Solitary wave fission and fusion in the $(2+1)$-dimensional generalized Broer–Kaup system*

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Abstract. Via a special Painlevé–Bäcklund transformation and the linear superposition theorem, we derive the general variable separation solution of the $(2 + 1)$-dimensional generalized Broer–Kaup system. Based on the general variable separation solution and choosing some suitable variable separated functions, new types of V-shaped and A-shaped solitary wave fusion and Y-shaped solitary wave fission phenomena are reported.

Keywords: variable separation solution, solitary wave fission and fusion, $(2 + 1)$-dimensional generalized Broer–Kaup system.

1 Introduction

Modern soliton theory is widely applied in many natural science such as chemistry, biology, mathematics, communication and particularly in almost all branches of physics like the fluid dynamics, plasma physics, field theory, nonlinear optics and condensed matter physics, and so on. In the past few decades, a vast variety of methods has been developed to obtain exact solitonic solutions to a given nonlinear partial differential equation (NLPDE), such as the the Painlevé expansion method [1, 2], the tanh-function method [3–5], the Jacobian elliptic function method [6], the the exp-function method [7, 8] and the similarity transformation method [9, 10] etc.

As one of the effective methods in linear physics, the variable separation approach (VSA) has been successfully extended to nonlinear domains. The multilinear variable separation approach (MLVSA) has also been established for various (1+1)-dimensional [11], $(2 + 1)$-dimensional [12] and $(3 + 1)$-dimensional [13] models. Recently, along with linear variable separation idea, the extended tanh-function method (ETM) [14–16] and the general projective Riccati equation method [17] based on mapping method have also

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been successfully generalized to obtain variable separation solutions for many (1+1)-dimensional, $(2+1)$-dimensional and $(3+1)$-dimensional models. Based on these variable separation solutions, abundant localized coherent structures such as lumps, dromions, peakons, compactons, foldons, ring solitons, were discussed.

Conventionally, a collision between solitons of integrable models is regarded to be completely elastic. That is to say, the amplitude, velocity and wave shape of a soliton do not undergo any change after the nonlinear interaction. However, for some special solutions of certain $(2+1)$-dimensional models, the interactions among solitonic excitations are not completely elastic since their shapes or amplitudes are changed after their collisions [18–22]. Moreover, the solitary wave fusion and fission phenomena have been reported for (1+1)-dimensional [23] and (2+1)-dimensional [24,25] models. Actually, the solitary wave fusion and fission phenomena have been observed in many physical fields like plasma physics, nuclear physics and hydrodynamics and so on [26]. In [18–25], it was only based on two variable separated functions that many localized coherent structures were discussed. To the best of our knowledge, the studies on the general solution with arbitrary number of variable separated functions, and especially on solitons with fusion and fission properties were not reported for the $(2+1)$-dimensional generalized Broer–Kaup (GBK) system [27–29]

$$\begin{aligned} H_t &= -4\big(H_{xx} + H^3 - 3HH_x + 3H\partial_y^{-1}V_x + 3\partial_y^{-1}(VH)_x\big)_x, \\ V_t &= -4\big(V_{xx} + 3HV_x + 3H^2V + 3V\partial_y^{-1}V_x\big)_x, \end{aligned} \tag{1}$$

which was first obtained from the inner parameter dependent symmetry constraints of the KP equation [27]. This equation (1) is a particular form of higher order Boussinesq equation. GBK is actually an extension of the so called Whitham–Broer–Kaup system (WBK) using Painlevé analysis. WBK is a valuable model for long waves by incorporating or mimicking convective, dispersive and viscous effects. When we take $x = y$, the system (1) is reduced to the usual $(1+1)$-dimensional GBK system. In [28], authors obtained many new exact solutions for the $(2+1)$-dimensional HBK system, including new soliton-like solutions and periodic solutions. Moreover, some special types of the kink soliton solution, periodic soliton solutions and lattice soliton solutions of this system (1) were discussed in [29].

2 Variable separation solution for the $(2+1)$-dimensional GBK system

In order to get the special solution of model (1), by making a transformation $W = \partial_y^{-1}V_x$, $P = \partial_y^{-1}(VH)_x$, and by cosmetic transformations as rescaling the time t and the dependent variables V, W and P to get rid of the factors "4" and "3", we are able to rewrite (1) in the following potential form:

$$\begin{aligned} &H_t + \big(H_{xx} + H^3 - 3HH_x + HW + P\big)_x = 0, \\ &V_t + \big(V_{xx} + 3V_xH + 3H^2V + VW\big)_x = 0, \\ &W_y - V_x = 0, \quad P_y - (VH)_x = 0. \end{aligned} \tag{2}$$

Via the standard truncated Painlevé expansion [12], we have a special Painlevé–Bäcklund transformation for differentiable functions H, V, W and P in (2)

$$\begin{aligned} H &= (\ln f)_x + H_0, \quad & V &= (\ln f)_{xy} + V_0, \\ W &= (\ln f)_{xx} + W_0, \quad & P &= (\ln f)_x (\ln f)_{xx} + P_0, \end{aligned} \tag{3}$$

where $f = f(x, y, t)$ is an arbitrary differentiable function of variables $\{x, y, t\}$ to be determined, and H_0, V_0, W_0, P_0 are arbitrary seed solutions satisfying the GBK system. In usual cases, by choosing some special trivial solutions, we can directly obtain the seed solutions. In the present case, via some simple calculations, it is evident that Eq. (1) possesses trivial seed solutions

$$H_0 = V_0 = 0, \quad W_0 = W_0(x, t), \quad P_0 = P_0(x, t), \tag{4}$$

where $W_0(x, t)$ and $P_0(x, t)$ are an arbitrary function of $\{x, t\}$.

Substituting (3) with the seed solutions (4) into (2) yields

$$f_t + f_{xxx} + W_0 f_x + P_0 f = 0. \tag{5}$$

Since (5) is only a linear equation, one can certainly utilize the linear superposition theorem. For instance

$$f = Q_0(y) + \sum_{i=1}^{N} P_i(x, t) Q_i(y, t), \tag{6}$$

where the variable separated functions $P_i(x, t) \equiv P_i$ and $Q_i(y, t) \equiv Q_i$ $(i = 1, 2, \ldots, N)$ are only the functions of $\{x, t\}$ and $\{y, t\}$, respectively, and $Q_0(y) \equiv Q_0$. Inserting the ansatz (6) into Eq. (5), we can obtain the following simple variable separated equations:

$$P_{it} + P_{ixxx} + W_0 P_{ix} + \bigl[b_i(t) + P_0\bigr] P_i = 0, \tag{7}$$

$$Q_{it} - b_i(t) Q_i = 0, \tag{8}$$

where $b_i(t)$ $(i = 1, 2, \ldots, N)$ are arbitrary functions of indicated variable.

Substituting all the results into (3), we obtain the corresponding exact solution of (2). Especially we are interested in the structure of the solution for the field V which has the final form

$$V = \frac{\sum_{i=1}^{N} P_{ix} Q_{iy}}{Q_0 + \sum_{i=1}^{N} P_i Q_i} - \frac{\sum_{i=1}^{N} P_{ix} Q_i [Q_{0y} + \sum_{i=1}^{N} P_i Q_{iy}]}{[Q_0 + \sum_{i=1}^{N} P_i Q_i]^2}, \tag{9}$$

where P_i and Q_i satisfy (7) and (8).

Case 1. If we consider the simplest case: $N = 1, Q_0 = \alpha_1(y)$, $\{P_1, Q_1\} = \{\beta(x, t), \alpha_2(y)\}$, then the result in [30] can be recovered. If selecting $N = 1$, $Q_0 = q(y)$, $\{P_1, Q_1\} = \{p(x, t), 1\}$, then (9) changes into the simple one

$$V = -\frac{p_x q_y}{(p + q)^2},$$

which is the same one in [30].

Case 2. In a similar way, when we consider the case: $N = 3$, $Q_0 = a_0$, $Q_1 = a_1$, $Q_2 = a_2Q$, $Q_3 = a_3Q$, ($a_i = consts$, $i = 0, \ldots, 3$), $P_1 = P_3 = P$, $P_2 = 1$ with $P \equiv P(x,t)$ and $Q \equiv Q(y,t)$, then (7) and (8) change into

$$P_t + P_{xxx} + W_0 P_x + \big[b(t) + P_0\big]P = 0, \tag{10}$$

$$Q_t - b(t)Q = 0. \tag{11}$$

Solving Eq. (10) is very difficult. However, because of the arbitrariness of W_0 and P_0, we can treat the problem alternatively. Actually, we can consider the functions P as arbitrary functions while W_0 and P_0 is fixed by Eq. (10). Therefore, W_0, P_0 and Q can be derived by solving (10) and (11), respectively, while the function P is considered as arbitrary function of $\{x, t\}$. Therefore we obtain another special exact excitation

$$V = -\frac{(a_0a_3 - a_1a_2)P_xQ_y}{[a_0 + a_1P + a_2Q + a_3PQ]^2},$$

which is valid for many other types of known $(2 + 1)$-dimensional integrable models like the Davey–Stewartsen (DS) equation, the dispersive long wave (DLW) equation, the Broer–Kaup–Kupershimidt (BKK) system, the Nizhnik–Novikov–Veselov (NVV) equation, etc. [12].

Evidently, in the variable separation solution (9), there are arbitrary number of variable separated functions. Usually, these variable separated functions are not all arbitrary. All the N functions Q_i are the solutions of (8). While for the other $N + 2$ functions, W_0, P_0 and P_i, only one of them can be considered to be arbitrary but the others should be the solutions of the equation system (7).

Here we do not discuss all the possible exact solutions of the equation system (7). Actually even in some special cases, $W_0 = P_0 = b_i = 0$, this is still very interesting for the exact variable solutions of the HBK system. In this special case, the P_i and Q_i in (9) have the form

$$P_i = \exp\big(k_i x - k_i^3 t + c_i\big), \quad Q_0 = Q_i = Q(y) \quad \forall i,$$

where Q are the arbitrary function of $\{y\}$.

3 Soliton fission and fusion phenomena

It is known that based on special exact solution (8), many localized coherent structures, such as lumps, dromions, peakons, compactons, foldons, ring solitons, were discussed in [12–25, 30, 31]. However, it was based on two variable separated functions that these localized coherent structures above mentioned were discussed. To the best of our knowledge, the studies on the general solution with more variable separated functions, and especially on V-shaped and A-shaped soliton fusion and Y-shaped soliton fission for the $(2+1)$-dimensional GBK system (1) were not reported. Here we will discuss these special fusion and fission phenomena in the $(2 + 1)$-dimensional GBK system.

3.1 Y-shaped soliton fission

In [31], the authors said: “Though we have not yet found the Y soliton fission phenomenon for the $(2+1)$-dimensional Burgers system, we do believe that it may exist in other $(2+1)$-dimensional models”. It is interesting to mention that for the $(2+1)$-dimensional GBK system, the resonant Y-shaped solitons may also display the soliton fission phenomenon. In Fig. 1, the profile of Y-shaped solitons fission is displayed for the field with the selections

$$P_i = \exp\left(k_i x - k_i^3 t + c_i\right), \quad Q_0 = 0.5\cos(y), \quad Q_i = \exp(l_i y + C_i) \quad \forall i, \tag{12}$$

and

$$\begin{aligned} &N = 4, \quad k_1 = l_1 = c_1 = C_1 = 0, \quad 2k_2 = c_2 = -C_2 = 1, \\ &2k_3 = l_4 = 2, \quad 2k_4 = -l_2 = c_3 = C_3 = 3, \quad c_4 = -C_4 = 5. \end{aligned} \tag{13}$$

From Fig. 1, one can see that with the increasing time the single Y-shaped soliton propagates from the positive x-axis to the negative x-axis. Then a new branch gradually separates from the point of intersection of the three branches for Y-shaped soliton and the single Y-shaped soliton fissions into three Y-shaped solitons, which constitute a simple spider-web-like soliton.

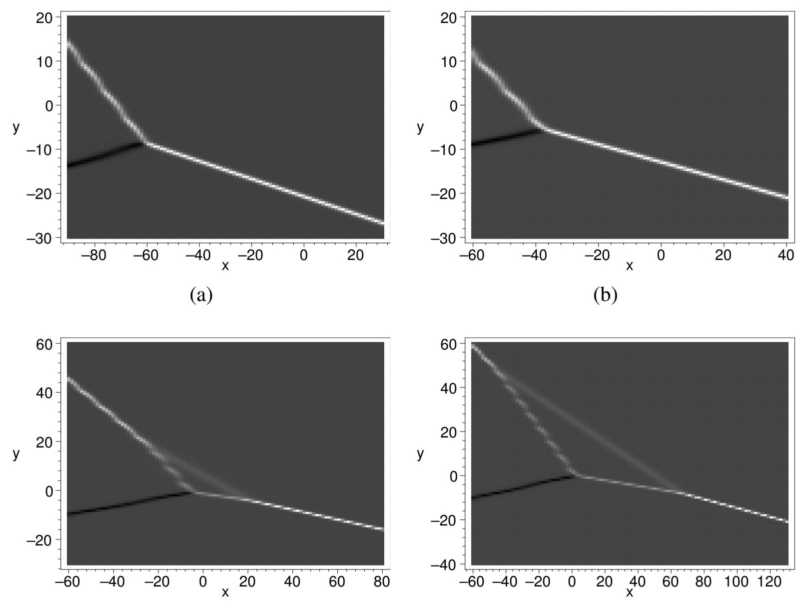

Fig. 1. Profile of the Y-shaped soliton fission interaction for the field V expressed by (9) with (12) and (13) at different time: (a) $t = -8$, (b) $t = -5$, (c) $t = 0$, and (d) $t = 2$, respectively.

3.2 V-shaped and A-shaped solitons fusion

Now we focus our attention on the intriguing fusion phenomenon. In Fig. 2, the plots of V-shaped and A-shaped solitons fusion are revealed for the field expressed by (9) with the selections (12) and

$$\begin{aligned} &N=5, \quad k_1=l_1=c_1=C_1=0, \quad 2k_2=2k_5=-l_3=-2l_4=c_2=-C_2=1, \\ &l_2=-l_5=-2k_4=c_3=C_3=C_5=3, \quad c_5=4, \quad c_4=C_4=5. \end{aligned} \tag{14}$$

From Fig. 2, one can see that the fusion phenomenon possesses apparently different evolutional property compared with fission phenomenon in Fig. 1. Before interaction, there are two separated V-shaped and A-shaped solitons. Then, after interaction, they fuse into to a single web-like soliton constituted by two Y-shaped solitons.

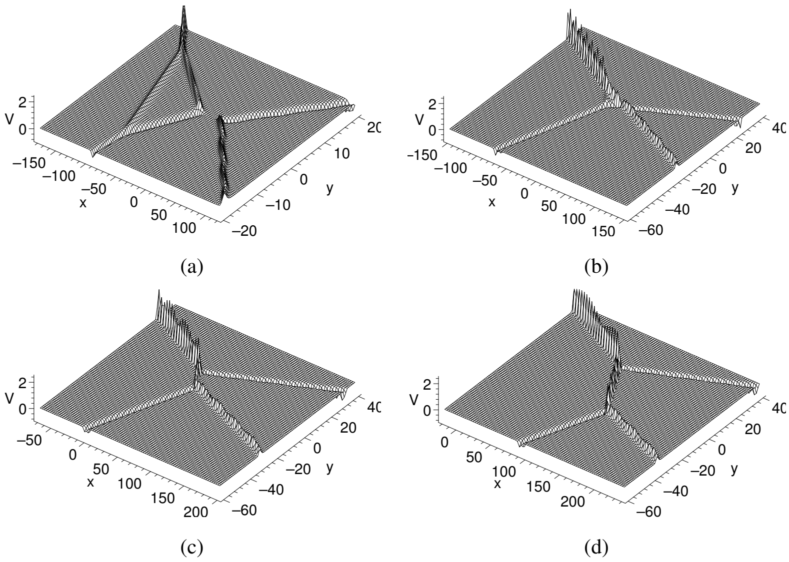

Fig. 2. Plot of the V-shaped and A-shaped solitons fusion interaction for the field V expressed by (9) with (12) and (14) at different time: (a) $t = -5$, (b) $t = -0.4$, (c) $t = 5$, and (d) $t = 12$, respectively.

4 Summary and discussion

In short, with the help of a special Painlevé–Bäcklund transformation and the linear superposition theorem, the general variable separation solution of arbitrary number of variable separated functions of the GBK system is obtained. Based on the general variable separation solution with some suitable variable separated functions, new types of V-shaped and A-shaped soliton fusion and Y-shaped soliton fission are reported. However, lacking theoretical and experiments related to the GBK system, we could not further say something about the real physical meaning of our exact solutions. We hope that in future experimental studies on these soliton fusion and fission phenomena can be realized in some fields.

Though the field possesses zero boundary conditions for expression (7), different selections of the arbitrary functions and correspond to different selections of boundary conditions of field with nonzero initial conditions. The changes of these boundary conditions propagate along the characteristic and then yield the changes of the localized excitations. That means, in some sense, the fusion and fission phenomena for some physical quantities are remote controlled by some other quantities which have nonzero boundary conditions. From the brief analysis in our present paper, we can see that these intriguing soliton fission and fusion phenomena can occur in a higher dimensional soliton system if we choose appropriate initial conditions or boundary conditions.

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