{\mathring{\mathcal{W}}(\sigma^T)} \leq c\big(\big\|f^{(N)}\big\|_{W^1_2(\sigma^T)} + \big\|v^{(N)}_0\big\|_{\mathring{X}(\sigma)}\big) \tag{17}$$

holds. Here c does not depend on N.

Proof. Multiply equalities (14) by $y^{(N)}_k(t)$ and sum by k from 1 until N:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\sigma} \big(\big|v^{(N)}\big|^2 + \alpha\big|\nabla v^{(N)}\big|^2\big)\,\mathrm{d}x' + \nu \int\limits_{\sigma} \big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'$$
$$= \int\limits_{\sigma} f^{(N)} v^{(N)}\,\mathrm{d}x' - \alpha \int\limits_{\sigma} \big(\mathbf{U}'\cdot\nabla\big)\Delta v^{(N)} v^{(N)}\,\mathrm{d}x'$$
$$\leq \frac{1}{2\varepsilon} \int\limits_{\sigma} \big|f^{(N)}\big|^2\,\mathrm{d}x' + c\varepsilon \int\limits_{\sigma} \big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'$$
$$+ \alpha \sup_{x'\in\sigma} |\mathbf{U}'| \big\|v^{(N)}\big\|_{L_2(\sigma)} \big\|\nabla\Delta v^{(N)}\big\|_{L_2(\sigma)}$$
$$\leq \frac{1}{2\varepsilon} \int\limits_{\sigma} \big|f^{(N)}\big|^2\,\mathrm{d}x' + \frac{c\delta_0^2}{\varepsilon} \big\|\nabla\Delta v^{(N)}\big\|^2_{L_2(\sigma)} + c\varepsilon \int\limits_{\sigma} \big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'.$$

Here we applied Cauchy inequality with ε and the inequality

$$\sup_{x'\in\sigma} |\mathbf{U}'| + \sup_{x'\in\sigma} |\nabla\mathbf{U}'| \\ \le c\|\mathbf{U}'\|_{W_2^3(\sigma)} \le c\big(\|\mathbf{U}'\|_{W_2^1(\sigma)} + \big\|\operatorname{curl}(\mathbf{U}' - \alpha\Delta\mathbf{U}')\big\|_{L_2(\sigma)}\big) \le c\delta_0, \tag{18}$$

which follows from the Sobolev embedding theorem, (8) and (12). Taking $\varepsilon = \nu/(2c)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma \big(\big|v^{(N)}\big|^2 + \alpha\big|\nabla v^{(N)}\big|^2\big)\,\mathrm{d}x' + \nu\int_\sigma \big|\nabla v^{(N)}\big|^2\,\mathrm{d}x' \\ \le c\delta_0^2\big\|\nabla\Delta v^{(N)}\big\|^2_{L_2(\sigma)} + c\int_\sigma \big|f^{(N)}\big|^2\,\mathrm{d}x'. \tag{19}$$

Denote

$$\Phi^{(N)}(x',t) = (\mathbf{U}'\cdot\nabla)\big(v^{(N)} - \alpha\Delta v^{(N)}\big) - \nu\Delta v^{(N)} - f^{(N)}. \tag{20}$$

Since the eigenfunctions $w_k \in W_2^4(\sigma)$, it follows that $\Phi^{(N)} \in W_2^1(\sigma)$.

Let us rewrite equalities (14) in the form:

$$\int_\sigma \big(\partial_t v^{(N)} w_k + \alpha\nabla\partial_t v^{(N)}\cdot\nabla w_k\big)\,\mathrm{d}x' + \int_\sigma \Phi^{(N)} w_k\,\mathrm{d}x' = 0. \tag{21}$$

Denote by $W^{(N)}(\cdot,t) \in \mathring{W}_2^1(\sigma)\cap W_2^3(\sigma)$ the solution of the following problem:

$$\begin{aligned} &-\alpha\Delta W^{(N)} + W^{(N)} = \Phi^{(N)},\\ &W^{(N)}\big|_{\partial\sigma} = 0. \end{aligned} \tag{22}$$

Then

$$\int_\sigma \big(\alpha\nabla W^{(N)}\cdot\nabla\eta + W^{(N)}\eta\big)\,\mathrm{d}x' = \int_\sigma \Phi^{(N)}\eta\,\mathrm{d}x' \quad \forall\eta\in\mathring{W}{}_2^1(\sigma). \tag{23}$$

Taking in (22) $\eta = w_k$ we obtain from (21) the relations

$$\int_\sigma \big(\partial_t v^{(N)} w_k + \alpha\nabla\partial_t v^{(N)}\cdot\nabla w_k\big)\,\mathrm{d}x' + \int_\sigma \big(W^{(N)} w_k + \alpha\nabla W^{(N)}\cdot\nabla w_k\big)\,\mathrm{d}x' = 0.$$

By the definition of the eigenfunctions w_k (see (9)) we can rewrite the last equalities in the form

$$\frac{1}{\lambda_k}\int_\sigma \big(\partial_t v^{(N)} w_k + \alpha\nabla\partial_t v^{(N)}\cdot\nabla w_k + \nabla\big(\partial_t v^{(N)} - \alpha\Delta\partial_t v^{(N)}\big)\cdot\nabla(w_k - \alpha\Delta w_k)\big)\,\mathrm{d}x' \\ + \frac{1}{\lambda_k}\int_\sigma \big(W^{(N)} w_k + \alpha\nabla W^{(N)\nabla w_k} + \nabla\big(W^{(N)} - \alpha\Delta W^{(N)}\big)\cdot\nabla(w_k - \alpha\Delta w_k)\big)\,\mathrm{d}x' \\ = 0.$$

Multiplying these relation by $\lambda_k y_k^{(N)}(t)$ and summing from 1 to N yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\sigma \big(\big|v^{(N)}\big|^2+\alpha\big|\nabla v^{(N)}\big|^2+\big|\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big|^2\big)\,\mathrm{d}x' + \int\limits_\sigma \big(W^{(N)}v^{(N)}+\alpha\nabla W^{(N)}\cdot\nabla v^{(N)}\big)\,\mathrm{d}x' + \int\limits_\sigma \nabla\big(W^{(N)}-\alpha\Delta W^{(N)}\big)\cdot\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\,\mathrm{d}x' = 0.$$

From (22), (23) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\sigma \big(|v^{(N)}|^2+\alpha\big|\nabla v^{(N)}\big|^2+\big|\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big|^2\big)\,\mathrm{d}x' + \int\limits_\sigma \Phi^{(N)}v^{(N)}\,\mathrm{d}x' + \int\limits_\sigma \nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\cdot\nabla\Phi^{(N)}\,\mathrm{d}x' = 0. \tag{24}$$

Substituting the expression (20) of the function $\Phi^{(N)}$ into (24) gives:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\sigma \big(\big|v^{(N)}\big|^2+\alpha\big|\nabla v^{(N)}\big|^2+\big|\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big|^2\big)\,\mathrm{d}x' \\ &\quad+\alpha\int\limits_\sigma \mathbf{U}'\cdot\nabla v^{(N)}\Delta v^{(N)}\,\mathrm{d}x' + \nu\int\limits_\sigma \big|\nabla v^{(N)}\big|^2\,\mathrm{d}x' \\ &\quad+\int\limits_\sigma \nabla\big[(\mathbf{U}\cdot\nabla)\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big]\cdot\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\,\mathrm{d}x' \\ &\quad+\frac{\nu}{\alpha}\int\limits_\sigma \big|\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big|^2\,\mathrm{d}x' - \frac{\nu}{\alpha}\int\limits_\sigma \nabla v^{(N)}\cdot\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\,\mathrm{d}x' \\ &=\int\limits_\sigma f^{(N)}v^{(N)}\,\mathrm{d}x' + \int\limits_\sigma \nabla f^{(N)}\cdot\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\,\mathrm{d}x'.\end{aligned} \tag{25}$$

The right-hand side of (25) contains the term with the fourth order derivative. However, this term can be estimated by the integral containing only derivatives up to the third order. The operations below are correct because the functions w_k belong to the space $W_2^4(\sigma)$. Denote for simplicity $v^{(N)}-\alpha\Delta v^{(N)}=u$. We have

$$\begin{aligned}&\int\limits_\sigma \nabla\big[(\mathbf{U}'\cdot\nabla)\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\big]\cdot\nabla\big(v^{(N)}-\alpha\Delta v^{(N)}\big)\,\mathrm{d}x' \\ &\quad=\int\limits_\sigma \nabla\big[(\mathbf{U}'\cdot\nabla)u\big]\cdot\nabla u\,\mathrm{d}x' = \int\limits_\sigma \{\nabla\mathbf{U}'\cdot\nabla\}u\cdot\nabla u\,\mathrm{d}x' + \int\limits_\sigma (\mathbf{U}'\cdot\nabla)\nabla u\cdot\nabla u\,\mathrm{d}x'\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_\sigma \{\nabla \mathbf{U}' \cdot \nabla\} u \cdot \nabla u \, \mathrm{d}x' + \frac{1}{2} \int\limits_\sigma (\mathbf{U}' \cdot \nabla) |\nabla u|^2 \, \mathrm{d}x' \\
&= \int\limits_\sigma \{\nabla \mathbf{U}' \cdot \nabla\} u \cdot \nabla u \, \mathrm{d}x' - \frac{1}{2} \int\limits_\sigma \nabla \cdot \mathbf{U}' |\nabla u|^2 \, \mathrm{d}x' = \int\limits_\sigma \{\nabla \mathbf{U}' \cdot \nabla\} u \cdot \nabla u \, \mathrm{d}x' \\
&\le c \int\limits_\sigma |\nabla \mathbf{U}'| |\nabla u|^2 \, \mathrm{d}x' \le c \sup_{x' \in \sigma} |\nabla \mathbf{U}'| \int\limits_\sigma |\nabla u|^2 \, \mathrm{d}x' \le c\delta_0 \int\limits_\sigma |\nabla u|^2 \, \mathrm{d}x' \\
&= c\delta_0 \int\limits_\sigma \big|\nabla v^{(N)} - \alpha \nabla \Delta v^{(N)}\big|^2 \, \mathrm{d}x'. \qquad (26)
\end{aligned}
$$

Here we have denoted $\{\nabla \mathbf{U}' \cdot \nabla\} = \nabla U_1 \partial_{x_1} + \nabla U_2 \partial_{x_2}$ and applied (18).

Using (26) from (25) we get the estimate

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_\sigma \big(\big|v^{(N)}\big|^2 + \alpha \big|\nabla v^{(N)}\big|^2 + \big|\nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big|^2\big) \, \mathrm{d}x' \\
&\quad + \nu \int\limits_\sigma \big|\nabla v^{(N)}\big|^2 \, \mathrm{d}x' + \frac{\nu}{\alpha} \int\limits_\sigma \big|\nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big|^2 \, \mathrm{d}x' \\
&\quad = \alpha \int\limits_\sigma \mathbf{U}' \cdot \nabla \Delta v^{(N)} v^{(N)} \, \mathrm{d}x' + \frac{\nu}{\alpha} \int\limits_\sigma \nabla v^{(N)} \cdot \nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big) \, \mathrm{d}x' \\
&\quad + \int\limits_\sigma \nabla\big[(\mathbf{U}' \cdot \nabla)\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big] \cdot \nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big) \, \mathrm{d}x' \\
&\quad + \int\limits_\sigma f^{(N)} v^{(N)} \, \mathrm{d}x' + \int\limits_\sigma \nabla f^{(N)} \cdot \nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big) \, \mathrm{d}x' \\
&\quad \le \alpha \sup_{x' \in \sigma} |\mathbf{U}'| \big\|\nabla \Delta v^{(N)}\big\|_{L_2(\sigma)} \big\|v^{(N)}\big\|_{L_2(\sigma)} \\
&\quad + \frac{\nu}{\alpha} \big\|\nabla v^{(N)}\big\|_{L_2(\sigma)} \big\|\nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big\|_{L_2(\sigma)} \\
&\quad + \frac{1}{2\varepsilon} \bigg(\int\limits_\sigma \big|f^{(N)}\big|^2 \, \mathrm{d}x' + \int\limits_\sigma \big|\nabla f^{(N)}\big|^2 \, \mathrm{d}x' \bigg) \\
&\quad + \bigg(\frac{\varepsilon}{2} + c\delta_0\bigg) \int\limits_\sigma \big|\nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big|^2 \, \mathrm{d}x' + \frac{\varepsilon}{2} \int\limits_\sigma \big|\nabla v^{(N)}\big|^2 \, \mathrm{d}x' \\
&\quad \le \frac{c\delta_0^2}{\varepsilon} \int\limits_\sigma \big|\nabla \Delta v^{(N)}\big|^2 \, \mathrm{d}x' + \varepsilon \int\limits_\sigma \big|\nabla v^{(N)}\big| \, \mathrm{d}x' + \frac{c}{\varepsilon} \int\limits_\sigma \big|\nabla v^{(N)}\big| \, \mathrm{d}x' \\
&\quad + (\varepsilon + c\delta_0) \int\limits_\sigma \big|\nabla\big(v^{(N)} - \alpha \Delta v^{(N)}\big)\big|^2 \, \mathrm{d}x'
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2\varepsilon}\int\limits_{\sigma}\big|f^{(N)}\big|^2\,\mathrm{d}x' + \frac{1}{2\varepsilon}\int\limits_{\sigma}\big|\nabla f^{(N)}\big|^2\,\mathrm{d}x' \\
&\le \left(\frac{c_1\delta_0^2}{\varepsilon} + \varepsilon + c_2\delta_0\right)\int\limits_{\sigma}\big|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\big|^2\,\mathrm{d}x' + \varepsilon\int\limits_{\sigma}\big|\nabla v^{(N)}\big|\,\mathrm{d}x' \\
&+\frac{c(1+\delta_0^2)}{\varepsilon}\int\limits_{\sigma}\big|\nabla v^{(N)}\big|\,\mathrm{d}x' + \frac{1}{2\varepsilon}\int\limits_{\sigma}\big|f^{(N)}\big|^2\,\mathrm{d}x' + \frac{1}{2\varepsilon}\int\limits_{\sigma}\big|\nabla f^{(N)}\big|^2\,\mathrm{d}x'.
\end{aligned}
$$

Taking $\varepsilon = \min\{\nu/2, \nu/(4\alpha)\}$ and assuming that δ_0 is sufficiently small, i.e.

$$
\frac{c_1\delta_0^2}{\varepsilon} + c_2\delta_0 \le \frac{\nu}{4\alpha}, \tag{27}
$$

from the latter inequality we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\sigma}\big(\big|v^{(N)}\big|^2 + \alpha\big|\nabla v^{(N)}\big|^2 + \big|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\big|^2\big)\,\mathrm{d}x' \\
&\quad + \nu\int\limits_{\sigma}\big|\nabla v^{(N)}\big|^2\,\mathrm{d}x' + \frac{\nu}{\alpha}\int\limits_{\sigma}\big|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\big|^2\,\mathrm{d}x' \\
&\le c\bigg(\int\limits_{\sigma}\big|f^{(N)}\big|^2\,\mathrm{d}x' + \int\limits_{\sigma}\big|\nabla f^{(N)}\big|^2\,\mathrm{d}x' + \int\limits_{\sigma}\big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'\bigg).
\end{aligned} \tag{28}
$$

Integrating inequality (19) by t gives the estimates

$$
\begin{aligned}
&\int\limits_{\sigma}\big(\big|v^{(N)}(x',t)\big|^2 + \alpha\big|\nabla v^{(N)}(x',t)\big|^2\big)\,\mathrm{d}x' + \nu\int\limits_0^t\int\limits_{\sigma}\big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'd\tau \\
&\le c\delta_0^2\int\limits_0^t\int\limits_{\sigma}\big|\nabla\Delta v^{(N)}\big|^2\,\mathrm{d}x'\,\mathrm{d}\tau + c\int\limits_0^t\int\limits_{\sigma}\big|f^{(N)}\big|^2\,\mathrm{d}x'\,\mathrm{d}\tau \\
&\quad + \int\limits_{\sigma}\big(\big|v_0^{(N)}\big|^2 + \alpha\big|\nabla v_0^{(N)}\big|^2\big)\,\mathrm{d}x' \\
&\le c\delta_0^2\int\limits_0^t\int\limits_{\sigma}\big|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\big|^2\,\mathrm{d}x'\,\mathrm{d}\tau + c_3\delta_0^2\int\limits_0^t\int\limits_{\sigma}\big|\nabla v^{(N)}\big|^2\,\mathrm{d}x'\,\mathrm{d}\tau \\
&\quad + c\int\limits_0^t\int\limits_{\sigma}\big|f^{(N)}\big|^2\,\mathrm{d}x'\,\mathrm{d}\tau + \int\limits_{\sigma}\big(\big|v_0^{(N)}\big|^2 + \alpha\big|\nabla v_0^{(N)}\big|^2\big)\,\mathrm{d}x'.
\end{aligned}
$$

If

$$
c_3\delta_0^2 \le \frac{\nu}{2}, \tag{29}
$$

the last inequality yields the estimate

$$\int_0^t \int_\sigma \left|\nabla v^{(N)}\right|^2 \mathrm{d}x' \,\mathrm{d}\tau \le c\delta_0^2 \int_0^t \int_\sigma \left|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\right|^2 \mathrm{d}x' \,\mathrm{d}\tau + c\int_0^t \int_\sigma \left|f^{(N)}\right|^2 \mathrm{d}x' \,\mathrm{d}\tau + \int_\sigma \big(\left|v_0^{(N)}\right|^2 + \alpha\left|\nabla v_0^{(N)}\right|^2\big) \,\mathrm{d}x'. \tag{30}$$

Integrating inequality (28) with respect to t and estimating the last term in right-hand side by (30) we obtain

$$\begin{aligned}
&\int_\sigma \big(\left|v^{(N)}(x',t)\right|^2 + \alpha\left|\nabla v^{(N)}(x',t)\right|^2 + \left|\nabla(v^{(N)}(x',t) - \alpha\Delta v^{(N)}(x',t))\right|^2\big) \,\mathrm{d}x' \\
&\quad + \nu\int_0^t \int_\sigma \left|\nabla v^{(N)}\right|^2 \mathrm{d}x' \,\mathrm{d}\tau + \int_0^t \int_\sigma \left|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\right|^2 \mathrm{d}x' \,\mathrm{d}\tau \\
&\le c\bigg(\int_0^t \int_\sigma \left|f^{(N)}\right|^2 \mathrm{d}x' \,\mathrm{d}\tau + \int_0^t \int_\sigma \left|\nabla f^{(N)}\right|^2 \mathrm{d}x' \,\mathrm{d}\tau\bigg) \\
&\quad + c_4\delta_0^2 \int_0^t \int_\sigma \left|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\right|^2 \mathrm{d}x' \,\mathrm{d}\tau \\
&\quad + c\int_\sigma \big(\left|v_0^{(N)}\right|^2 + \alpha\left|\nabla v_0^{(N)}\right|^2 + \left|\nabla(v_0^{(N)} - \alpha\Delta v_0^{(N)})\right|^2\big) \,\mathrm{d}x'.
\end{aligned} \tag{31}$$

Assuming that

$$c_4\delta_0^2 \le \frac{1}{2}, \tag{32}$$

from (31) we derive the estimate

$$\begin{aligned}
&\int_\sigma \big(\left|v^{(N)}(x',t)\right|^2 + \alpha\left|\nabla v^{(N)}(x',t)\right|^2 + \left|\nabla\big(v^{(N)}(x',t) - \alpha\Delta v^{(N)}(x',t)\big)\right|^2\big) \,\mathrm{d}x' \\
&\quad + \int_0^t \int_\sigma \big(\nu\left|\nabla v^{(N)}\right|^2 + \left|\nabla\big(v^{(N)} - \alpha\Delta v^{(N)}\big)\right|^2\big) \,\mathrm{d}x' \,\mathrm{d}\tau \\
&\le c\int_0^t \int_\sigma \big(\left|f^{(N)}\right|^2 + \left|\nabla f^{(N)}\right|^2\big) \,\mathrm{d}x' \,\mathrm{d}\tau \\
&\quad + c\int_\sigma \big(\left|v_0^{(N)}\right|^2 + \alpha\left|\nabla v_0^{(N)}\right|^2 + \left|\nabla\big(v_0^{(N)} - \alpha\Delta v_0^{(N)}\big)\right|^2\big) \,\mathrm{d}x'.
\end{aligned} \tag{33}$$

Let us multiply equalities (14) by $\mathrm{d}y_k^{(N)}(t)/\mathrm{d}t$ and sum them by k from 1 to N:

$$\int_\sigma \left(\left|\partial_t v^{(N)}\right|^2 + \alpha\left|\nabla\partial_t v^{(N)}\right|^2\right)\mathrm{d}x' + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma \left|\nabla v^{(N)}\right|^2 \mathrm{d}x'$$
$$= \int_\sigma (\mathbf{U}'\cdot\nabla)\left(v^{(N)} - \alpha\Delta v^{(N)}\right)\partial_t v^{(N)}\,\mathrm{d}x' + \int_\sigma f^{(N)}\partial_t v^{(N)}\,\mathrm{d}x'$$
$$\le \frac{1}{2\varepsilon}\sup_{x'\in\sigma}\left|\mathbf{U}'\right|^2 \int_\sigma \left|\nabla\left(v^{(N)} - \alpha\Delta v^{(N)}\right)\right|^2 \mathrm{d}x'$$
$$+ \varepsilon\int_\sigma \left|\partial_t v^{(N)}\right|^2 \mathrm{d}x' + \frac{1}{2\varepsilon}\int_\sigma \left|f^{(N)}\right|^2 \mathrm{d}x'.$$

Taking $\varepsilon = 1/2$, integrating with respect to t and applying inequalities (18) and (33) we obtain

$$\nu\int_\sigma \left|\nabla v^{(N)}(x',t)\right|^2 \mathrm{d}x' + \int_0^t\int_\sigma \left(\left|\partial_\tau v^{(N)}\right|^2 + \alpha\left|\nabla\partial_\tau v^{(N)}\right|^2\right)\mathrm{d}x'\,\mathrm{d}\tau$$
$$\le c\int_0^t\int_\sigma \left(\left|f^{(N)}\right|^2 + \left|\nabla f^{(N)}\right|^2\right)\mathrm{d}x'\,\mathrm{d}\tau$$
$$+ c\int_\sigma \left(\left|v_0^{(N)}\right|^2 + \alpha\left|\nabla v_0^{(N)}\right|^2 + \left|\nabla\left(v_0^{(N)} - \alpha\Delta v_0^{(N)}\right)\right|^2\right)\mathrm{d}x'. \tag{34}$$

Estimate (17) follows from (33) and (34) and the definitions of the norms. □

5 Existence and uniqueness of the solution

Theorem 2. *Suppose that $\partial\sigma \in C^4$, $f \in W_2^1(\sigma^T)$, $v_0 \in W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$ and $\mathbf{U}' \in \mathring{\mathcal{V}}(\sigma^T)$ satisfies condition* (12) *with δ_0 subject to inequalities* (27), (29), (32). *Then problem* (1) *admits a unique week solution $v \in \mathring{\mathcal{W}}(\sigma^T)$ and the following estimate holds*:

$$\sup_{t\in[0,T]}\|v\|_{\mathring{X}(\sigma)} + \|v\|_{\mathring{\mathcal{W}}(\sigma^T)} \le c\left(\|f\|_{W_2^1(\sigma^T)} + \|v_0\|_{\mathring{X}(\sigma)}\right). \tag{35}$$

Proof. Multiplying equations (15) by arbitrary function $\eta \in \mathring{W}_2^{1,0}(\sigma^T)$ and integrating by parts in σ and by t we get the following integral identity:

$$\int_0^t\int_\sigma \left(\partial_\tau v^{(N)}\eta + \alpha\nabla\partial_\tau v^{(N)}\cdot\nabla\eta\right)\mathrm{d}x'\,\mathrm{d}\tau + \nu\int_0^t\int_\sigma \nabla v^{(N)}\cdot\nabla\eta\,\mathrm{d}x'\,\mathrm{d}\tau$$
$$= \int_0^t\int_\sigma f^{(N)}\eta\,\mathrm{d}x'\,\mathrm{d}\tau + \int_0^t\int_\sigma \mathbf{U}'\cdot\nabla\eta\left(v^{(N)} - \alpha\Delta v^{(N)}\right)\mathrm{d}x'\,\mathrm{d}\tau \quad \forall t\in[0,T]. \tag{36}$$

From estimates (33), (34) it follows that there exists a subsequence $\{v^{(N_l)}\}$ such that

$$v^{(N_l)}(\cdot,t) \rightharpoonup v(\cdot,t) \quad \text{in } \mathring{X}(\sigma)\ \forall t\in[0,T], \quad v^{(N_l)} \rightharpoonup v \quad \text{in } \mathring{\mathcal{W}}(\sigma^T).$$

Passing in (36) to a limit as $N_l \to \infty$ we obtain for v the integral identity (13). Obviously, v satisfies the initial condition. Moreover, from the inequality (17) follows the estimate (35).

Let us prove the uniqueness. Let $v^{[1]}$ and $v^{[2]}$ be two weak solutions of problem (1). The difference $V = v^{[1]} - v^{[2]}$ satisfies the integral identity

$$\begin{aligned}\int_0^t\int_\sigma (\partial_\tau V\eta + \alpha\partial_\tau\nabla V\cdot\nabla\eta)\,\mathrm{d}x'\,\mathrm{d}\tau + \nu\int_0^t\int_\sigma \nabla V\cdot\nabla\eta\,\mathrm{d}x'\,\mathrm{d}\tau \\ = \int_0^t\int_\sigma \mathbf{U}'\cdot\nabla\eta(V-\alpha\Delta V)\,\mathrm{d}x'\,\mathrm{d}\tau \quad \forall\eta\in \mathring{W}_2^{1,0}(\sigma^T).\end{aligned}$$

Taking $\eta = V$ yields

$$\frac{1}{2}\int_\sigma \bigl(|V|^2+\alpha|\nabla V|^2\bigr)\,\mathrm{d}x' + \nu\int_0^t\int_\sigma |\nabla V|^2\,\mathrm{d}x'\,\mathrm{d}\tau = -\alpha\int_0^t\int_\sigma \mathbf{U}'\cdot\nabla V\Delta V\,\mathrm{d}x'\,\mathrm{d}\tau.$$

Integrating by parts in the right-hand side term we get

$$\begin{aligned}&-\alpha\int_0^t\int_\sigma \mathbf{U}'\cdot\nabla V\Delta V\,\mathrm{d}x'\,\mathrm{d}\tau \\ &= \alpha\int_0^t\int_\sigma\Biggl[-\frac{1}{2}\sum_{i=1}^2\nabla\cdot\mathbf{U}'(\partial_{x_i}V)^2 + \sum_{i=1}^2\sum_{j=1}^2\partial_{x_i}U_j\partial_{x_j}V\partial_{x_i}V\Biggr]\,\mathrm{d}x'\,\mathrm{d}\tau \\ &\le \alpha\int_0^t \sup_{x'\in\sigma}|\nabla\mathbf{U}'|\int_\sigma|\nabla V|^2\,\mathrm{d}x'\,\mathrm{d}\tau \le c\int_0^t\|\mathbf{U}'\|_{W_2^3(\sigma)}\int_\sigma|\nabla V|^2\,\mathrm{d}x'\,\mathrm{d}\tau.\end{aligned}$$

Therefore,

$$\int_\sigma|\nabla V|^2\,\mathrm{d}x' \le c\int_0^t\|\mathbf{U}'\|_{W_2^3(\sigma)}\int_\sigma|\nabla V|^2\,\mathrm{d}x'\,\mathrm{d}\tau.$$

By Gronwall's inequality $\int_\sigma|\nabla V|^2\,\mathrm{d}x' \le 0$ and, since $V|_{\partial\sigma} = 0$, we conclude $V(x',t) = 0$. Thus, $v^{[1]} = v^{[2]}$. □

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